Notes on the Moduli of L_∞ Connections

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To begin, as per usual, let's motivate some of the definitions of the paper. After this motivation, the definitions become almost formal rephrasings.

We recall the definition of the Chevalley-Eilenberg algebra of a Lie algebra $\mathfrak{g}$:

Definition: Given a Lie algebra $\mathfrak{g}$ the Chevalley-Eilenberg algebra on $\mathfrak{g}$, denoted $\mathrm{CE}(\mathfrak{g})$, has underlying algebra the Grassmann algebra on $\mathfrak{g}^*$: $\wedge^\bullet \mathfrak{g}^*$, and differential is given on generators by the dual of the Lie bracket and extended by linearity+Leibniz. That is,

$$\mathrm{d}_{\mathrm{CE}} := -[-,-]^* : \mathfrak{g}^* \to \mathfrak{g}^* \wedge \mathfrak{g}^*$$

$$\mathrm{d}_{\mathrm{CE}}(\omega)(\alpha, \beta) = -\omega([\alpha, \beta])$$

For later computational reference I'll include the formula, given $f \in \wedge^{n-1}\mathfrak{g}^*$:

$$(\mathrm{d}_{\mathrm{CE}} f)(x_1, ..., x_n) = \sum_{1 \le i < j \le n} (-1)^{i+j} f([x_i, x_j], x_1, ..., \hat{x_i}, ..., \hat{x_j}, ..., x_n)$$

where the hats denote removal from the list.

Immediately, what should come to mind since we are taking the Grassmann algebra on the dual of $\mathfrak{g}$, is that this is something like the algebra of differential forms on the vector space $\mathfrak{g}$. In fact, we can do better than this:

Proposition: Let G be a Lie group, and denote by $(\Omega^\bullet(G)^G, \mathrm{d}_{\mathrm{dR}})$ the differential graded algebra of invariant differential forms on the Lie group G. Then $(\mathrm{CE}(\mathfrak{g}), \mathrm{d}_{\mathrm{CE}}) \cong (\Omega^\bullet(G)^G, \mathrm{d}_{\mathrm{dR}})$ as differential graded algebras.

Proof: We sketch an isomorphism of cochain complexes. To begin, recall that a differential form $\omega \in \Omega^n(G)$ is invariant precisely when

$$\omega = \omega \circ dL_g$$

where L_g is the left action of G on itself by multiplication on the left by $g \in G$, and dL_g is the derivative (tangent) map:

$$dL_g : TG \to TG$$

Now, observe that this property implies that the value of such a form ω is determined by it's values on vectors tangent to the identity. Take a collection of tangent vectors $(v_1, ..., v_n) \in (T_h G)^{\otimes n}$ to a point $h \in G$. Then $(dL_{h^{-1}} v_1, ..., dL_{h^{-1}} v_n) \in (T_e G^{\otimes n}) \cong \mathfrak{g}^{\otimes n}$, and invariance says that:

$$\omega(v_1, ..., v_n) = \omega(dL_{h^{-1}} v_1, ..., dL_{h^{-1}} v_n)$$

thus, any invariant n-form ω defines an element in $(\wedge^n \mathfrak{g})^* \cong \wedge^n \mathfrak{g}^*$ by first translating to the identity. Now, to see that $\mathrm{d}_{\mathrm{dR}} = \mathrm{d}_{\mathrm{CE}}$, recall we have the following formula for the de Rham differential of an n-form ω:

$$\mathrm{d}_{\mathrm{dR}} \omega(\chi_1, ..., \chi_{n+1}) = \sum_{i=1}^{n+1} (-1)^i \mathcal{L}_{\chi_i} \omega(\chi_1, ..., \hat{\chi_i}, ..., \chi_{n+1}) + \sum_{1 \le i < j \le n+1} (-1)^{i+j} \omega([\chi_i, \chi_j], \chi_1, ..., \hat{\chi_i}, ..., \hat{\chi_j}, ..., \chi_{n+1})$$

where $\chi_1, ..., \chi_{n+1} \in \mathfrak{X}(G)$ and $\hat{(-)}$ denotes the removal of $(-)$ from the list. Immediately, one should note that the rightmost summand is $\mathrm{d}_{\mathrm{CE}}\omega$, thus we must argue that the leftmost summand vanishes. Notice that in the leftmost summand, since we contract ω with as many vector fields as its degree, we are taking the directional derivative of an invariant smooth function. The fact that this sum vanishes follows from the fact that for any f, an invariant smooth function on G, f is constant: given $g \neq h$

$$f(g) = f(g(g^{-1}h)) = f(h)$$

and thus the Lie derivatives in the leftmost sum must vanish. One can conclude after verifying that multiplication is respected (a fairly tautological step) that $(\mathrm{CE}(\mathfrak{g}), \mathrm{d}_{\mathrm{CE}}) \cong (\Omega^\bullet(G)^G, \mathrm{d}_{\mathrm{dR}})$ as differential graded algebras.

While this is a remarkable fact all on its own, $\mathrm{CE}(\mathfrak{g})$ is capable of encoding more than forms on G.

