CONSTRUCTING THE VIRASORO GROUP USING DIFFERENTIAL COHOMOLOGY

ARUN DEBRAY
JANUARY 25, 2022

Abstract.

The main theorem, joint with Yu Leon Liu and Christoph Weis, is the construction of the Virasoro groups, a family of central extensions of $\Gamma := \mathrm{Diff}^+(S^1)$ by the circle group $\mathbb{T}$, by transgression of a characteristic class in differential cohomology. I'll explain each of these pieces (the Virasoro groups; differential cohomology; differential lifts of characteristic classes; the transgression) in turn, but first I'll say what the idea of the construction is:

- Begin with $p_1 \in H^*(B\mathrm{GL}_n^+(\mathbb{R}); \mathbb{Z})$. Lift it to a differential cohomology group – not the usual Chern-Weil lift that Cheeger-Simons did. There is an $\mathbb{R}$ worth of lifts.
- Evaluate this class on the vertical tangent bundle on the universal oriented circle bundle $E \to B_\bullet\Gamma$ (this notation means the classifying stack).
- Then integrate along the fiber, obtaining an $\mathbb{R}$ worth of differential cohomology classes for $B_\bullet\Gamma$. These can be interpreted as cocycles defining central extensions, and sure enough these are the Virasoro extensions.

It's possible that this came out of left field to you: it certainly did when Dan Freed and Mike Hopkins suggested it to us. The theorem is similar (albeit with a different proof) to Brylinski-McLaughlin's construction of the Kac-Moody central extensions of loop groups using a different transgression in differential cohomology. And there are close relationships between Kac-Moody groups and Virasoro groups, tied together with the Segal-Sugawara formula (or more fundamentally, the appearance of both as symmetries of the Wess-Zumino-Witten conformal field theories). From that perspective, it's reasonable to conjecture that some sort of transgression procedure can also get you the Virasoro extensions.

1. The Virasoro extensions

Definition 1.1. The Virasoro group with central charge λ is the central extension of Γ by $\mathbb{T}$ which as a space is $\mathbb{T} \times \Gamma$ with multiplication

$$(z_1, \gamma_1) \cdot (z_2, \gamma_2) := (z_1 \dot{z}_1 \cdot B_\lambda(\gamma_1, \gamma_2), \gamma_1 \circ \gamma_2), \tag{1.2}$$

where $B_\lambda \colon \Gamma \times \Gamma \to \mathbb{T}$ is the Bott-Thurston cocycle

$$B_\lambda(\gamma_1, \gamma_2) := \exp\left(-\frac{i}{48} \int_{S^1} \log(\gamma_1' \circ \gamma_2) \, \mathrm{d}(\log \gamma_2')\right). \tag{1.3}$$

The derivative of an orientation-preserving diffeomorphism is a map to $\mathbb{R}$, except the derivative is never 0 (because it's a diffeomorphism) and it's never negative (the diffeomorphism is orientation-preserving). So it lands in $\mathbb{R}^+$, which we regard as a group under multiplication; then $\log \colon \mathbb{R}^+ \to \mathbb{R}$ is the usual natural logarithm.

Now the geography of central extensions of Γ. If you have a central extension, you can do two things to it.

- You can pull it back to $i \colon \mathrm{PSL}_2(\mathbb{R}) \hookrightarrow \Gamma$ (the real fractional linear transformations of $\mathbb{RP}^1 = S^1$). Ultimately because $\pi_1(\mathrm{PSL}_2(\mathbb{R})) = \mathbb{Z}$, there is a $\mathbb{T}$ worth of central extensions of $\mathrm{PSL}_2(\mathbb{R})$ by $\mathbb{T}$.
- You can differentiate to a central extension of Lie algebras. The Lie algebra of Γ is called the Witt algebra $\mathfrak{w}$, and is the algebra of (polynomial) vector fields on S^1. Here I'm being a little imprecise about completions, which does not matter for the groups of central extensions. There is an $\mathbb{R}$ worth of central extensions of the Witt algebra by $\mathbb{R}$.

Theorem 1.4 (Segal). *The map* $(i^*, d) \colon \mathrm{CExt}_{\mathbb{T}}(\Gamma) \to \mathrm{CExt}_{\mathbb{T}}(\mathrm{PSL}_2(\mathbb{R})) \times \mathrm{CExt}_{\mathbb{R}}(\mathfrak{w}) = \mathbb{T} \times \mathbb{R}$ *is an isomorphism.*

That is, a central extension is uniquely determined by its restriction to $\mathrm{PSL}_2(\mathbb{R})$ and its derivative, and all combinations occur. The Virasoro extensions are trivial when restricted to $\mathrm{PSL}_2(\mathbb{R})$; on the Witt algebra, they are the *Virasoro algebra* extensions. These are better-studied.

