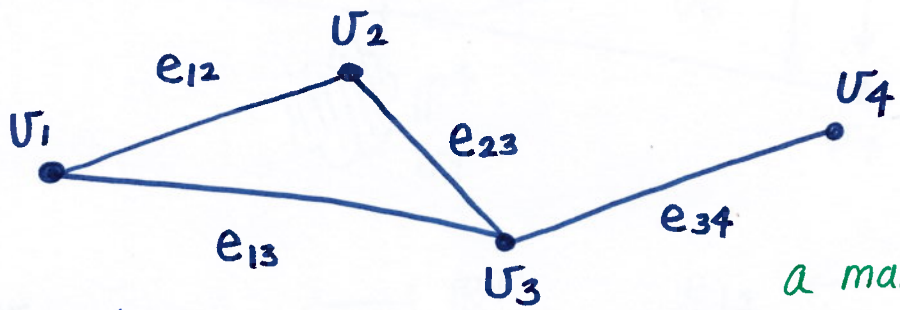
Mathematical Sheaves

SHEAVES

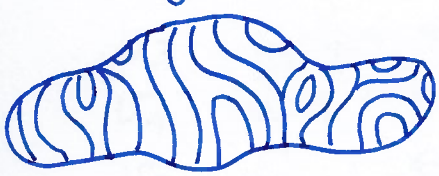
NOTES ON THE NOTES "A VERY ELEMENTARY INTRODUCTION TO SHEAVES" BY MARK AGRIOS (2022)

a graph...

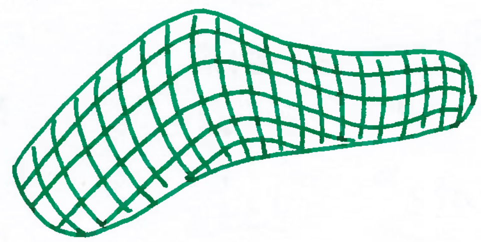
"SPACE"



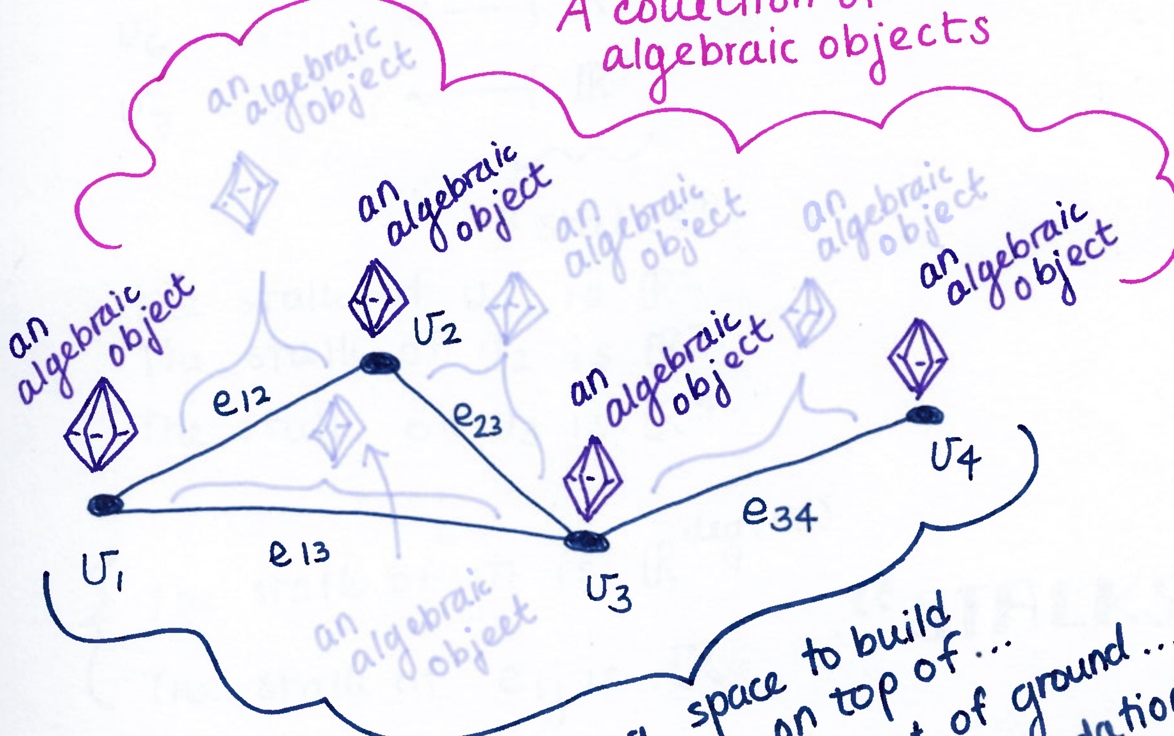
an abstract topological space...



a manifold...



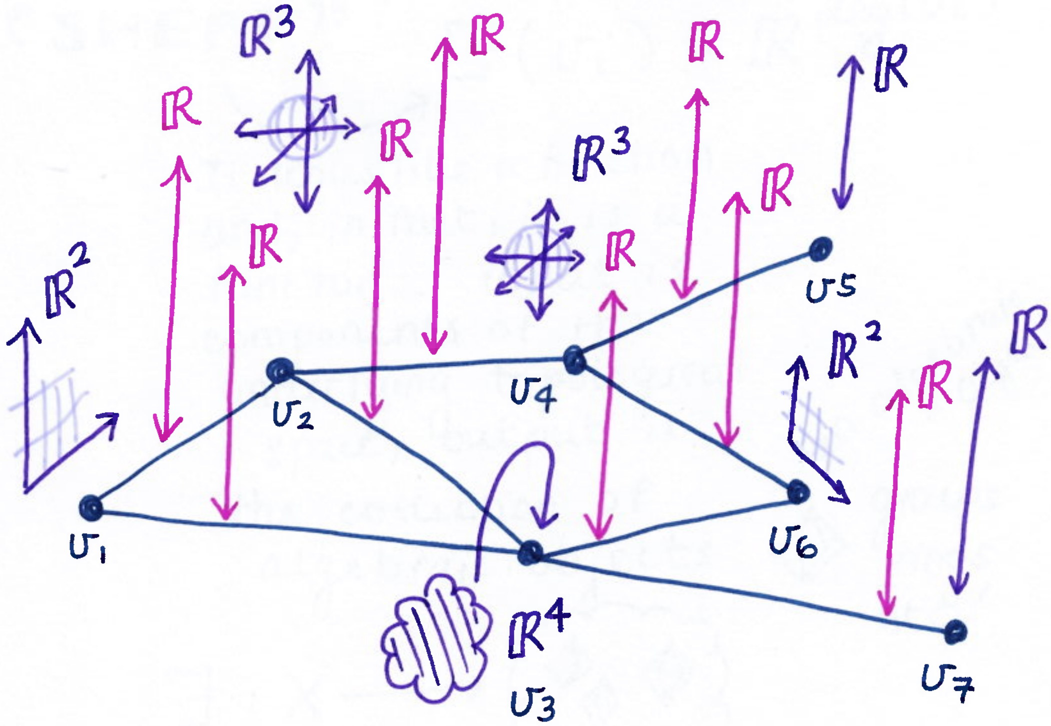
A collection of algebraic objects



a space to build on top of...

a plot of ground...

a foundational topological object on which to build!



v_1 deg 2 $\mapsfrom \mathbb{R}^2$

v_2 deg 3 $\mapsfrom \mathbb{R}^3$

v_3 deg 4 $\mapsfrom \mathbb{R}^4$

v_4 deg 3 $\mapsfrom \mathbb{R}^3$

v_5 deg 1 $\mapsfrom \mathbb{R}^1$

v_6 deg 2 $\mapsfrom \mathbb{R}^2$

v_7 deg 1 $\mapsfrom \mathbb{R}^1$

"stalks"

$e_{12} \mapsfrom \mathbb{R}$

$e_{13} \mapsfrom \mathbb{R}$

⋮

etc!

"stalks"

The stalk of v_1 is $\mathbb{R}^2$

The stalk of v_2 is $\mathbb{R}^3$

The stalk of v_3 is $\mathbb{R}^4$

⋮

$$\left\{ \begin{array}{l} \text{The stalk of } v_i \text{ is } \mathbb{R}^{\deg(v_i)} \\ \text{The stalk of } e_{ij} \text{ is } \mathbb{R} \end{array} \right.$$

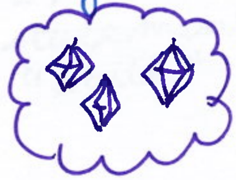
"STALKS"

"SHEAF" $\mathcal{F}(v_i) = \mathbb{R}^{\deg(v_i)}$

It looks like a function and, in fact, it is a functor... input is components of the underlying topological space, output is the collection of algebraic objects

an algebraic object

groups
rings
etc.!

$\mathcal{F} : X \longrightarrow$ 

But, like most things in mathematics, there is a lot more to say about the technical details involved in justifying every interaction...

TECHNICAL COMPATIBILITY & gluing properties!