Proposition: : Let $p : Y \to X$ be a principal G-bundle and denote by $\Omega^1_\flat(Y, \mathfrak{g})$ the set of $\mathfrak{g}$-valued one forms on Y with vanishing curvature. Then there is a bijection:

$$\Omega^1_\flat(Y, \mathfrak{g}) \cong \mathrm{Hom}_{\mathrm{dgca}}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet(Y))$$

Proof: We construct the datum of one side from the other in both directions. To begin, take a $\mathfrak{g}$-valued one form ω with vanishing curvature, on a principal G-bundle $Y \to X$. This defines a morphism of modules

$$\psi : \mathfrak{g}^* \to \Omega^1(Y)$$

$$\alpha \mapsto \alpha \circ \omega$$

which thus defines a morphism of graded commutative algebras $\psi : \mathrm{CE}(\mathfrak{g}) \to \Omega^\bullet(Y)$ (in degree 0 the extension is the "inclusion of constants") since $\mathrm{CE}(\mathfrak{g})$ is generated by $\mathfrak{g}^*$. What remains is to see that this morphism respects differentials, this we again check on generators. Observe that:

$$\psi(\mathrm{d}_{\mathrm{CE}}\alpha) = (\mathrm{d}_{\mathrm{CE}}\alpha) \circ (\omega \wedge \omega) = -\alpha([\omega \wedge \omega])$$

Now, it is a subtly non-trivial fact that given any vector space valued form $\omega : \wedge^k TM \to V$, the de Rham differential commutes with postcomposition by any linear map $\varphi : V \to W$, that is, $\mathrm{d}(\varphi \circ \omega) = \varphi \circ \mathrm{d}\omega$. Thus in our case:

$$\mathrm{d}_{\mathrm{dR}}(\psi(\alpha)) = \mathrm{d}_{\mathrm{dR}}(\alpha \circ \omega) = \alpha \circ \mathrm{d}_{\mathrm{dR}}\omega$$

We obtain then that

$$\mathrm{d}_{\mathrm{dR}}(\psi(\alpha)) - \psi(\mathrm{d}_{\mathrm{CE}}\alpha) = \alpha(\mathrm{d}_{\mathrm{dR}}\omega + [\omega \wedge \omega]) = \alpha(F_\omega) = \alpha(0) = 0$$

so this map does indeed respect the differentials. In the other direction, we start with a homomorphism of differential graded algebras:

$$\varphi : \mathrm{CE}(\mathfrak{g}) \to \Omega^\bullet(Y)$$

we tensor both the domain and codomain with the dga $\mathfrak{g}[0]$ with trivial differential to obtain a map:

$$\varphi \otimes \mathrm{id}_{\mathfrak{g}} : \mathrm{CE}(\mathfrak{g}) \otimes \mathfrak{g}[0] \to \Omega^\bullet(Y) \otimes \mathfrak{g}[0]$$

restricting our attention to degree 1, we see a homorphism of modules:

$$\varphi_1 : \mathfrak{g}^* \otimes \mathfrak{g} \to \Omega^1(Y) \otimes \mathfrak{g}$$

we can canonically produce a $\mathfrak{g}$-valued one form by evaluating the following composition on the identity endomorphism:

$$\mathrm{End}(\mathfrak{g}) \xrightarrow{\cong} \mathfrak{g}^* \otimes \mathfrak{g} \xrightarrow{\varphi_1 \otimes \mathrm{id}_{\mathfrak{g}}} \Omega^1(Y) \otimes \mathfrak{g}$$

that this one form has vanishing curvature follows from the fact that ψ respects differentials.

The above proposition immediately begs the question, what sort of differential graded algebraic gadget will parameterize arbitrary $\mathfrak{g}$-valued one forms on a principal bundle? The answer, we will see, is the Weil algebra of the Lie algebra. First a reminder:

Definition: The free graded commutative algebra on an $\mathbb{N}$ graded vector space $\mathfrak{g}$, denoted by $\mathrm{Sym}^\bullet\mathfrak{g}$, is given by:

$$\mathrm{Sym}^\bullet\mathfrak{g} = \mathrm{Sym}(\bigoplus_p \mathfrak{g}_{2p}) \otimes \bigwedge^\bullet(\bigoplus_p \mathfrak{g}_{2p+1})$$

With a slight abuse of notation, on the right hand side we have the ordinary symmetric and exterior algebras of vector spaces. The grading of an element in $\mathrm{Sym}^r(\mathfrak{g}_{2p}) \otimes \bigwedge^r \mathfrak{g}_{2p+1}$ is $2pr + 2pr + r$. We could write down the direct sum decomposition by grading:

$$\mathrm{Sym}^\bullet\mathfrak{g} = k \oplus (\wedge^1\mathfrak{g}_1) \oplus (\mathrm{Sym}^1(\mathfrak{g}_2) \oplus \wedge^2\mathfrak{g}_1) \oplus (\wedge^3\mathfrak{g}_1 \oplus \wedge^1\mathfrak{g}_3 \oplus (\mathrm{Sym}^1(\mathfrak{g}_2) \otimes \wedge^1\mathfrak{g}_1)) \oplus ...$$

where we suppress tensor factors of $\mathrm{Sym}(\mathfrak{g}_0)$ (the entire symmetric algebra of $\mathfrak{g}_0$) in every degree which has degree 0 for the sake of readability. We introduce a notation:

$$\bigwedge^\bullet \mathfrak{g} \coloneqq \mathrm{Sym}^\bullet\mathfrak{g}[1]$$

where $\mathfrak{g}[1]$ denotes $\mathfrak{g}$ with degrees shifted by one. It is obvious from the above definition that the entire exterior algebra on every odd homogeneous component, and the entire symmetric algebra on every even homogeneous component, are recovered. However, the algebra structure allows one to form products of these which are witnessed by the mixed tensor products in the above decomposition. Take note that if $\mathfrak{g}$ is concentrated in degree 1 we obtain the free graded commutative algebra on a vector space, i.e., the exterior or Grassmann algebra.