2. Differential cohomology

The basic idea of differential cohomology is to encode the idea of closed differential forms obeying some sort of integrality condition. For example: Chern-Weil forms corresponding to integer cohomology classes. Also quantization often involves differential forms (field strength, which knows e.g. the electric and magnetic fields), but where the possible values are discrete.

It's convenient, but not quite true, to think of a differential cocycle as data of a closed differential form and some sort of integrality data. This is not literally true at the level of cocycles, but it is true at the level of cohomology theories.

Definition 2.1. A *generalized differential cohomology theory* is (the cohomology theory given by) a sheaf of spectra on the site of smooth manifolds.

"Ordinary differential cohomology" is therefore given by a sheaf of chain complexes of abelian groups. To compute this on a manifold, take hypercohomology of the double complex. (Generalized) differential cohomology groups are not homotopy-invariant, which is what allows them to say useful things about geometry.

Definition 2.2. TODO: $\mathbb{R}(n)$, then $\mathbb{Z}(n)$. $\mathbb{Z}(n)$ is the homotopy pullback.

On a manifold M, an element of $H^n(M;\mathbb{R}(n))$ is a closed n-form. So on the level of complexes of sheaves, the Deligne complex is something like "closed forms, integer cohomology classes, and an identification in de Rham cohomology."

Remark 2.3. There are several other definitions of differential cohomology (Cheeger-Simons, Simons-Sullivan, Hopkins-Singer, ...). The thing people usually call "differential cohomology" is $H^n(-;\mathbb{Z}(n))$, the "on-diagonal" part of what we're looking at, but a major theme of today's talk is that the "off-diagonal" differential cohomology groups are interesting too. ◀

Fiber integration goes here.

Finally, though we defined differential cohomology on manifolds, we can also make sense of it on (some) stacks on $\mathcal{M}an$, by which we mean simplicial sheaves on $\mathcal{M}an$. We can present a stack by a simplicial Fréchet manifold $X_\bullet$: for M a manifold, we want a simplicial set $\mathrm{Map}(M, X_\bullet)$, and we declare that its n-simplices are $\mathrm{Map}(M, X_n)$. The value of a (generalized) differential cohomology theory E on a stack $X_\bullet$ is the hypercohomology of the triple complex $E^*(X_\bullet)$.

Example 2.4. Let G be a Fréchet Lie group. Then $B_\bullet G$ denotes the stack pt/G of principal G-bundles, which admits a presentation by the bar construction $\mathrm{pt} \leftleftarrows\leftleftarrows G\times G\ldots$. The quotient $\mathrm{pt}\to\mathrm{pt}/G = B_\bullet G$ is the universal principal G-bundle in the setting of stacks on $\mathcal{M}an$; it is also possible to present the total space $E_\bullet G$ as a simplicial manifold. ◀

Example 2.5. Now let G be a finite-dimensional Lie group. There is a stack $B_\nabla G$ of principal G-bundles with connection, and a universal principal G-bundle with connection $E_\nabla G\to B_\nabla G$ in the world of stacks. ◀

We can use these ingredients to (finally?) connected differential cohomology to central extensions. Recall that for discrete groups G, central extensions by an abelian group A are classified by $H^2(G;A) \coloneqq H^2(BG;A)$. But for Γ, we want to remember the smooth structure: the Virasoro groups are Fréchet Lie groups. There are various notions of smooth group cohomology which give the right answer, including Segal-Mitchison cohomology; see Wagemann-Wockel for a general comparison theorem. Letting G be a Fréchet Lie group and A be a (finite-dimensional) abelian Lie group, Segal-Mitchison cohomology is naturally isomorphic to $H^2(B_\bullet G;\underline{A})$, where $\underline{A}$ denotes the sheaf of A-valued functions where A carries its Lie group topology.

Now, there is an equivalence of complexes $\mathbb{T}[-1]\simeq\mathbb{Z}(1)$, given by the diagram (TODO). The upshot is that $H^3(B_\bullet G;\mathbb{Z}(1))$ classifies Fréchet Lie group central extensions of G by $\mathbb{T}$!

Our goal is to produce classes in $H^3(B_\bullet\Gamma;\mathbb{Z}(1))$ using differential lifts of characteristic classes.