A SHEAF OF  ___ ON X ___

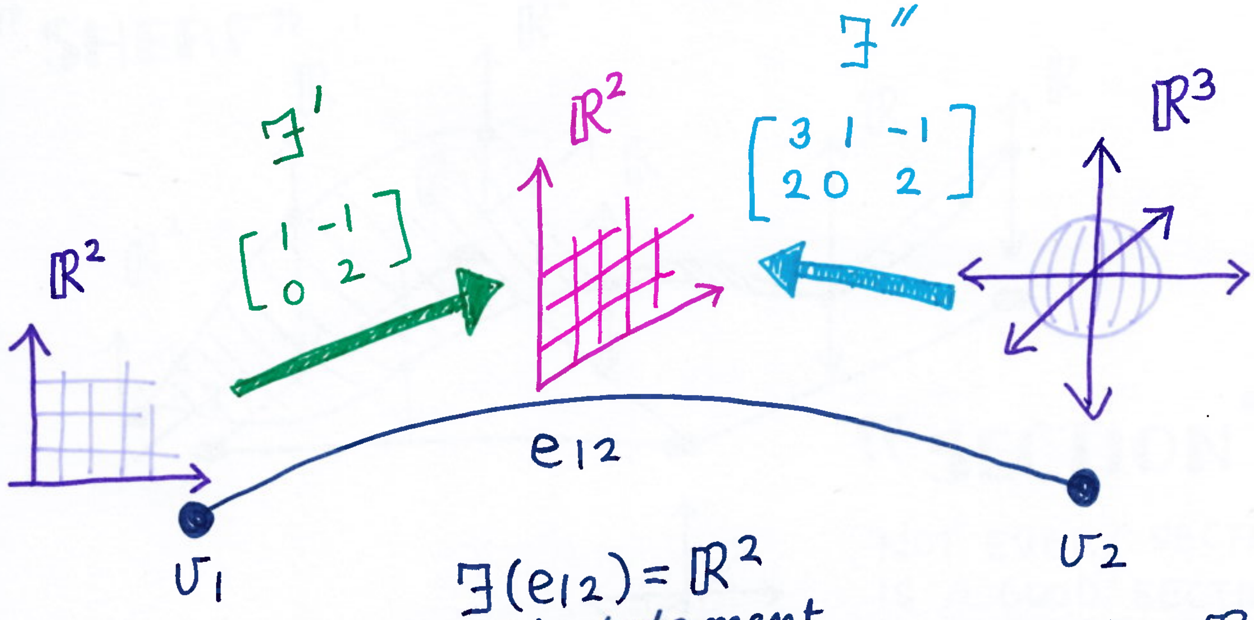
vector spaces

rings

a graph

an abstract topological space

w/technical compatibility & gluing properties...



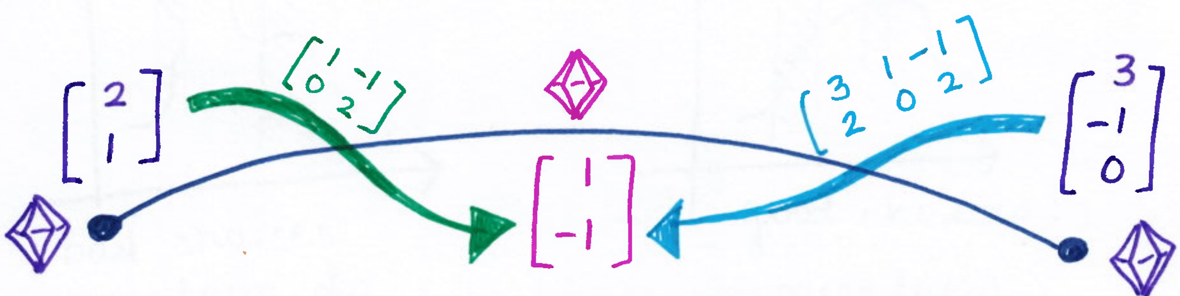
$\mathcal{F}(e_{12}) = \mathbb{R}^2$
sheaf statement
"stalk"

$\mathcal{F}(v_1) = \mathbb{R}^2$
sheaf statement
"stalk"

$\mathcal{F}(v_2) = \mathbb{R}^3$
sheaf statement
"stalk"

$\mathcal{F}' : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$
2x2 matrix

$\mathcal{F}'' : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$
2x3 matrix



IN **SHEAF**

$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$

$\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

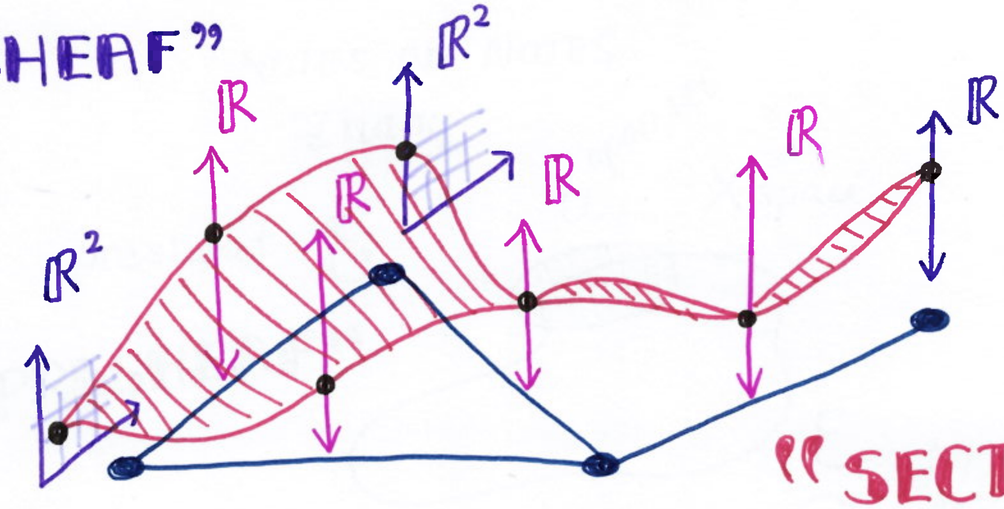
IN **SHEAF**

$\begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 3 & 1 & -1 \\ 2 & 0 & 2 \end{bmatrix}$

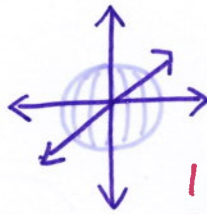
$\begin{bmatrix} 3 & 1 & -1 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 8 \\ -6 \end{bmatrix}$

"SHEAF"



"SECTION"

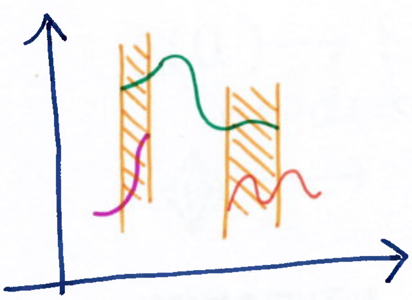


NOT EVERY SECTION IS A GOOD SECTION...

IN FACT, MANY ARE DOWNRIGHT AWFUL WITH TERRIBLE PROPERTIES!!

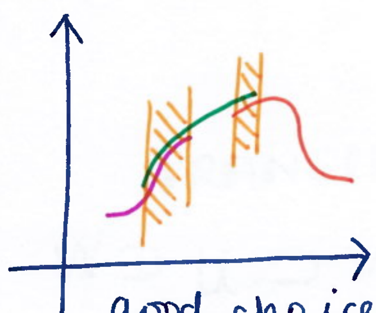
Choose your sections wisely!

"GLOBAL SECTION" = good section!



bad choices

overlaps do NOT agree!



good choices!

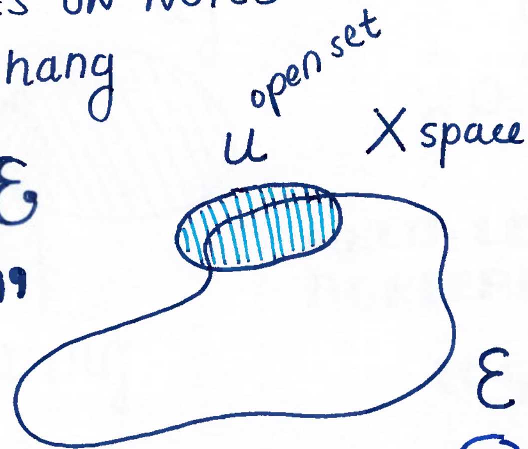
overlaps agree!

NOW we move to the DIFFICULT DEFINITIONS...

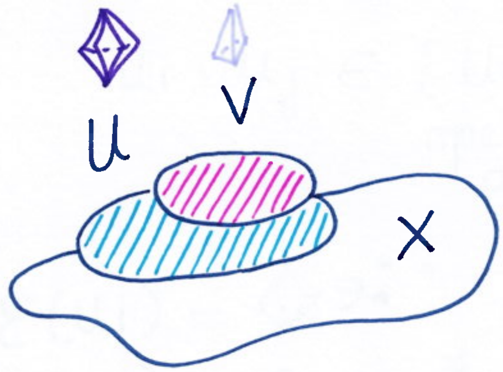
NOTES ON NOTES

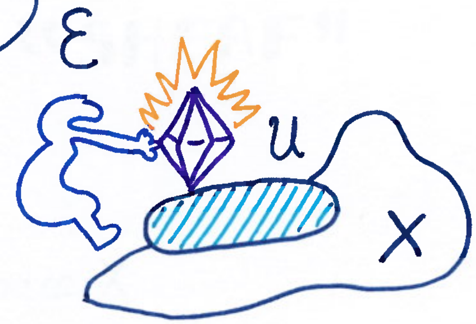
Zhang

presheaf $\mathcal{E}$



"PRESHEAF"





$\mathcal{E}(U)$

$U \supset V$

$\mathcal{E}(U) \to \mathcal{E}(V)$

restriction!

$\mapsto$

IDENTITY

$\mathcal{E}(U) \longrightarrow \mathcal{E}(U)$

$\xmapsto{\ \mathrm{Id}\ }$

identity - no strange spatial twisting or special mapping!