Definition: Given a Lie algebra $\mathfrak{g}$, the Weil algebra of $\mathfrak{g}$, denoted by $W(\mathfrak{g})$ is the semi-free differential graded algebra whose underlying graded algebra is given by $\bigwedge^\bullet(\mathfrak{g}^* \oplus \mathfrak{g}^*[1])$, and whose differential is defined on generators as follows:

$$\mathrm{d}_{W(\mathfrak{g})}\big|_{\mathfrak{g}^*} = \mathrm{d}_{\mathrm{CE}} \otimes \mathrm{d} : \mathfrak{g}^* \to (\mathfrak{g}^* \wedge \mathfrak{g}^*) \otimes \mathfrak{g}^*[1]$$

$$\mathrm{d}_{W(\mathfrak{g})}\big|_{\mathfrak{g}^*[1]} = -[-,-]^*_\otimes : \mathfrak{g}^*[1] \cong \mathfrak{g}^* \to \mathfrak{g}^* \otimes \mathfrak{g}^*[1] \cong \mathfrak{g}^* \otimes \mathfrak{g}^*$$

where on the unshifted generators we have the Chevalley-Eilenberg differential and the free differential which simply sends an element of $\mathfrak{g}^*$ to itself shifted in degree one more and vanishes on $\mathfrak{g}^*[1]$. On the shifted generators we have the dual of the Lie bracket considered as simply a map on the tensor product and not the second exterior power:

$$[-,-]_\otimes : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$$

I'm often annoyed that I have to look at sometimes terse definitions and wonder what the author meant and whether or not I should really be doing mathematics because oh my god I've stared at this definition for an hour now and have I really understood it what on earth do they mean I computed this but is it right am I really going to call my thesis advisor at 9:00 PM on a Wednesday to ask why isn't this on the nLab I can't find ... Let me save you the trouble, the Weil algebra explicitly looks like:

$$W(\mathfrak{g}) = \bigoplus_{r\geq 0} \bigoplus_{2p+q=r} \mathrm{Sym}^p\mathfrak{g}^* \otimes \wedge^q\mathfrak{g}^* = k \oplus (\wedge^1\mathfrak{g}^*) \oplus (\mathrm{Sym}^1(\mathfrak{g}^*) \oplus \wedge^2\mathfrak{g}^*) \oplus ((\mathrm{Sym}^1\mathfrak{g}^* \otimes \wedge^1\mathfrak{g}^*) \oplus ...) \oplus ...$$

where in the expansion we can see the shifted generators in degree 2 as $\mathrm{Sym}^1\mathfrak{g}^*$, and we can see the tensor square of $\mathfrak{g}^*$ as $\mathrm{Sym}^1 \otimes \wedge^1\mathfrak{g}^*$ where the degree difference is made clear.

Proposition: Let G be a Lie group and $p : Y \to X$ a principal G-bundle. There exists a bijection

$$\Omega^1(Y, \mathfrak{g}) \cong \mathrm{Hom}_{\mathrm{dgca}}(W(\mathfrak{g}), \Omega^\bullet(Y))$$

PROOF: This follows the same as the case for the Weil algebra, the difference here is that the generators we've added in degree 2 will account for non-trivial curvature. Let $\omega \in \Omega^1(Y, \mathfrak{g})$ be a $\mathfrak{g}$-valued one form and F_ω its curvature, we define a morphism:

$$\psi : W(\mathfrak{g}) \to \Omega^\bullet(Y)$$

by defining it on generators:

$$\psi\big|_{\mathfrak{g}^*} : \mathfrak{g}^* \to \Omega^1(Y)$$
$$\psi\big|_{\mathfrak{g}^*}(\alpha) = \alpha \circ \omega$$
$$\psi\big|_{\mathfrak{g}^*[1]} : \mathfrak{g}^*[1] \to \Omega^2(Y)$$
$$\psi\big|_{\mathfrak{g}^*[1]}(\beta) = \beta \circ F_\omega$$

Now we verify that ψ respects differentials, again on generators. Let $\alpha \in \mathfrak{g}^*$ and $\beta \in \mathfrak{g}^*[1]$. Then we have the following:

$$\psi(\mathrm{d}_W(\alpha)) = \psi(\mathrm{d}_{\mathrm{CE}}\alpha + \mathrm{d}\alpha) = \psi(\mathrm{d}_{\mathrm{CE}}) + \psi(\mathrm{d}\alpha) = -\alpha \circ [\omega \wedge \omega] + \alpha \circ F_\omega = \alpha(F_\omega - [\omega \wedge \omega])$$

and

$$\mathrm{d}_{\mathrm{dR}}(\psi\alpha) = \alpha(\mathrm{d}_{\mathrm{dR}}\omega)$$

We see that these are equal since $F_\omega = \mathrm{d}_{\mathrm{dR}}\omega + [\omega \wedge \omega]$. Now on degree 2 we have:

$$\psi(\mathrm{d}_W\beta) = -\psi(\beta([-,-]_\otimes)) = -\beta([\omega \wedge F_\omega])$$

where here we have used that the dual of the bracket must have an entry in degree 1 and degree 2 and ψ was defined accordingly. Lastly we have

$$\mathrm{d}_{\mathrm{dR}}(\psi(\beta)) = \beta(\mathrm{d}_{\mathrm{dR}}F_\omega)$$

and we see that the commutator vanishes as a result of the Bianchi identity:

$$\mathrm{d}_{\mathrm{dR}}F_\omega + [\omega \wedge F_\omega] = 0$$

I'll leave the other direction to you.

As if this were not enough, one may further prod at this structure. Take notice in what exactly differs between $\mathrm{CE}(\mathfrak{g})$ and $W(\mathfrak{g})$. The latter contains the entire symmetric algebra on $\mathfrak{g}^*$ concentrated in even degrees, these are of course, polynomial functions on the Lie algebra $\mathfrak{g}$. Any idea where this is going?