3. Differential characteristic classes

3.1. The on-diagonal story.

Quick review of Chern-Weil theory: given $P,\Theta\to M$, get a ring homomorphism $I^*(G)\to\Omega^*(M)$ doubling the degree. Here $I^*(G) = \mathrm{Sym}^*(\mathfrak{g}^\vee)^G$, the ring of G-invariant polynomials on $\mathfrak{g}$, where G acts by the adjoint action. Curvature Ω is in $\Omega^2_P(\mathfrak{g})$; $\Omega^{\wedge k}$ in $\Omega^{2k}_P(\mathfrak{g}^{\otimes k})$, evaluate using the polynomial, then Ad-invariance means whatever you get descends to M.

This is natural in P,Θ, which ultimately means it refines to the universal object: a ring homomorphism $CW\colon I^*(G)\to\Omega^*(B_\nabla G)$.

Theorem 3.1 (Freed-Hopkins)**.** *If G is a compact Lie group, the Chern-Weil map defines an isomorphism of DGAs $I^*(G) \to \Omega^*(B_\nabla G)$.*

Note: because we've doubled the degree and put 0s in odd components, $d = 0$.

Where differential cohomology enters the story: suppose that $c \in H^*(BG;\mathbb{Z})$. Its image in $H^*(BG;\mathbb{R})$ is canonically associated to an invariant polynomial by the Chern-Weil map. The data of the lift to integer cohomology gives us: a closed form, an integer cohomology class, and data of an identification of their images in de Rham cohomology. This suggests to us that we should get a differential cohomology class, and:

Theorem 3.2 (Cheeger-Simons, Bunke-Nikolaus-Völkl)**.** *For G a compact Lie group and $c \in H^{2n}(BG;\mathbb{Z})$, there is a unique lift of c to a class $\check{c} \in H^{2n}(B_\nabla G;\mathbb{Z}(2n))$ whose curvature form is the Chern-Weil form associated to the image of c in $H^*(BG;\mathbb{R})$. This lift is natural in c.*

If time, describe how to use this to define Chern-Simons invariants.

What we need today, though, is a different lift, to off-diagonal differential cohomology. People who worked on this in various contexts: Beilinson in an algebro-geometric setting; Bott and Waldorf in the smooth setting.

Theorem 3.3 (Bott)**.** *There is a natural isomorphism $\phi\colon I^n(G) \xrightarrow{\cong} H^{2n}(B_\bullet G;\mathbb{R}(n))$ such that the composition*

$$I^n(G) \xrightarrow{\phi} H^{2n}(B_\bullet G;\mathbb{R}(n)) \xrightarrow{t} H^{2n}(B_\bullet G;\mathbb{R}) = H^{2n}(BG;\mathbb{R}) \tag{3.4}$$

is the Chern-Weil homomorphism.

Note: Bott stated and proved a different-looking theorem, identifying $H^p(B_\bullet G;\Omega^q)$ with the smooth cohomology H^{p-q} of G with coefficients in the G-representation $\mathrm{Sym}^q(\mathfrak{g}^\vee)$. The reinterpretation in terms of off-diagonal differential cohomology is due to Mike Hopkins.

We can rephrase Bott's theorem as follows: for any Lie group G with $\pi_0(G)$ finite, there is a pullback square

$$\begin{array}{ccc} H^{2n}(B_\bullet G;\mathbb{Z}(n)) & \xrightarrow{t} & H^{2n}(BG;\mathbb{Z}) \\ \downarrow & & \downarrow \\ I^n(G) & \xrightarrow{CW} & H^{2n}(BG;\mathbb{R}). \end{array} \tag{3.5}$$

(A priori it's a Mayer-Vietoris sequence, but $H^{2n+1}(BG;\mathbb{R}) = 0$, so it breaks into a bunch of pullback squares.)

If G is compact, the bottom map is an isomorphism, and so the top map is too: every class in $H^{2n}(BG;\mathbb{Z})$ lifts uniquely to $\mathbb{Z}(n)$-cohomology!

And even if G is noncompact, we're still in business. For example, if $G = \mathrm{GL}_n^+(\mathbb{R})$, we can compute the affine space of lifts of $p_1 \in H^4(BG;\mathbb{Z})$ to $H^4(B_\bullet G;\mathbb{Z}(2))$ by computing the affine space of preimages of its image in $H^*(BG;\mathbb{R})$ under the Chern-Weil map. This is $\mathbb{R}\oplus\mathbb{R}$ when $n > 1$ (spanned by $\mathrm{tr}(A^2)$ and $\mathrm{tr}(A)^2$) and $\mathbb{R}$ for $n = 1$ (preimages of 0).

So to summarize, we've identified the $\mathbb{R}$ worth of differential lifts of p_1; now we have to transgress them.

4. Transgression

We have all the ingredients, and now we put them together.