TRANSITIVITY

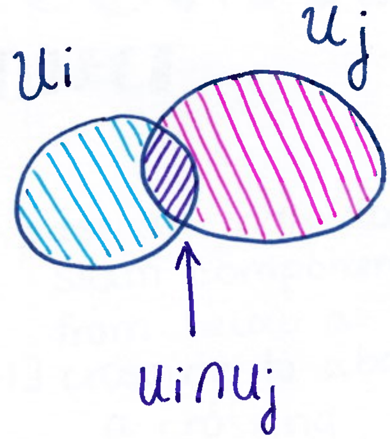
$W \supset U \supset V$

$\mathcal{E}(W) \to \mathcal{E}(U) \to \mathcal{E}(V)$

$\mapsto \quad \mapsto$

restriction has to stay the same no matter how you restrict it!

so what makes a sheaf different from a presheaf ??? →



MULTI-LEVEL AGREEABILITY

"SHEAF"

$u_i, u_j \in \{U_k\}_{k \in K}$ open cover of the space X

$\mathcal{E}(u_i) =$  $\ni \bullet \bullet \bullet$

$\mathcal{E}(u_j) =$  $\ni * * *$

$\mathcal{E}(u_i \cap u_j) =$  $\ni + + +$

$\mathcal{E}(u_i \cap u_j) \longrightarrow \mathcal{E}(u_i)$

$\mathcal{E}(u_i \cap u_j) \longrightarrow \mathcal{E}(u_j)$

$\mathcal{E}(u_i) \longrightarrow \mathcal{E}(u_i \cap u_j)$

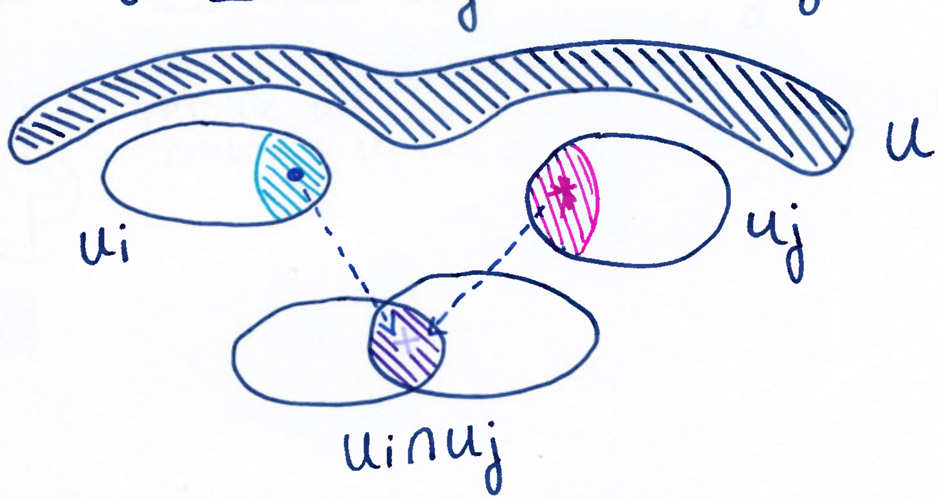
$\bullet \longmapsto +$

$u_i \supset u_i \cap u_j$

$\mathcal{E}(u_j) \longrightarrow \mathcal{E}(u_i \cap u_j)$

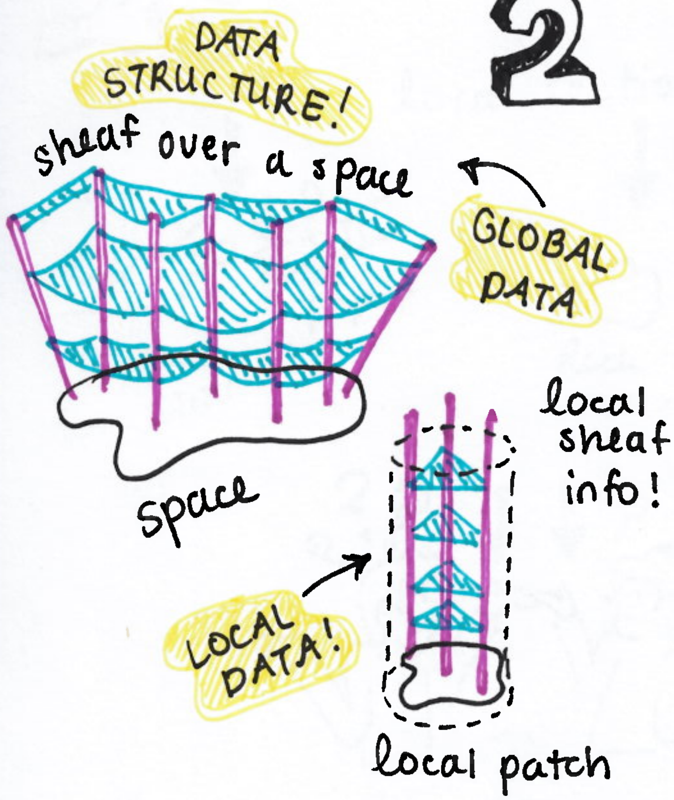
$* \longmapsto +$

$u_j \supset u_i \cap u_j$

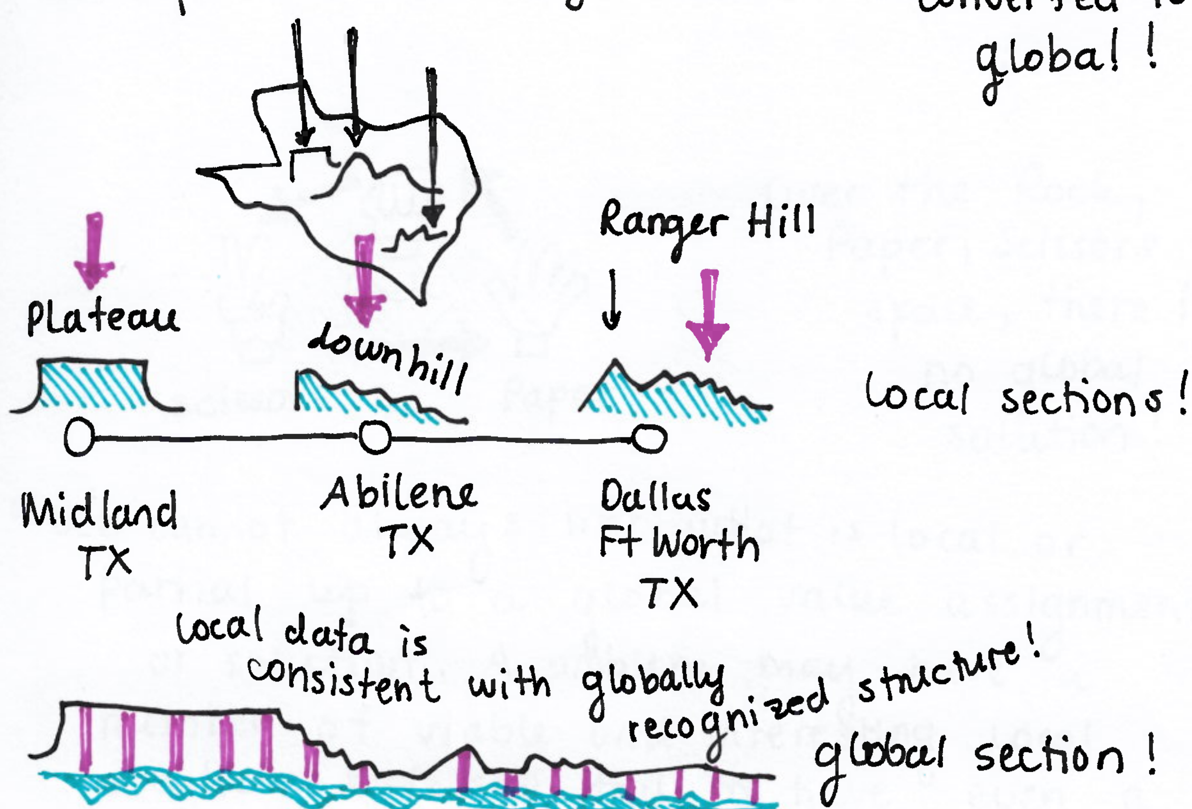


SHEAVES 2

taken from the Introductory Notes by Daniel Rosiak (MIT 2020) "Introduction to sheaves Through Examples"

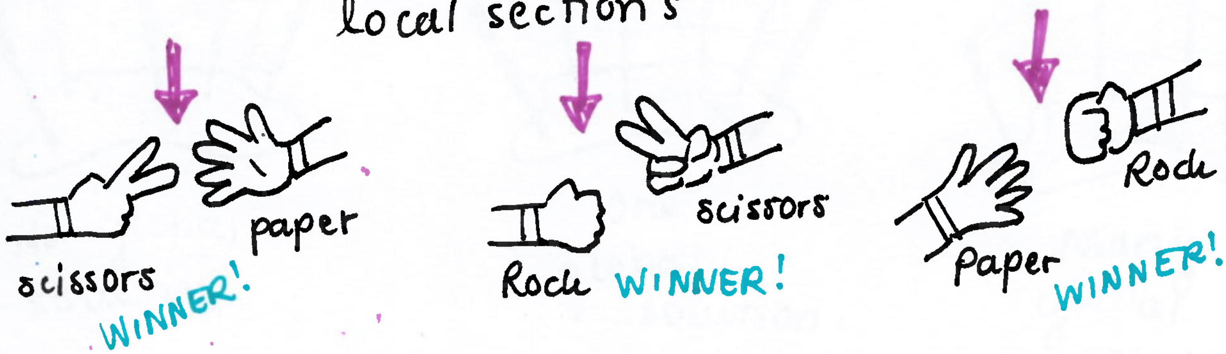


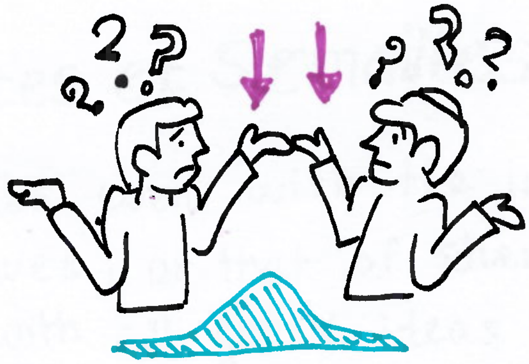
Example of constructing a sheaf → local can be converted to global!



Rock, Paper, Scissors

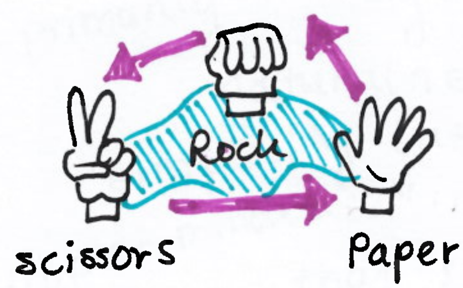
local sections





global section

How do you wih Rock, Paper, scissors?
There is no global solution!