DEFINITION: Given a Lie group G, we can *define* an invariant polynomial P on the Lie algebra as a wedge product of elements in the shifted generators $P \in \wedge^\bullet\mathfrak{g}^*[1] \hookrightarrow W(\mathfrak{g})$ such that $\mathrm{d}_W P = 0$. These form a graded sub-algebra of $W(\mathfrak{g})$ and we write $\mathrm{inv}(\mathfrak{g})$ for the dga given by this sub-algebra equipped with the trivial differential.

Let's see exactly why this definition is equivalent to the usual one. The fact that $P \in \wedge^\bullet\mathfrak{g}^*[1]$ can be interpretted as a polynomial should be no shock, the exterior and symmetric algebras can be obtained from each other by degree shift and in this case we see that $P \in \wedge^\bullet\mathfrak{g}^*[1] \cong \mathrm{Sym}(\mathfrak{g}^*)$. This means that our invariance statement comes from the closedness property, if we write P as a function:

$$P : \mathfrak{g}^{\otimes n} \to k$$

then

$$\mathrm{d}_W P = -P([-,-]) = -P([-,-],-,...,-) - P(-,[-,-],...,-) - ... - P(-,...,-,[-,-]) = 0$$

is the usual invariance statement for polynomials (albiet moved to the wrong side because of sign conventions). We will see that this definition needs a bit of upgrading in order for it to be ported to L_∞ algebras, but that is not our concern now. At this juncture we are ready to state something rather interesting:

Theorem: Let G be a compact connected Lie group, in what is written below, $\mathbf{B}_\nabla G$ is the moduli stack (simplicial presheaf) of G-bundles with connection, $\mathbf{E}_\nabla G$ the moduli stack of trivial G-bundles with connection, G is replaced by its representable simplicial presheaf, and $\Omega^\bullet$ is the moduli stack of differential forms, all as defined in Freed-Hopkins' *Chern-Weil forms and Abstract Homotopy theory.* All presheaves are taken to be defined over cartesian spaces. We have a commutative diagram in which the horizontal arrows are equivalences:

$$\begin{tikzcd}
\mathrm{CE}(\mathfrak{g}) \arrow[r] & \Omega^\bullet(G) \\
W(\mathfrak{g}) \arrow[r] \arrow[u] & \Omega^\bullet(\mathbf{E}_\nabla G) \arrow[u] \\
\mathrm{inv}(\mathfrak{g}) \arrow[r] \arrow[u] & \Omega^\bullet(\mathbf{B}_\nabla G) \arrow[u]
\end{tikzcd}$$

Now, in the above we've explicitly studied bijections between hom-sets of dga's and $\mathfrak{g}$-valued forms on total spaces of principal bundles with arbitrary or vanishing curvature. Of course, these need not form connections. In order for such a differential form to comprise principal connection data we need the following two "Cartan-Ehresmann" conditions to be satisfied:

Cartan-Ehresmann Conditions: Let G be a Lie group, $\rho : Y \times G \to Y$ be the action of G on the total space of a principal G-bundle X. Further, for any $v \in \mathfrak{g}$ let $\rho_*(v)$ be left invariant the vector field on Y which at a point $y \in Y$ is the derivative of $\rho(y, -) : G \to Y$ at the identity evaluated at v. A $\mathfrak{g}$-valued 1-form $A \in \Omega^1(Y, \mathfrak{g})$ comprises a principal connection if:

1. For any $v \in \mathfrak{g}$, $\iota_{\rho_*(v)} A = v$.
2. For any $v \in \mathfrak{g}$, $\iota_{\rho_*(v)} F_A = 0$.

These conditions can be packaged into our framework rather easily, but first some housekeeping:

Recall: For any submersion $p : Y \to X$ we can define the vertical de Rham complex of Y with respect to p, denoted by $\Omega^\bullet_{\mathrm{vert}}(Y)$, as a quotient of dga's:

$$\Omega^\bullet_{\mathrm{vert}}(Y) \coloneqq \Omega^\bullet(Y)/\langle \Omega^\bullet_{\mathrm{hor}}(Y)\rangle$$

where $\langle \Omega^\bullet_{\mathrm{hor}}(Y)\rangle$ is the dg-ideal generated by those forms which vanish on the kernel of $dp : TY \to TX$, this is the ideal of *horizontal forms.* This is just another way of thinking about vertical forms in the usual sense (forms on the fibers of p), but with a dg-algebraic sensibility.

Proposition: Given a surjection submersion with simply connected fibers $p : Y \to X$, and a Lie group G, cocycles

$$g : Y \times_X Y \to G$$

are in bijection with morphisms of differential graded algebras:

$$A_{\mathrm{vert}} : \mathrm{CE}(\mathfrak{g}) \to \Omega^\bullet_{\mathrm{vert}}(Y)$$

Proof: From the above discussion, it shouldn't require too much convincing that a morphism A_{vert} as described above defines a flat $\mathfrak{g}$-valued 1-form, unsurprisingly the fact that we are mapping into the vertical de Rham complex means this form is vertical. Thus, on each fiber we have a connection 1-form, which we denote abusively by A_{vert}. Given two points in the same fiber $(y_1, y_2) \in Y \times_X Y$, the simple connectedness allows us to consider the parallel transport of A_{vert} along any path $\gamma : y_1 \to y_2$, and the flatness of A_{vert} allows us to disregard dependence on the choice of path. The parallel transport is an element of G:

$$\text{tra}_\gamma A_{\text{vert}} = P\text{exp}(\int_\gamma A_{\text{vert}}) \in G$$

which defines for us a map

$$g : Y \times_X Y \to G$$
$$g(y_1, y_2) = P\text{exp}(\int_\gamma A_{\text{vert}})$$

that this map satisfies the cocycle condition is a sort of multiplicativity of the path ordered exponential. In the other direction, given any principal G-bundle, $p : Y \to X$, we can define the canonical invariant, vertical, $\mathfrak{g}$-valued 1-form by taking the identity map $TY \to TY$ and then projecting to the vertical vectors and composing with the isomorphism $TY \to TY \to \ker(p) \cong \mathfrak{g}$.

