

The quantomorphism  
 $\infty$ -group extensions

# Quantomorphisms of a higher connection (recollection)

(A)  $U(1)\text{-}u\text{-Conn}$  is a stack equipped w/ a

refinement map  $B^u U(1)\text{-}u\text{-Conn} \xrightarrow{P} U(1)\text{-}u\text{-Conn}$

which "projects onto  $X$ " :  $\{X, B^u U(1)\text{-}u\text{-Conn}\} \rightarrow U(1)\text{-}u\text{-Conn}(X)$

(B)  $\text{Quant Morph}(\nabla) \rightarrow \text{Aut}(X)$

$$\begin{array}{ccc}
 & \xrightarrow{h} & \\
 \downarrow & & \downarrow \\
 * & \xrightarrow{p \nabla} & U(1)\text{-}u\text{-Conn}(X)
 \end{array}
 \quad
 \begin{array}{c}
 \\
 \\
 p[X, \nabla]
 \end{array}$$

# Hamiltonian symplectomorphisms (recollection)

Def:  $\text{Ham Symp}(\nabla) := \text{im}_1(\text{Quant Morph}(\nabla) \rightarrow \text{Aut}(X))$

i.e.  $\pi_0(\text{Quant Morph}(\nabla)) \twoheadrightarrow \pi_0(\text{Ham Symp}(\nabla))$

$\pi_0(\text{Ham Symp}(\nabla)) \hookrightarrow \pi_0(\text{Aut}(X))$

$\pi_{i>0}(\text{Ham Symp}(\nabla)) \xrightarrow{\cong} \pi_i(\text{Aut}(X))$

Quant Morph ( $\nabla$ ) as a  
 (higher) central extension of  
 Ham Symp ( $\nabla$ )

Thm (3.3.1)  $\nabla : X \rightarrow \mathbb{B}U(1)_{\text{conn}}$  - prin. conn. on  $X$

We have a l.e. fib. s. of smooth  $\infty$ -groups

$$\begin{array}{ccccc}
 \text{Flat } U(1)\text{-}(n\text{-})\text{-Conn}(X) & \rightarrow & \text{Quant Morph } (\nabla) & \xrightarrow{\quad} & \pi \\
 \downarrow & & \downarrow & & \downarrow \\
 \pi & \xrightarrow{\quad} & \text{Ham Symp } (\nabla) & \xrightarrow{\quad} & \mathbb{B}(\text{Flat } U(1)\text{-}(n\text{-})\text{-Conn}(X)) \\
 & & \nabla_X & & 
 \end{array}$$

# Proof

(1) Consider 1-image factorizations

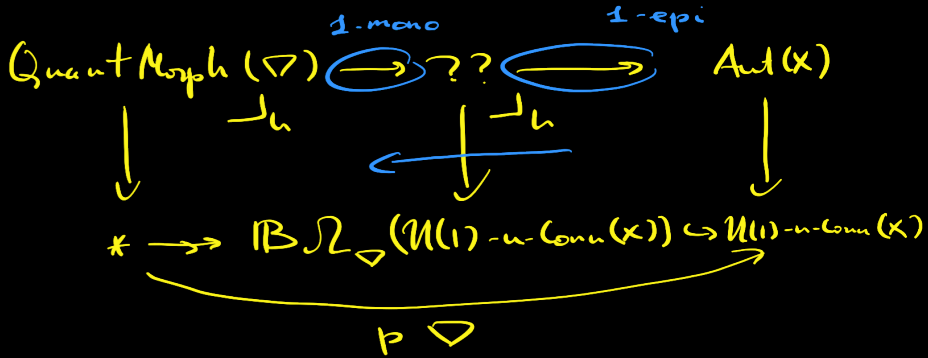
$$\begin{array}{ccccc} \text{Quant Morph}(\mathcal{D}) & \rightarrow & \text{Hom Symp}(\mathcal{D}) & \xleftrightarrow{\quad} & \text{Aut}(X) \\ \downarrow & & \downarrow & & \downarrow \\ \text{pt} & \rightarrow & \text{BR}_{\mathcal{D}}(\mathcal{U}(1) \text{-} n\text{-Conn}(X)) & \xleftrightarrow{\quad} & \mathcal{U}(\mathcal{D} \text{-} n\text{-Conn}(X)) \end{array}$$

$\curvearrowright$

(2) 1-image factorization is ess. unique

# Proof II

1) Consider a pasting diagram of string pull-backs



# Proof III

Δ) By uniqueness ??  $\cong$   $\text{Hann Symp}(\nabla)$

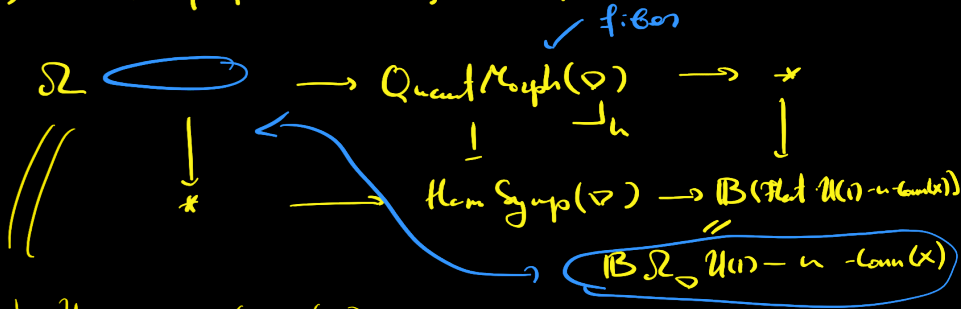
ℜ) Note that  $\Omega_{\nabla} \mathcal{U}(1)-n\text{-Conn}(X) \cong \text{Flat } \mathcal{U}(1)-(n-1)\text{-Conn}(X)$

$$\begin{array}{ccc}
 \Omega_{\nabla} \mathcal{U}(1)-n\text{-Conn}(X) & \xrightarrow{\quad} & * \\
 \downarrow & \searrow & \downarrow p_{\nabla} \\
 * & \xrightarrow{p_{\nabla}} & \mathcal{U}(1)-n\text{-Conn}(X)
 \end{array}$$