Over the Rock, Paper, Scissors space, there is no global solution!

"We cannot always lift what is local or partial up to a global value assignment or solution. A problem may have a number of viable and interesting local solutions but still fail to have even a single global solution"

No global solution | One global solution | Many global solutions!

Récoltes et Semailles, Promenade 13

Grothendieck

"as even with the idea of sheaves, or that of schemes, as well as with all grand ideas that overthrow the established vision of things, the idea of the topos had everything one could hope to cause a disturbance, primarily through its "self-evident" naturalness, through its simplicity (at the limit naive, simple-minded, "infantile") – through that special quality which so often makes us cry out: "Oh, that's all there is to it!" in a tone mixing betrayal with envy, that innuendo of the "extravagant", the "frivolous"; that one reserved for all things that are unsettling by their unforseen simplicity, causing us to recall, perhaps, the long-buried days of our infancy."

CATEGORY THEORY ABSTRACTION!

Ref: Category theory Notes (2018) D. Pavlov

OBJECTS | **OPERATIONS**

Antiquity, Middle Ages

Numbers + Figures

Mathematical objects

Addition, Multiplication, Division

Compass + straightedge constructions

▶ Abstraction: Some numbers + figures need not be present in nature!

18th & 19th century:

Functions are mathematical objects

Axiomatizing sequences of operations

polynomials, analytic functions, smooth functions, continuous functions

Operations on Functions!

Addition, Multiplication, Limits, Infinite sums!

▶ Abstraction: some functions may not be specified by an explicit formula

Early 20th century:

Abstract Math structures

Axiomatizing operations on functions:

sets, groups, rings, fields, vector spaces, topological spaces, Banach spaces, C*-algebras, measurable spaces, Lie Groups

Operations on structures!

Direct sum, product, Direct + inver. limits

▶ Abstraction: Some mathematical structures might not have functions as their elements continued!

→ Middle 20th Century:

Categories!

abelian categories, toposes, regular categories, sites, Grothendieck topologies

Operations on categories!

Coproducts + products, functor categories

▸ Abstraction: Some categories need not arise as categories of mathematical structures!

21st Century!

Higher Categories!

2-categories, model categories, ∞-categories, (∞, n)-categories, Wow! Oh my!

Operations on higher categories are the same as on categories!

Categories of categories!

▸ Abstraction: some higher categories need not arise from specific classes of categories!!

"Houston, we have a PATTERN!"

Numbers exist even when I don't see them!

Categories exist even when I don't see them!

"Acquiring a working understanding of category theory resembles climbing the Tibetan Plateau: one first has to expend a substantial amount of effort simply to climb 5 kilometers (3 miles) to the top of the plateau (i.e. to learn and understand the relevant notions such as categories, functors, adjunctions, Kan extensions, etc.) After this, one still has to spend a considerable amount of time acclimating to the high altitude of the plateau (i.e. the high level of abstraction associated with the categorical language). The first few days one is guaranteed to have altitude sickness (i.e. difficulty managing the high level of abstraction and using the associated notions and tools), which eventually disappears once one spends a sufficient amount of time on the plateau."

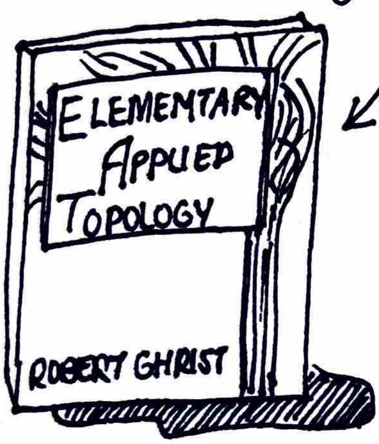
— Dr. Dmitri Pavlov
(Texas Tech University, Lubbock TX)

QUANTUM HOMOTOPY SEMINAR

Texas Tech University Seminar Discussion
3- OCTOBER 2023 delivered in-person
while on leave in Lubbock TX

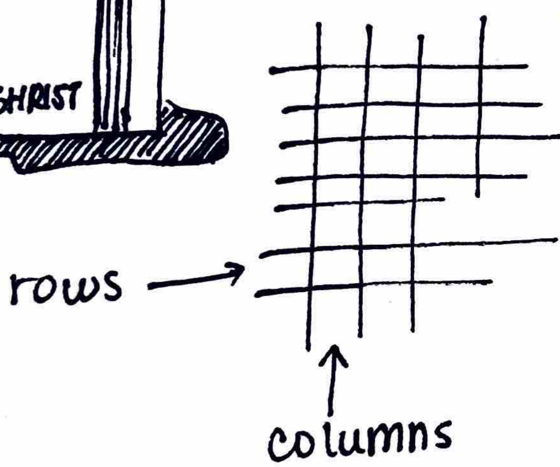
Let me start with a figure by
Dr. Robert Ghrist (University of Pennsylvania)

"Elementary Applied Topology" (2014)



TOPOLOGY → SHEAVES!

Sheaves as a Data Structure



TABLES

ARRAYS

$A = [1, 0, 2, -1, 0, 1]$

single dim
or
multi-dim array

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 5 \end{bmatrix}$$ 2×2 array

MATRICES

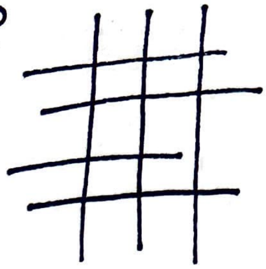
$$M = \begin{bmatrix} 3 & 2 & -1 \\ 0 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix}$$ 3×3 matrix

Array vs. Matrix * [Arr ≥ Mat]
(according to geeksforgeeks.org)

- data storage
- homogeneous
- singular vector

- data transformation
- homogeneous
- multiple equal length vectors

Tab



- non-homogenous (multiple data types ...)

dog	3
cat	2

And the pedagogy could continue ... but, since we're here to talk about sheaves ...

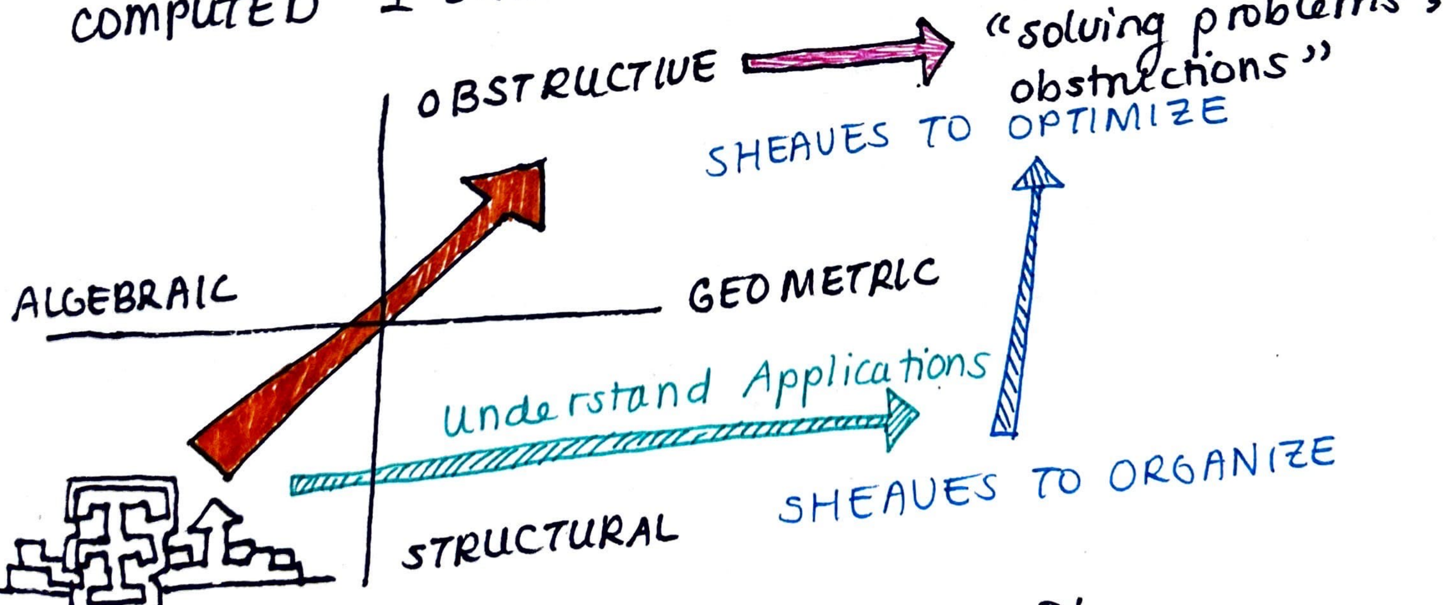
SHEAVES ARE A DATA STRUCTURE

I will point to several recent! papers with some great examples of sheaves used in this context.

*(stanford.edu says
Matrix is 2-dim (r x c) object
stacked vectors
Array is 3-dim (r x c x h) object
stacked matrices)

But now, its time to be brutally honest:

WHO IN THIS ROOM HAS ACTUALLY, HONEST-TO-GOODNESS COMPUTED 1 EXAMPLE OF A SHEAF OVER A GRAPH?



WHAT CAN WE DO WITH SHEAVES ?!

We tend to study sheaves for their utility in tracking data associated to open sets of a topological space.

|||

We use sheaves to as an advanced data framework for manipulation $ optimization of a complex systems of interrelated data

TWO TYPES:

① Sheaves for Data Organization (Existing Data) Manage

② Sheaves to Gain Additional Information (New Data) Expose

* tutorialspoint.com

"The arrays in Python are ndarray objects. The matrix objects are strictly 2-dimensional, whereas the ndarray objects can be multi-dimensional"

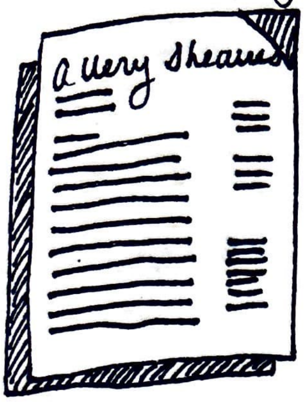
"Matrix is a special case of 2-dimensional array where each data element is of strictly the same size.