Disclaimer: The above proposition is perhaps a bit misleading. Morphisms of graded algebras like this don't coincide with any classification data for higher bundles when we replace $\mathfrak{g}$ by an L_∞ algebra, but I've chosen to include it here because the proof makes obvious that these morphisms of graded algebras correspond to something like fiberwise Maurer-Cartan forms on Y. These "fiber-wise Maurer-Cartan objects" are necessary for the formulation of L_∞ connections as we will see.

Proposition: For a principal G-bundle $Y \to X$ with simply connected fibers, let $A_{\text{vert}} : \text{CE}(\mathfrak{g}) \to \Omega^\bullet_{\text{vert}}(Y)$ be the canonical $\mathfrak{g}$-descent object. If there exists data $(A, F_A) : W(\mathfrak{g}) \to \Omega^\bullet_{\text{vert}}(Y)$ and $\{K_i\} : \text{inv}(\mathfrak{g}) \to \Omega^\bullet_{\text{vert}}(Y)$ which lift the descent object, i.e., such that the following diagram commutes:

$$\begin{tikzcd}
\Omega^\bullet_{\text{vert}}(Y) & \text{CE}(\mathfrak{g}) \arrow[l, "A_{\text{vert}}"'] \\
\Omega^\bullet(Y) \arrow[u, "q"] & W(\mathfrak{g}) \arrow[l, "(A,F_A)"'] \arrow[u] \\
\Omega^\bullet(X) \arrow[u, "p^*"] & \text{inv}(\mathfrak{g}) \arrow[l, "\{K_i\}"'] \arrow[u, hook]
\end{tikzcd}$$

Then A is a principal connection with curvature F_A.

The commutativity of the top square is equivalent to the first Cartan-Ehresmann condition, the bottom square the second. Let's see how:

Proof: The commutativity of the top square says that when A is restricted to the fibers of $p : Y \to X$, i.e., when only evaluated on vertical vector fields, that we obtain A_{vert}. If the square commutes, then Cartan-Ehresmann (1) holds because all left invariant vector fields are vertical. Suppose that Cartan-Ehresmann (1) holds and let χ be a vertical vector field, if $\chi = \rho_*(v)$ for some $v \in \mathfrak{g}$ then the commutativity of the top square and Cartan-Ehresmann (1) are manifestly equivalent. Suppose that χ is not of thi form, explicitly, suppose that there does not exist a single $v \in \mathfrak{g}$ such that for all points $y \in Y$, $\chi_y = \rho_*(v)_y$. Observe that if we restrict $T\rho(y, -) : TG \to TY$ to the tangent space at the identity we obtain an isomorphism $\mathfrak{g} \to V_yY$, where V_yY is the vertical tangent space of Y at y. Pulling back χ_y along this isomorphism gives an element $v_{\chi_y} \in \mathfrak{g}$, and by definition, $\rho_*(v_{\chi_y})_y = \chi_y$. By the previous case then, Cartan-Ehresmann (1) implies commutativity of the top square at the point y. However, we can repeat this process for any y and the equivalence is obtained globally for the arbitrary vertical vector field χ.

Commutativity of the bottom square amounts to saying that for any invariant polynomial $P_i \in \wedge^n \mathfrak{g}^*[1]$, the characteristic form $P_i(F_A)$ is pulled back from a $2n$-form K_i. This in turn is equivalent to the characteristic forms $P_i(F_A)$ being basic. Recall that basic forms, defined for any submersion, are those forms that vanish on vertical vector fields, and whose differentials also vanish on vertical vector fields. When the submersion under consideration is a principal bundle, the condition of being basic reduces to a pair of conditions in terms of the Cartan calculus:

LEMMA: A form ω is basic relative to a principal G-bundle if and only if for every $v \in \mathfrak{g}$

1. $\iota_{\rho_*(v)}\omega = 0$
2. $\mathcal{L}_{\rho_*(v)}\omega = [\mathrm{d_{dR}}, \iota_{\rho_*(v)}]\omega = 0$

$P_i(F_A)$ is basic if and only if F_A is basic, so we reduce to proving that Cartan-Ehresmann (2) holds if and only the F_A is basic. Assuming Cartan-Ehresmann (2) holds gives us half of the proof trivially, the second condition comes from the fact that principal curvature forms are invariant under the G-action on the total space. The opposite direction is trivial.