If time: what Brylinski-McLaughlin did:

- LG is the free loop group of G, which is a Fréchet Lie group satisfying $B_\bullet LG \simeq LB_\bullet G$.
- There is an evaluation map $q\colon S^1 \times LB_\bullet G \to B_\bullet G$.
- There is a projection map $p\colon S^1 \times LB_\bullet G \to LB_\bullet G = B_\bullet LG$.

Assume G compact, simple, and simply connected. The data of a "level" $h \in H^4(BG;\mathbb{Z})$ defines a Kac-Moody central extension of LG by $\mathbb{T}$; let $h_{KM} \in H^3(B_\bullet G;\mathbb{Z}(1)) = H^2(B_\bullet G;\mathbb{T})$ be its cohomology class.

Theorem 4.1 (Brylinski-McLaughlin)**.** *Let $\widetilde{h} \in H^4(B_\bullet G;\mathbb{Z}(2))$ be the unique differential lift of h guaranteed by Bott's theorem. Then $p_*q^*\widetilde{h} = h_{KM}$.*

Here p_* is integration along the S^1 fiber, which gets us from $H^4(–;\mathbb{Z}(2))$ to $H^3(–;\mathbb{Z}(1))$ as desired.

What we have to do is a little different. Let $E \coloneqq E_\bullet\Gamma \times_\Gamma S^1 \to B_\bullet\Gamma$ be the universal oriented circle bundle, and let $V \to E$ be the vertical tangent bundle. Then V defines a classifying map $p\colon E \to B_\bullet\mathrm{GL}_1^+(\mathbb{R})$, and let $q\colon E \to B_\bullet\Gamma$ be the quotient map.

Theorem 4.2 (D.-Liu-Weis, '21)**.** *The transgression p_*q^* maps the line of off-diagonal differential lifts of p_1 isomorphically onto the line of Virasoro central extensions of Γ.*

Recall that $H^3(B_\bullet\Gamma; \mathbb{Z}(1)) \cong \mathbb{R} \times \mathbb{T}$, and this means we obtained $\mathbb{R} \times \{0\}$.

Now let's do better, and actually identify which differential lift hits which Virasoro group.

- Which Virasoro group is canonical is mostly a matter of convention. Let's take $\lambda = 1$, which is the Virasoro extension for the CFT of bosonic periodic scalars. This is a common convention (i.e. that this Virasoro group has central charge 1).
- We choose as our distinguished differential lift of p_1 the unique differential lift $\hat{p}_1$ which satisfies the Whitney sum formula.

  (But wait, you say! Pontrjagin classes don't satisfy the Whitney sum formula! Yes, but p_1 does for $\mathrm{GL}_n^+(\mathbb{R})$.)

Theorem 4.3 (D.-Liu-Weis, '21)**.** *The transgression $p_* q^*$ maps $\hat{p}_1$ to the Virasoro group with central charge -12.*

That is, we ended up with -12 times the generator.

If time remains, say something about the proof: there are a few different ways we can think about E.

- Computing q^* is easiest when we simplicially resolve the $\mathrm{GL}_1^+(\mathbb{R})$-action on E, in parallel to the bar construction simplicially resolving $B_\bullet\mathrm{GL}_1^+(\mathbb{R})$: $\Gamma\backslash\mathrm{Fr}^+(S^1) \times \mathrm{GL}_1^+(\mathbb{R})^{\times\bullet} \to \mathrm{GL}_1^+(\mathbb{R})^{\times\bullet}$. Then the pullback is simple: take the cocycle $x_1\,\mathrm{d}x_2 \in \Omega^2(\mathbb{R}^2)$ ($\mathbb{R}$ identified with $\mathrm{GL}_1^+(\mathbb{R})$) and pull it back to a cocycle constant along $\Gamma\backslash F$.
- However, p_* is very difficult to compute with this presentation. Here it makes most sense to resolve the Γ-action on the left, in parallel with the bar resolution on $B_\bullet\Gamma$: $\Gamma^{\times\bullet} \times \mathrm{Fr}^+(S^1) \to \Gamma^{\times\bullet}$. Now integration along the fiber is actually integrating along an S^1 (on each simplicial level).
- We have to get from one description to another. So we resolve in *both* directions at once, producing a bisimplicial object $\Gamma^{\times\bullet} \times \mathrm{Fr}^+(S^1) \times \mathrm{GL}_1^+(\mathbb{R})^\bullet$. Then, chase $x_1\,\mathrm{d}x_2$ across the double complex for Ω^1 of this bisimplicial object, which turns into a very explicit single-variable calculus computation.