... Every Matrix is also a 2-dimensional array, but not vice-versa."

Rachel Notes "Mathematical Sheaves"

Introduction

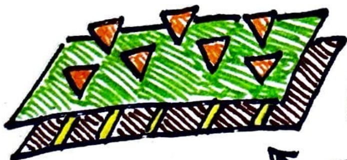
Resource #1: A Very Elementary Introduction to Sheaves (Mark Agrios)



Shoutouts!

- Dr. Robert Ghrist
- Dr. Michael Robinson
- Dr. Jakob Hansen
- Dr. Justin Curry

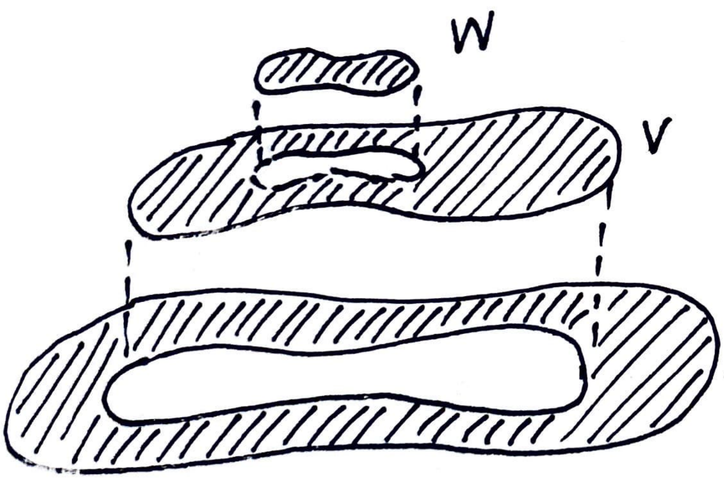
"This paper is very NON-RIGOROUS..."



Garden (sheaf structure)

Plot of Land (Base space)

- Sheaves store data over a topological space...
- Sheaves have to respect the structure of the underlying Topological space...



Local $\longrightarrow$ global

$W \subseteq V \subseteq U$

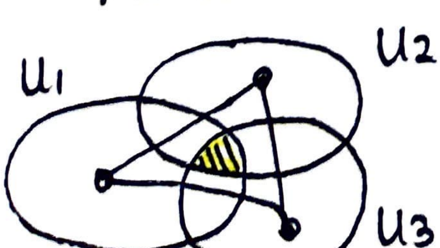
$W \xhookrightarrow{\text{subset}} V \xhookrightarrow{\text{subset}} U$

$W \xleftarrow{\text{data}} V \xleftarrow{\text{data}} U$

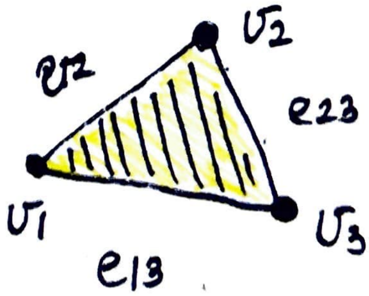
U global $\longrightarrow$ local

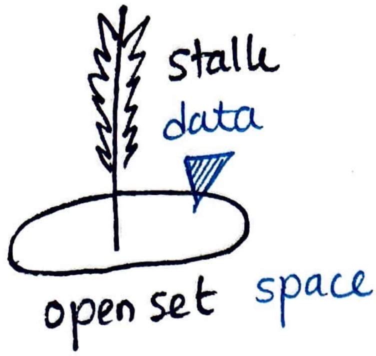
"Data attached to a larger part of the object must be consistent when restricted to a smaller part of the object"

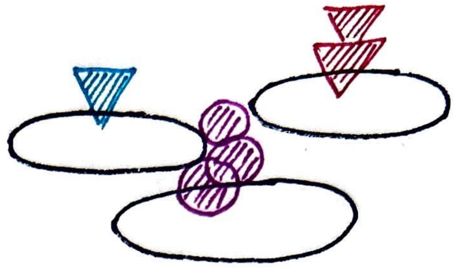
"Refinement"



"Increasing Resolution"

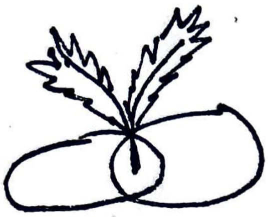






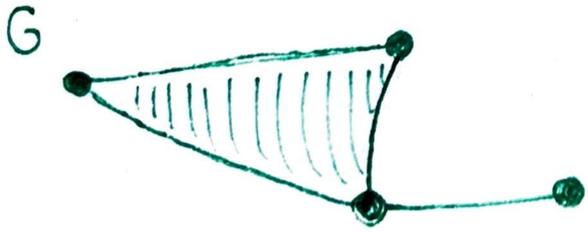
Mechanism # 1
Assign each component a "stalk"

RECALL!
Sheaves

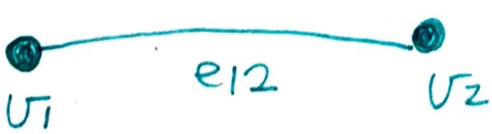


Mechanism # 2
Restriction Maps &
Multi-level Agreement

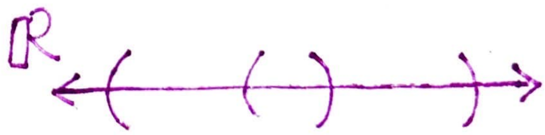
Example # 1
Graph

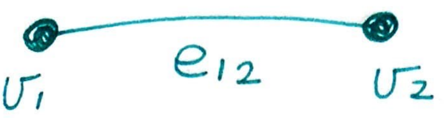


Cellular Sheaves



Example # 2
Open sets on the Real Line





$$\mathcal{F}(v_1) = \mathbb{R}^2 \ni \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\mathcal{F}(v_2) = \mathbb{R}^3 \ni \begin{bmatrix} -2 \\ 10 \\ 3 \end{bmatrix} \text{ or } \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$$

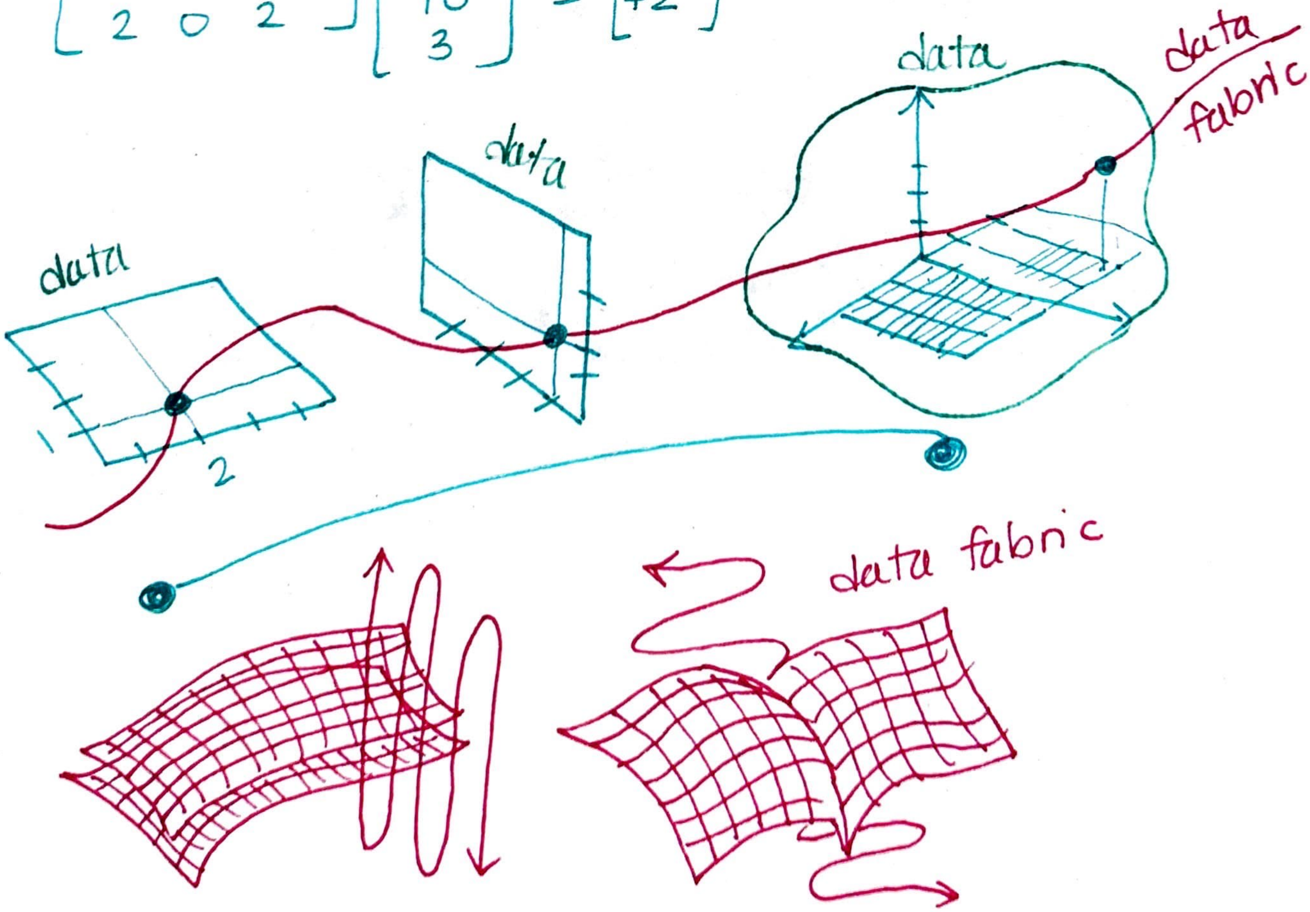
$$\mathcal{F}(e_{12}) = \mathbb{R}^2$$

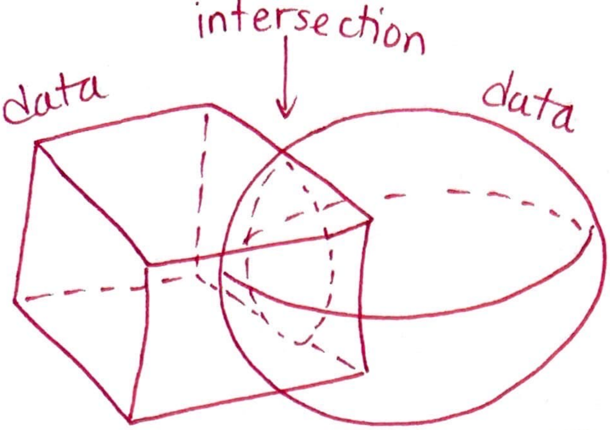
Map #1 $\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$ $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$