We're almost ready to generalize all of this, first, let's recall the basic notions of an L_∞ algebra:

DEFINITION: An L_∞ algebra is a graded vector space $\mathfrak{g}$ together with skew symmetric multilinear maps $\{l_n : \mathfrak{g}^{\otimes n} \to \mathfrak{g}\}_{n\in\mathbb{N}}$, the nth such map having degree $2-n$, subject to the generalized Jacobi identity for all choices of n:

$$\mathcal{J}_n \coloneqq \sum_{p=1}^{n}(-1)^{p(n-p)} \sum_{\sigma\in \mathrm{Shuff}(p,n-p)} \varepsilon(\sigma, v_1, ..., v_n) l_{n-p+1}(l_p(v_{\sigma(1)}, ..., v_{\sigma(p)}), v_{\sigma(p+1)}, ..., v_{\sigma(n)}) = 0$$

Where $\varepsilon(\sigma, v_1, .., v_n)$ is the total effect of signs by both σ and the permuting of graded elements by the Koszul sign rule. Unpacking the above identity for some special cases is enlightening:

$$\mathcal{J}_1 = l_1 \circ l_1 = 0$$

says that l_1 is a differential.

$$\mathcal{J}_2 = -l_2(l_1(v_1), v_2) + (-1)^{|v_1||v_2|} l_2(l_1(v_2), v_1) + l_1(l_2(v_1, v_2)) = 0$$

To clean this up observe that:

$$(-1)^{|v_1||v_2|} l_2(l_1(v_2), v_1) = -(-1)^{(|v_2|+1)|v_1|}(-1)^{|v_1||v_2|} l_2(v_1, l_1(v_2)) = -(-1)^{|v_1|} l_2(v_1, l_1(v_2))$$

and thus

$$l_1(l_2(v_1, v_2)) = l_2(l_1(v_2), v_1) + (-1)^{|v_1|} l_2(v_1, l_1(v_2))$$

so l_1 is a graded derivation of l_2. Let's have a look at $n = 3$, with some cleaning up:

$$\mathcal{J}_3 = l_3(l_1(v_1), v_2, v_3) + (-1)^{|v_1|} l_3(v_1, l_1(v_2), v_3) + (-1)^{|v_1+|v_2|} l_3(v_1, v_2, l_1(v_3))$$

$$+l_2(l_2(v_1, v_2), v_3) - (-1)^{|v_3||v_2|} l_2(l_2(v_1, v_3), v_2) + (-1)^{|v_1|(|v_2|+|v_3|)} l_2(l_2(v_2, v_3), v_1) + l_1(l_3(v_1, v_2, v_3)) = 0$$

we see then that

$$l_2(l_2(v_1, v_2), v_3) - (-1)^{|v_3||v_2|} l_2(l_2(v_1, v_3), v_2) + (-1)^{|v_1|(|v_2|+|v_3|)} l_2(l_2(v_2, v_3), v_1) =$$

$$-l_1(l_3(v_1, v_2, v_3)) - l_3(l_1(v_1), v_2, v_3) - (-1)^{|v_1|} l_3(v_1, l_1(v_2), v_3) - (-1)^{|v_1+|v_2|} l_3(v_1, v_2, l_1(v_3))$$

on the left hand side we have the sum present in the graded Jacobi identity. If we let

$$f = \sum_{\sigma\in \mathrm{Shuff}(2,1)} \varepsilon(\sigma, -, -) l_2(l_2(-, -), -)$$

(the left hand side of the above), then viewing l_1 as a differential, and l_3 as a degree $2 - 3 = -1$ cochain map, we can draw the following diagram of cochain complexes:

$$\begin{tikzcd}
(\mathfrak{g}^{\otimes 3})_{n-1} \arrow[r, "l_1^{\otimes 3}"] \arrow[d, "f"'] & (\mathfrak{g}^{\otimes 3})_{n} \arrow[r, "l_1^{\otimes 3}"] \arrow[d, "f"'] \arrow[dl, "l_3"'] & (\mathfrak{g}^{\otimes 3})_{n+1} \arrow[d, "f"] \arrow[dl, "l_3"'] \\
\mathfrak{g}_{n-1} \arrow[r, "l_1^{\otimes 3}"'] & \mathfrak{g}_{n} \arrow[r, "l_1^{\otimes 3}"'] & \mathfrak{g}_{n+1}
\end{tikzcd}$$

The equality $\mathcal{J}_3 = 0$ states precisely that l_3 witnesses f as cochain homotopic to 0. For higher values of n, the equations $\mathcal{J}_n = 0$ are the manifestations of higher homotopy coherences.

Observations: If $l_n = 0$ for all $n > 2$, an L_∞ algebra is just a differential graded Lie algebra. If in addition, $\mathfrak{g}$ is concentrated in degree 1, this is an ordinary Lie algebra.

We can characterize the degree-wise finite dimensional L_∞ algebras with a more familiar structure with a bit of work.

Definition: A co-derivation $D : C \to C$ on a co-algebra C with co-multiplication $\Delta : C \to C \otimes C$ is a k-linear map with the property that

$$\Delta \circ D = (D \otimes \mathrm{id} + \mathrm{id} \otimes D) \circ \Delta$$

Definition: The free graded commutative co-algebra on a graded vector space $\mathfrak{g}$, denoted $\bigvee^\bullet \mathfrak{g}$, has the same underlying graded vector space as $\bigwedge^\bullet \mathfrak{g}$ with co-mulitplication:

$$\Delta(t_1 \vee ... \vee t_n) = \sum_{p+q=n} \sum_{\sigma \in \mathrm{Shuff}(p,q)} \varepsilon(\sigma, t_1, ..., t_n)(t_{\sigma(1)} \vee t_{\sigma(3)} \vee ... \vee t_{\sigma(i)}) \otimes (t_{\sigma(i+1)} \vee ... \vee t_{\sigma(n)})$$

where here $\varepsilon(\sigma, t_1, ..., t_n)$ is the Koszul sign.