This is a version of equiv:  
 $\Omega_{\nabla} \mathcal{H}(X; \mathbb{B}^{n+1} \mathcal{U}(1)_{\text{Conn}}) \cong$   
 $\cong \mathcal{H}(X, \mathfrak{b} \mathbb{B}^n \mathcal{U}(1))$

# Proof IV

K) Now flip the diagram of our images:



Flat- $\mathcal{U}(1)$ - $(n-1)$ -Conn( $X$ )



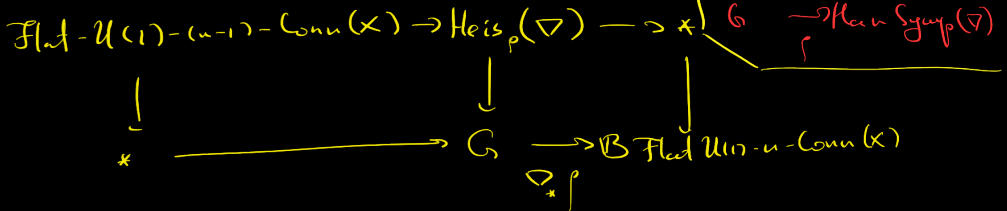


# Grading for Heisenberg groups

Cor (3.3.2)  $G \xrightarrow{\rho} \text{Ham Symp}(\nabla)$  - Hamiltonian

$G$ -action  $\Rightarrow$  we have

Recall:  $\text{Heis}_\rho(\nabla) \rightarrow \text{Quat Morph}(\nabla)$



## Alternative description of Flat- $U(1)$ - $n$ -Conn( $X$ )

Prop(3.3.3)  $X$  is  $(n-1)$ -conn. smooth  $n$ -fld (i.e.  $\pi_i(X) = 0$   
 $i \in n-1$ )

Then Flat- $U(1)$ - $(n-1)$ -Conn( $X$ )  $\cong B^{n-1}U(1)$ .

P-f: ( $n=2$ ) (i.e.  $X$  is simply connected)

An idea: if  $X$  is simply connected  $\Rightarrow$  all local systems are trivial  $\Rightarrow$  flat bundles = bundles w/ no connection

## Proof:

4) For  $(\Gamma + b)$ -adjunction we have

$H(X, b BU(n)) \leftarrow$  space of flat bundles

SI

$$H(\Gamma X, \underbrace{BU(n)}_{1\text{-Stack}}) \cong H((\Gamma X)_{\leq 1}; BU(n))$$

$$(\pi_i(\mathbb{C}P^\infty) = 0 \quad i > 2)$$

## Proof II

$$(2) H(U; \text{Flat-}\mathcal{U}(1)\text{-Conn}(X)) \simeq H(U \times \Gamma X, \mathbb{B}\mathcal{U}(1))$$

$$U \text{ is a } 0\text{-stack} \Rightarrow \simeq H(U \times (\Gamma X)_{\leq 1}, \mathbb{B}\mathcal{U}(1)) \textcircled{2}$$

$$1) (\Gamma X)_{\leq 1} \simeq * \Rightarrow \textcircled{2} H(U; \mathbb{B}\mathcal{U}(1)) \Rightarrow$$

$$\text{Flat-}\mathcal{U}(1)\text{-Conn}(X) \rightarrow \mathbb{B}\mathcal{U}(1)$$

3) For  $n > 2$  the proof is analogous  $\square$

# Corollary for (higher) conn. case

Cor(3.4) If  $X$  is  $(n-1)$ -conn. we have a fiber seq.:

$$\begin{array}{ccccc}
 \mathbb{B}^{n-1}\mathcal{U}(1) & \longrightarrow & \text{Quant Morph}(\nabla) & \longrightarrow & * \\
 \downarrow & & \downarrow & & \downarrow \\
 * & \longrightarrow & \text{Kum Symp}(\nabla) & \longrightarrow & \mathbb{B}^n\mathcal{U}(1)
 \end{array}$$

If  $G \xrightarrow{p} \text{Kum Symp}(\nabla)$

$$\begin{array}{ccccc}
 \mathbb{B}^{n-1}\mathcal{U}(1) & \longrightarrow & \text{Heis}_p(\nabla) & \longrightarrow & * \\
 \downarrow & & \downarrow & & \downarrow \\
 * & \longrightarrow & G & \longrightarrow & \mathbb{B}^n\mathcal{U}(1)
 \end{array}$$

## $n=1$ Application

If  $n=1$  ( $X$  is connected preq. symplectic manifold)

$$1 \rightarrow U(1) \rightarrow \text{Quant Morph}(\nabla) \rightarrow \text{Ham Symp}(D) \rightarrow 1$$

Central extension

Thank  
you!!!