Map #2 $\begin{bmatrix} 3 & 1 & -1 \\ 2 & 0 & 2 \end{bmatrix}$ $\mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$$\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} +1 \\ +2 \end{bmatrix}$$

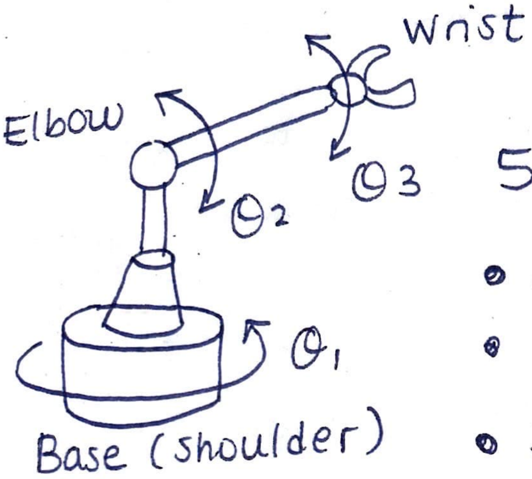
$$\begin{bmatrix} 3 & 1 & -1 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 10 \\ 3 \end{bmatrix} = \begin{bmatrix} +1 \\ +2 \end{bmatrix}$$





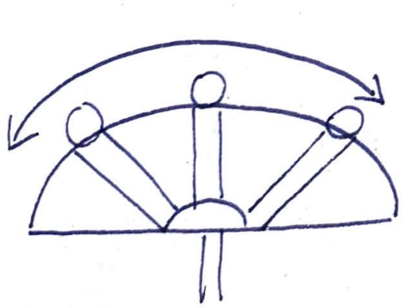
ROBOTICS

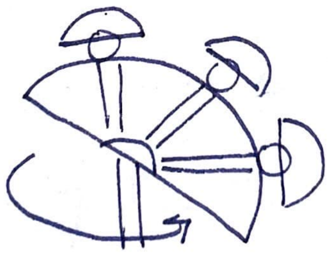
Configuration spaces



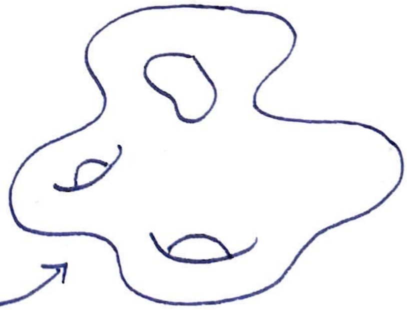
5 Types of Industrial Robots

- Cartesian
- Cylindrical
- SCARA
- 6-Axis
- Delta





HIGH-
DIMENSIONAL
OBJECT.



Elementary Applied Topology

Chapter 1 §2 De

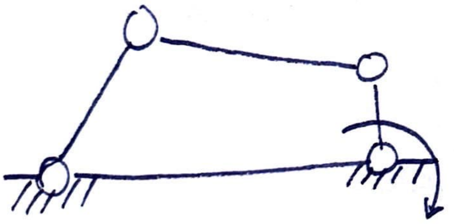
Configuration spaces of Linkages

"The configuration space of the linkage is a topological space that assigns a point to each →

→ configuration of the linkage ...
a neighborhood of a configuration is all configurations obtainable via a small perturbation of the mechanical linkage.

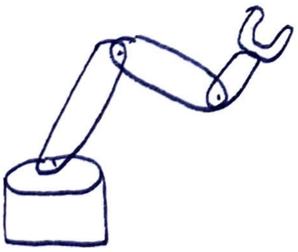
The configuration space of a planar linkage is almost always a manifold, the dimension of which conveys the number of mechanical degrees of freedom of the device" (pg 11–12)

Theorem 1.4
Any smooth compact manifold is diffeomorphic to the configuration space of some planar linkage.
(Kapovich, Millson, 2002)



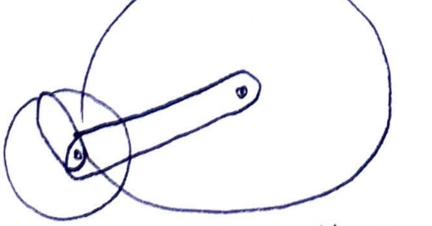
Crank-Rocker
$S^1 \sqcup S^1$

(Ignoring geometric collision)



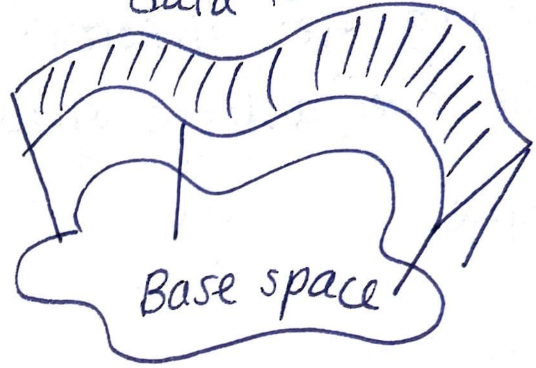
$$T^n = \prod_i^n S^1$$

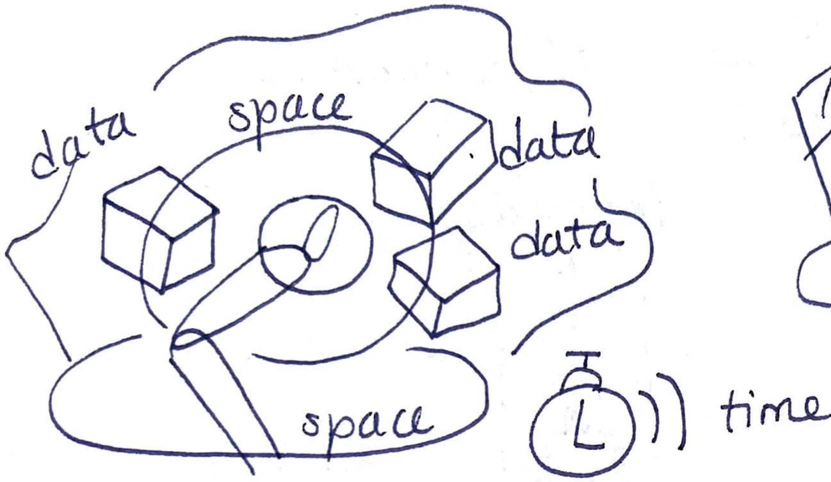
cartesian product of n circles,
n = # rotation joints



configuration space

open neighborhood

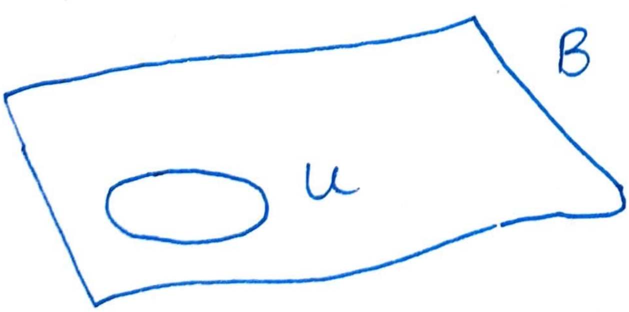




[Aside: Knots & Braids can also be associated w/ Robot planning, see Ghrist §1.5

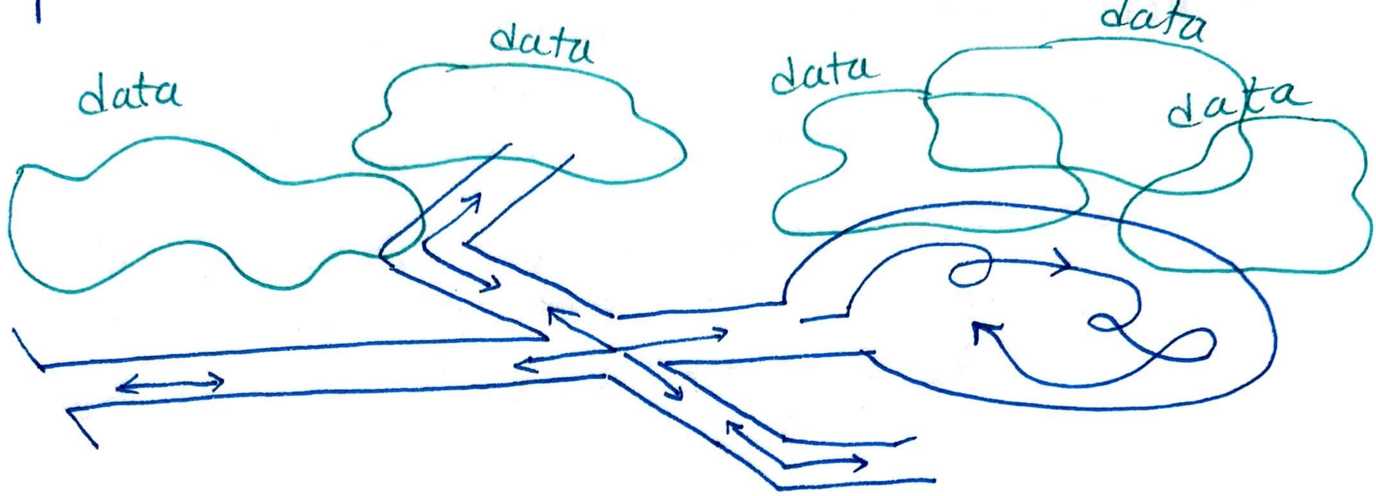
Example of Robots to get
Back to Sheaves ...