Remark: Given an L_∞ algebra $\mathfrak{g}$, we can define a differential graded co-algebra structure on $\vee^\bullet \mathfrak{g}$. Each bracket l_n is extended to indecomposables of wordlength k to graded co-derivations by the following formula:

$$l_n(t_1 \vee ... \vee t_k) = \sum_{\sigma \in \mathrm{Shuff}(n,k-n)} \varepsilon(\sigma, t_1, ..., t_k) l_n(t_{\sigma(1)} \vee ... \vee t_{\sigma(n)}) \vee t_{\sigma(n+1)} \vee ... \vee t_{\sigma(k)}$$

Once this extension has been made, we define a graded co-derivation $D : \vee^\bullet \mathfrak{g} \to \vee^\bullet \mathfrak{g}$ by

$$D = \sum_{n \geq 1} l_n$$

The take away at this point is that L_∞ algebras are equivalently semi-free differential graded co-algebras (although we only showed one direction here). In the case of (degreewise) finite dimensional L_∞ algebras, we can dualize everything to obtain the more familiar and friendly semi-free differential graded algebras.

Definition: An L_∞ algebra is a graded vector space $\mathfrak{g}$ with a degree -1 co-derivation

$$D : \vee^\bullet \mathfrak{g} \to \vee^\bullet \mathfrak{g}$$

When $\mathfrak{g}$ is degree-wise finite dimensional, this is equivalently a degree +1 derivation

$$D^* : \wedge^\bullet \mathfrak{g}^* \to \wedge^\bullet \mathfrak{g}^*$$

We define the Chevalley-Eilenberg algebra of a finite dimensional L_∞ algebra as the differential graded algebra

$$\mathrm{CE}(\mathfrak{g}) \coloneqq (\wedge^\bullet \mathfrak{g}^*, \mathrm{d}_{\mathrm{CE}} \coloneqq D^*)$$

THEOREM: The categories of degree-wise finite dimensional semi-free differential graded algebras is contravariantly equivalent to the category of L_∞ algebras of finite type.

From here on out, we will identify a finite dimensional L_∞ algebra $\mathfrak{g}$ with its Chevalley-Eilenberg algebra $\mathrm{CE}(\mathfrak{g})$.

EXAMPLE: This could serve almost as a definition, but in light of the theory of higher Lie integration, it really is an example. The line Lie n-algebra, denoted $\mathrm{b}^{n-1}\mathfrak{u}(1)$ is given by

$$\mathrm{CE}(\mathrm{b}^{n-1}\mathfrak{u}(1)) = (\wedge^\bullet(\mathbb{R}[n]), \mathrm{d}_{\mathrm{b}^{n-1}\mathfrak{u}(1)} = 0)$$

Although this won't be expounded upon here, in a rigorous sense this L_∞ algebra "integrates" to the ∞-groupoid $\mathbf{B}\mathrm{U}(1)$.

EXAMPLE: Perhaps first non-trivial example to consider is the string Lie 2-algebra. Consider $\mathfrak{so}(n)$, the Lie alebra of $\mathrm{SO}(n)$. On any finite dimensional Lie algebra $\mathfrak{g}$ we have a canonical symmetric bilinear invariant polynomial $\langle -, - \rangle : \mathfrak{g} \otimes \mathfrak{g} \to k$, the Killing form, given by

$$\langle x, y \rangle = \mathrm{tr}([x, [y, -]])$$

We define a degree 3 element of $\mathrm{CE}(\mathfrak{g})$ as:

$$\mu = \langle -, [-, -] \rangle$$

which is in fact closed:

$$\begin{aligned}
(\mathrm{d}_{\mathrm{CE}}\mu)(x_1, x_2, x_3, x_4) &= -\mu([x_1, x_2], x_3, x_4) + \mu([x_1, x_3], x_2, x_4) - \mu([x_1, x_4], x_2, x_3) - \mu([x_2, x_3], x_1, x_4) \\
&\quad + \mu([x_2, x_4], x_1, x_3) - \mu([x_3, x_4], x_1, x_2) \\
&= -\langle [x_1, x_2], [x_3, x_4] \rangle + \langle [x_1, x_3], [x_2, x_4] \rangle - \langle [x_1, x_4], [x_2, x_3] \rangle - \langle [x_2, x_3], [x_1, x_4] \rangle \\
&\quad + \langle [x_2, x_4], [x_1, x_3] \rangle - \langle [x_3, x_4], [x_1, x_2] \rangle = 0
\end{aligned}$$

You can see that this sum should vanish in pairs because $\langle -, - \rangle$ is invariant. Thus μ is a 3 cocycle in the Lie algebra cohomology of $\mathfrak{so}(n)$, and thus classifies a shifted extension:

$$\mathrm{b}\mathfrak{u}(1) \to \mathfrak{g}_\mu \to \mathfrak{so}(n)$$

the object $\mathfrak{g}_\mu$ is not a Lie algebra but the Lie 2-algebra $\mathfrak{string}(n)$. We give it's explicit description:

$$\mathrm{CE}(\mathfrak{string}(n)) = (\wedge^\bullet(\mathfrak{g}^* \oplus \langle s \rangle[1]), \mathrm{d}_{\mathfrak{string}(n)})$$

where $\langle s \rangle$ is a single generator placed in degree 2. The differential is given on generators by:

$$\mathrm{d}_{\mathfrak{string}(n)}\big|_{\mathfrak{g}^*} = \mathrm{d}_{\mathrm{CE}(\mathfrak{g})}$$

$$\mathrm{d}_{\mathfrak{string}(n)}(s) = \mu$$

This can all be generalized. For any $\mathfrak{g}$ an L_∞ algebra and μ a degree n cocycle in $\mathrm{CE}(\mathfrak{g})$, there is a corresponding extension

$$\mathrm{b}^{n-2}\mathfrak{u}(1) \to \mathfrak{string}_\mu(n) \to \mathfrak{g}$$

We can now extend all of these concepts and define the moduli stack of L_∞-connections.

Write Δ^k for the smooth k-simplex, a manifold with boundary and corners. Write $\Omega^\bullet_{\mathrm{si}}(\Delta^k)$ for the dga of differential forms on Δ^k with sitting instants (constant around boundaries/corners). Given a Cartesian space U, a form $\omega \in \Omega^n(U \times \Delta^k)$ has sitting instants if for every point $u : * \to U$, the pullback of ω along

$(u, \mathrm{Id}) : \Delta^k \to U \times \Delta^k$ has sitting instants. Finally we write $\Omega^n_{\mathrm{si}}(U \times \Delta^k)_{\mathrm{vert}}$ for those forms with sitting instants that are vertical with respect to the projection $U \times \Delta^k \to U$.

Definition: For an L_∞ algebra $\mathfrak{g}$ we define a simplicial presheaf on cartesian spaces:

$$\exp_\Delta(\mathfrak{g})(U, [k]) = \mathrm{Hom}_{\mathrm{dga}}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet_{\mathrm{si}}(U \times \Delta^k)_{\mathrm{vert}})$$

$\exp_\Delta(\mathfrak{g})(U)$ is a simplicial set of smooth U-familes of differential forms on $\Delta^1, \Delta 2, \ldots.$ $\exp_\Delta(\mathfrak{g})$ is fibrant in the projective model structure on simplicial presheaves so it takes values in Kan complexes, i.e., it is an ∞-groupoid.

Proposition: When G is a simply connected Lie group with Lie algebra $\mathfrak{g}$, we have an equivalence:

$$\exp_\Delta(\mathfrak{g}) \to \mathbf{B}G$$

Proof: Given a U-family of flat $\mathfrak{g}$-valued 1-form on the 1 simplex, we obtain a smooth function on U valued in G by for each point $u \in U$, fixing a flat 1-form A_u on $[0, 1]$, and integrating along $[0, 1]$ to obtain an element in G. For higher simplices we integrate along $[0, 1]$ and then along $[1, 2], [2, 3], ..., [n-1, n]$ to obtain an element in G^{n+1}. Since the A_u are flat, this assignment is well defined and we obtain an element in $\mathbf{B}G(U)_{n+1}$.

Proposition: There is a similar equivalence:

$$\exp_\Delta(\mathrm{b}^{n-1}\mathfrak{u}(1)) \to \mathbf{B}^n \mathrm{U}(1)$$

We have several differential refinements of this stack to classifying stacks of various connection data.

Definition: $\exp_\Delta(\mathfrak{g})_{\mathrm{diff}}$ is the classifying stack for $\mathfrak{g}$-valued forms that satisfy CE(1). On a pair $(U, [k])$

$$\exp_\Delta(\mathfrak{g})_{\mathrm{diff}}(U, [k]) =$$

$$\begin{tikzcd}
\Omega^\bullet_{\mathrm{si}}(U \times \Delta^k)_{\mathrm{vert}} & & \mathrm{CE}(\mathfrak{g}) \arrow[ll, "A_{\mathrm{vert}}"'] \\
\Omega^\bullet_{\mathrm{si}}(U \times \Delta^k) \arrow[u] & & W(\mathfrak{g}) \arrow[ll, "A"'] \arrow[u]
\end{tikzcd}$$

that is to mean, on a pair $(U, [k])$ we assign the set of commuting diagrams as above. This is essentially the same (weakly equivalent in the case of a Lie algebra for simply connected Lie group) as $\exp_\Delta(\mathfrak{g})$. The whole point of this resolution becomes apparent in the ∞-Chern-Weil theory as it "serves to model the canonical curvature characteristic map $\mathbf{B}G \to \flat_{\mathrm{dR}}\mathbf{B}G$". This is a bit outside the scope of these notes/talks, but this is the content of section 5 in *Cech cocycles for differential characteristic classes* by Schreiber, Fiorenza, and Stasheff for those interested.

Definition: $\exp_\Delta(\mathfrak{g})_{\mathrm{CW}} \subset \exp_\Delta(\mathfrak{g})_{\mathrm{diff}}$ is the moduli stack of $\mathfrak{g}$-valued forms satisfying CE(1) and CE(2), on a pair $(U,[k])$ we assign the set of commuting diagrams:

$$\begin{tikzcd}
\Omega^\bullet_{\mathrm{si}}(U\times\Delta^k)_{\mathrm{vert}} & \mathrm{CE}(\mathfrak{g}) \arrow[l, "A_{\mathrm{vert}}"'] \\
\Omega^\bullet_{\mathrm{si}}(U\times\Delta^k) \arrow[u] & W(\mathfrak{g}) \arrow[l, "A"'] \arrow[u] \\
\Omega^\bullet_{\mathrm{si}}(U) \arrow[u] & \mathrm{inv}(\mathfrak{g}) \arrow[l, "F_A"'] \arrow[u]
\end{tikzcd}$$

This is almost the moduli stack of L_∞ connections, and for a principal G bundle with Lie algebra $\mathfrak{g}$ this works, but in the general case of an L_∞ algebras we have to enforce a horizontality condition (see section 5.3 of L_∞ *connections and applications to string and Chern-Simons n-transport* by Schreiber, Sati, and Stasheff for the definition of vertical fields in the language of dg algebras).

Definition: $\exp_\Delta(\mathfrak{g})_{\mathrm{conn}} = \exp_\Delta(\mathfrak{g})_{\mathrm{CW}} +$ horizontality, is the moduli stack of L_∞-connections. The horizontality condition states that for F_A in the above diagram, and for any vertical vector field χ on $U\times\Delta^k$, $\iota_\chi F_A = 0$. For an L_∞ algebra this vector field and this contraction are defined in the language of dgas as in the above reference.