How are sheaves different from Fiber Bundles?



"A fiber bundle associates to sufficiently small open sets U in the base space B a product $U \times F$ with fiber F... ⟶

"The amalgamation of algebraic data along a space is at the heart of the notion of a sheaf".



"Sheaves are the canonical Data structure for sensor Integration"

(Robinson 2016)

"A sensor integration framework should be
(1) sufficiently general to accuratey represent all sensors of interest,

and

(2) be able to summarize information in a faithful way that emphasizes important actionable information"

(2021)

Green, Cardona, Cleveland, Ozbolt, Hylton, Short, Robinson

"Dude Where's My stars"
A Novel Topologically-Justified Approach to star Tracking

① Organizational Sheaves

2 "Path Optimization Sheaves" (2020)

Moy, Cardona, Green, Cleveland, Hylton, Short

Decision-Making Sheaves

Sheaf Theoretic Approach Recast Dijkstra's pathfinding algorithm by defining "path-finding sheaf"

Dude Where's My Stars?

$$P_n := \begin{tikzcd}[column sep=small]
x_1 \cdots & x_n & & & y_1 \cdots & y_n \\
& d \arrow[ul] \arrow[u] & & & a \arrow[u] \arrow[ur] & \\
& c \arrow[u] & & r & & \\
& & m \arrow[ul] \arrow[ur] \arrow[uurr] & & &
\end{tikzcd}$$

Rank 4 poset
Equation 1

$s(m) = S^2 \times [0, 2\pi]$

$S(c) = S^2$

$s(d) = (([0,\pi] \sqcup \{\infty\}) \times \mathbb{R}^n / \sim$

and more!

PATH OPTIMIZATION SHEAVES

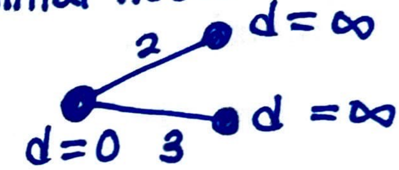
Dijkstra's Algorithm (shortest path between nodes in a weighted graph) [1956]

search Algorithm
w/ applications in network routing, traffic study etc.
[WIKIPEDIA ARTICLE]

1. Mark all nodes "unvisited"
   Create U = the set of unvisited nodes

2. Assign each node a "tentative distance value"
   Initial node, $d=0$ All other nodes, $d=\infty$

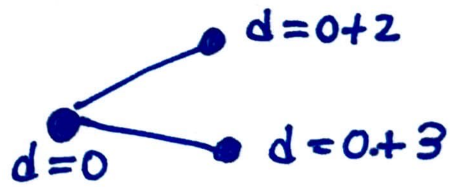
   During the algorithm (node v) d_v = Length of the shortest path discovered between v and initial node



3. For the current node, consider unvisited neighbors

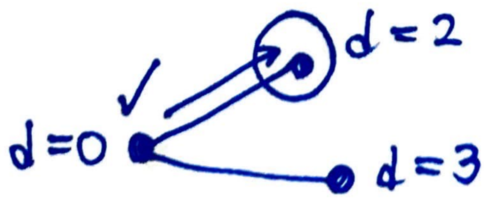
   Calculate distance to current node (use shortest distance when connected via 2 different weighted edges)

   If considering a node with a distance already assigned,

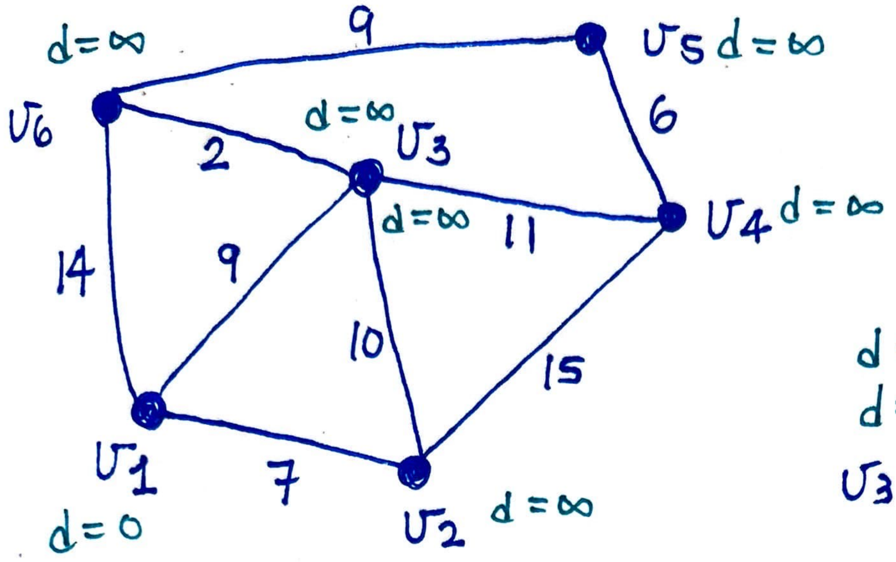


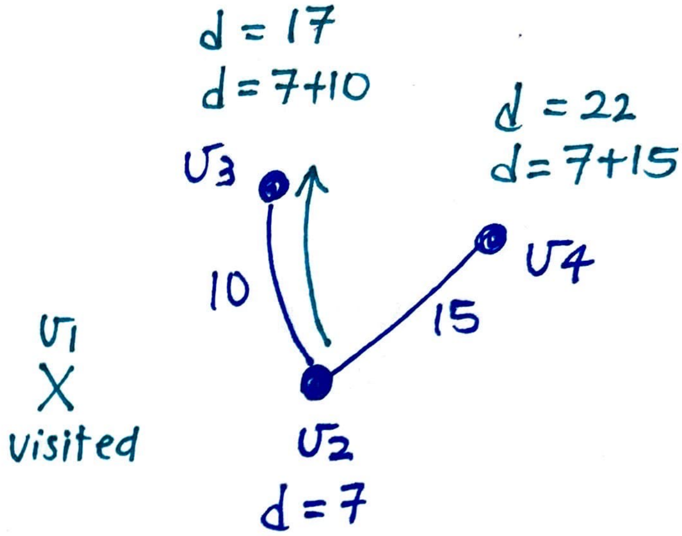
4. Mark current node as "Visited" (Remove from the unvisited set)

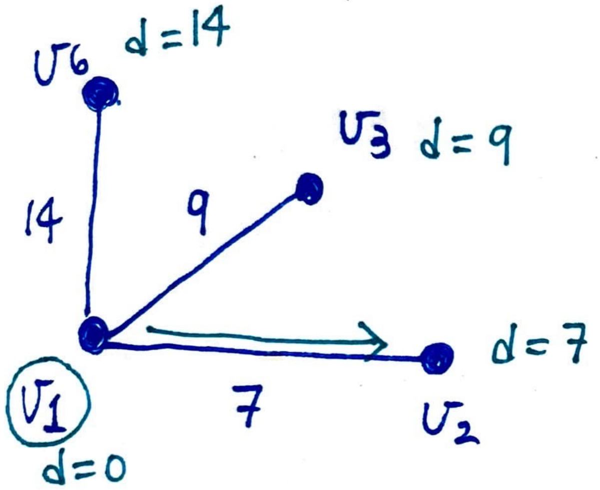
5. Continue until visited the destination node!

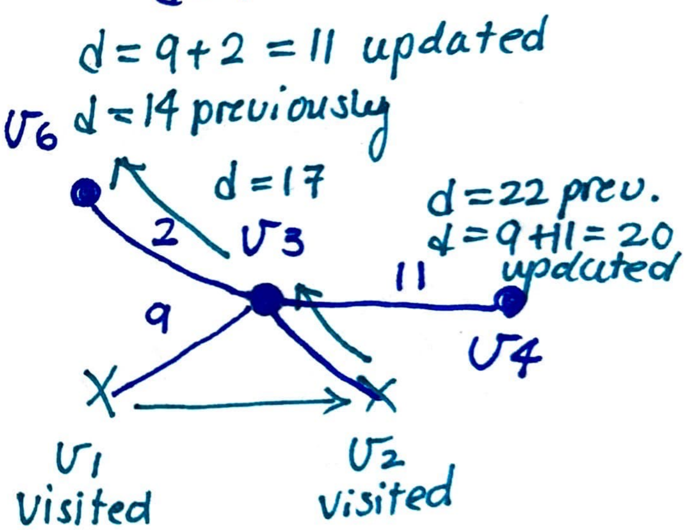


Select the unvisited node marked with smallest tentative distance, set as current node, Repeat!









and we continue until we get to the sink node!

We can define a Path Optimization Sheaf Structure on a graph, and then use it to define a Networking, shortest path algorithm!

"sheaves have arisen as a more rigorous way to describe local actions impacting global structure"

$G = (V, E)$ graph

nodes edges

$e = \{v_1, v_2\}$

edge

TERMS

$\deg(v)$ = # edges containing vertex v

degree of vertex

For weighted graphs, define map $w : E \longrightarrow \mathbb{R}^+$

$w(e)$ = weight of edge "e"

SHEAF OVER GRAPH

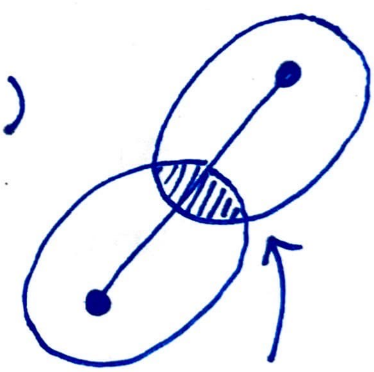
(Cellular Sheaves)

A functor $\mathcal{F}$ assigning to each vertex and edge of G a set $\mathcal{F}(v)$, $\mathcal{F}(e)$

vertices edges

<u>AND</u> $\mathcal{F}(v \rightsquigarrow e) : \mathcal{F}(v) \longrightarrow \mathcal{F}(e)$

(Restriction maps)



edge is thought of as the intersection between two vertices in the nerve construction

GLOBAL SECTION:

Choice of elements in each set over each vertex and edge of G such that

$v \in e$, $\mathcal{F}(v \rightsquigarrow e)(s(v) = s(e))$

Global section "s"

section value at vertex

section value at edge

PATH SHEAF

v_S = source vertex
v_T = sink vertex

* Require $\deg(v) \geq 2$ $\forall v \in V$ except v_S, v_T (source, sink)

For $E(v)$ (the set of all edges connected to vertex v)

define $H(v) = \{ \{e_i, e_j\} \subseteq E(v) \mid e_i \neq e_j \}$

(the set of two-element subsets of $E(v)$)

[used later, $H_0(v) = \{\{e_i, e_j\} \in E(v)^2 \mid e_i \neq e_j\}$, in be the set of all possible orderings on sets in $H(v)$, in $E(v)^2$ $\{e_i, e_j\}$ and $\{e_j, e_i\}$ are distinct]

T: active object $\perp$: inactive object

The Path sheaf is defined:

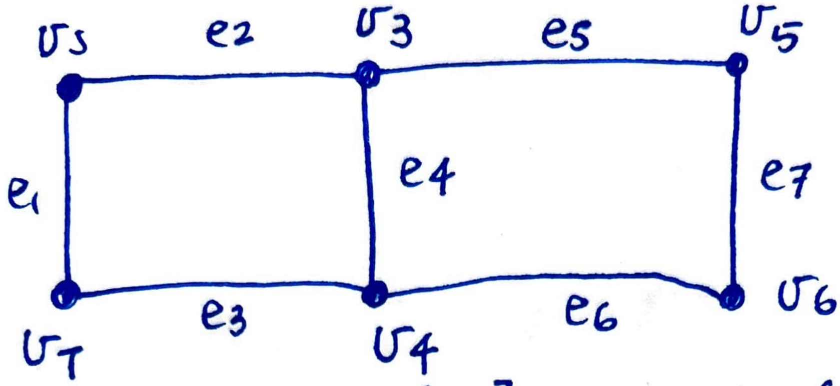
$$P(v) = \begin{cases} E(v) & \text{if } v = v_S, v_T \\ H(v) \cup \{\perp\} & \text{otherwise} \end{cases}$$

$$P(e) = \{\perp, T\}$$

$$\begin{matrix} P(v_S \rightsquigarrow e\}(e_i) \\ P(v_T \rightsquigarrow e\}(e_i) \end{matrix} = \begin{cases} T & \text{if } e = e_i \\ \perp & \text{otherwise} \end{cases}$$

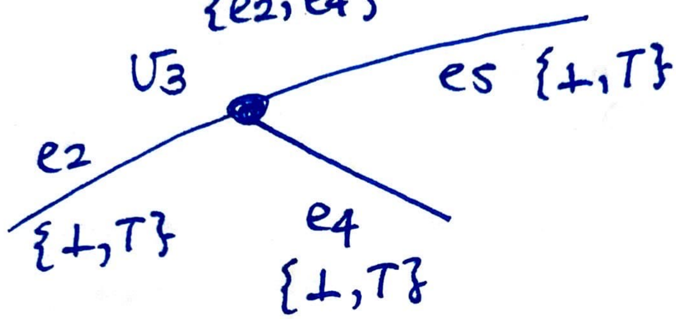
$$P(v \rightsquigarrow e)(\{e_i, e_j\}) = \begin{cases} T & e = e_i \text{ or } e = e_j \\ \perp & \text{otherwise} \end{cases}$$

When v is assigned $\perp$, $P(v \rightsquigarrow e\}(\perp) = \perp$



{⊥}
{e4, e5}
{e2, e5}
{e2, e4}

Sheaf Values over the graph



THEOREM 1

If s is a global section of P,
$\exists$ path $e_0, e_1, \ldots, e_n$ from v_S to v_T
such that $s(e_i) = T, \forall i$

"Follow the truth operator!"

THEOREM 2

For $e_0, \ldots, e_{n-1}$ (w/ vertex sequence $v_0, v_1, \ldots v_n$)
where $v_i \neq v_S, T_T$
$\exists$ local section of P, $s(e_i) = T, \forall i$
s
If $v_0 = v_S$, then s can be extended to a global section

In Theorem 1, all global sections have a path from source to sink, however, it is possible for global sections to have additional cycles disjoint from this path

WEIGHTED GRAPHS

Define the cost function

$$c(s) = \sum_{s(e)=T} w(e)$$

the sum over activated "T" edges

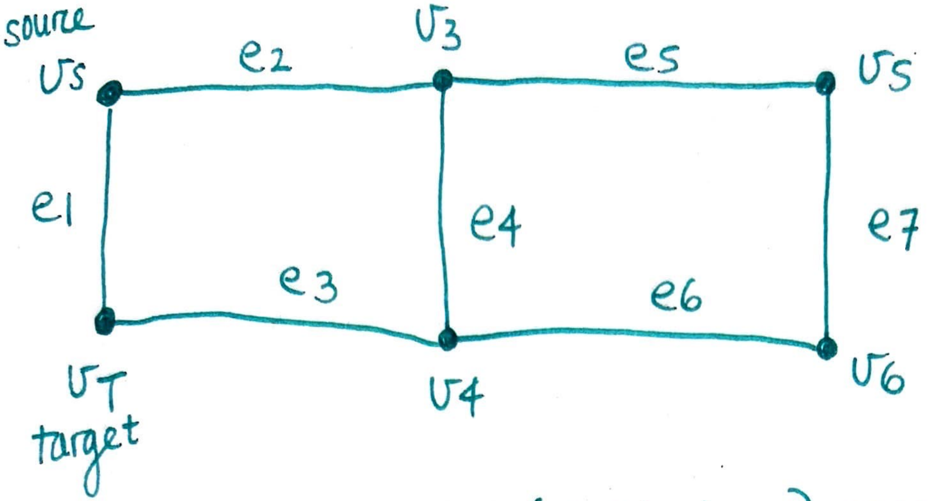
(the cost of traveling over all edges in the section)

DISTANCE PATH SHEAF

$$DP(v) = \begin{cases} E(v) \times \{0\} & \text{if } v = v_S \\ E(v) \times \mathbb{R}^+ & \text{if } v = v_T \\ (H_0(v) \times \mathbb{R}^+) \cup \{\perp\} & \text{otherwise} \end{cases}$$

$$DP(e) = \mathbb{R}^+ \cup \{\perp\}$$

* Note! Here, $H_0(v)$ is used, ordering will matter on 2-edge sets!

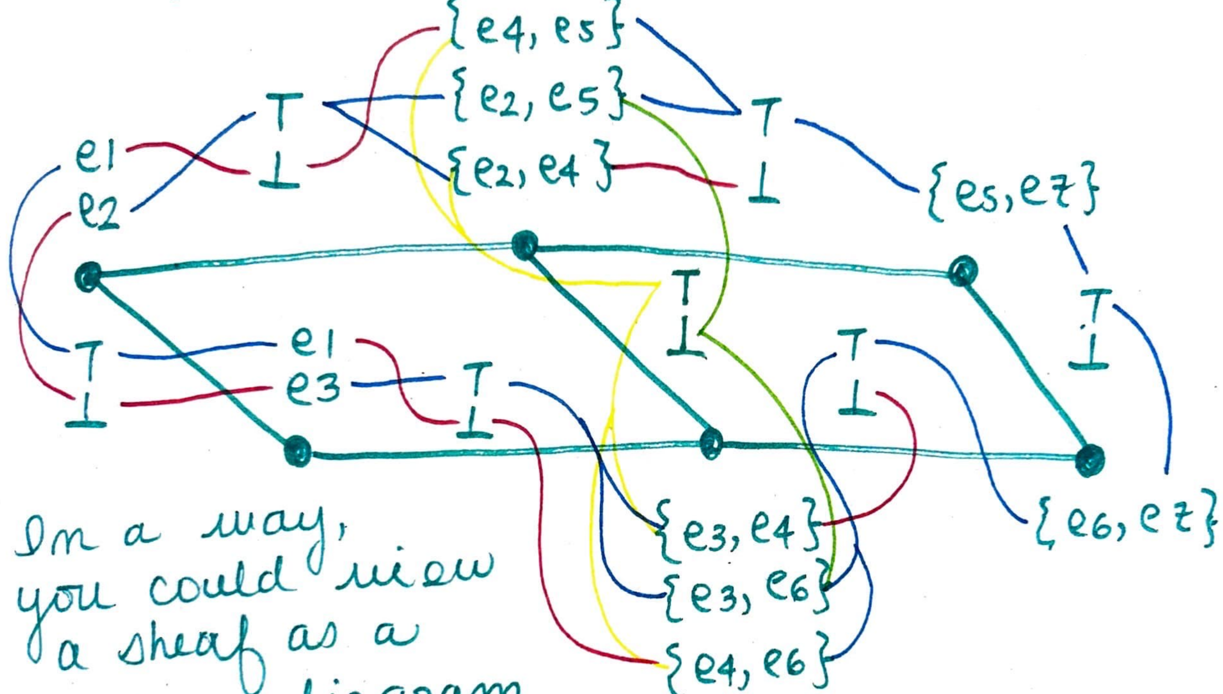


Graph

vertex v_3 attaches to edges e_2, e_4, e_5

so, edge e_2 "wires up" to $\{e_2, e_5\}$ and $\{e_2, e_4\}$ but not to $\{e_4, e_5\}$

Sheaf structure (cellular) over graph



In a way, you could view a sheaf as a wiring diagram

Section of the sheaf over the graph