The quantomorphism ∞-group extensions

Quantomorphisms of a higher connection (recollection)

(1) $U(1)$-n-Conn is a stack equipped w/ a refinement map $\mathbf{B}^n U(1)_{conn} \xrightarrow{p} U(1)$-$n$-Conn which "projects onto X": $[X, \mathbf{B}^n U(1)_{conn}] \to U(1)$-$n$Conn$(X)$

(2)
$$\begin{tikzcd}
\mathrm{QuantMorph}(\nabla) \arrow[r] \arrow[d] & \mathrm{Aut}(X) \arrow[d, "{p[X,\nabla]}"] \\
* \arrow[r, "p\nabla"] \arrow[ur, phantom, "\lrcorner h"] & U(1)\text{-}n\text{-}\mathrm{Conn}(X)
\end{tikzcd}$$

Hamiltonian symplectomorphisms

(recollection)

Def: $\mathrm{HamSymp}(\nabla) :\equiv \mathrm{im}_1\big(\mathrm{QuantMorph}(\nabla) \to \mathrm{Aut}(X)\big)$

i.e. $\pi_0\big(\mathrm{QuantMorph}(\nabla)\big) \twoheadrightarrow \pi_0\big(\mathrm{HamSymp}(\nabla)\big)$

$\pi_0\big(\mathrm{HamSymp}(\nabla)\big) \hookrightarrow \pi_0\big(\mathrm{Aut}(X)\big)$

$\pi_{i>0}\big(\mathrm{HamSymp}(\nabla)\big) \xrightarrow{\simeq} \pi_i\big(\mathrm{Aut}(X)\big)$

QuantMorph(∇) as a (higher) central extension of HamSymp(∇)

Thm (3.3.1) $\nabla : X \to \mathbf{B}^n U(1)_{\mathrm{conn}}$ - prin. conn. on X

We have a l.e. fiber s. of smooth ∞-groups

$$
\begin{tikzcd}
\mathrm{Flat}\, U(1)\text{-}(n-1)\text{-}\mathrm{Conn}(X) \arrow[r] \arrow[d] & \mathrm{QuantMorph}(\nabla) \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] & \mathrm{HamSymp}(\nabla) \arrow[r, "\nabla_*"'] & \mathbf{B}(\mathrm{Flat}\, U(1)(n-1)\mathrm{Conn}(X))
\end{tikzcd}
$$

Proof

α) Consider 1-image factorization

$$\begin{tikzcd}
\mathrm{QuantMorph}(\triangledown) \arrow[r, two heads] \arrow[d] & \mathrm{HomSymp}(\triangledown) \arrow[r, hook] \arrow[d] & \mathrm{Aut}(X) \arrow[d] \\
* \arrow[r, two heads] \arrow[rr, bend right, "\triangledown"'] & \mathbf{B}\Omega_{\triangledown}(U(1)\text{-}n\text{-}\mathrm{Conn}(X)) \arrow[r, hook] & U(1)\text{-}n\text{-}\mathrm{Conn}(X)
\end{tikzcd}$$

β) 1-image factorization is ess. unique

Proof II

1) Consider a pasting diagram of htpy pull-backs

$$\begin{tikzcd}
\mathrm{QuantMorph}(\triangledown) \arrow[r, "\text{1-mono}"] \arrow[d] \arrow[dr, phantom, "\lrcorner_h"] & ?? \arrow[r, "\text{1-epi}"] \arrow[d] \arrow[dr, phantom, "\lrcorner_h"] & \mathrm{Aut}(X) \arrow[d] \\
* \arrow[r] \arrow[rr, bend right, "p\,\triangledown"'] & \mathbf{B}\Omega_{\triangledown}(U(1)\text{-}n\text{-}\mathrm{Conn}(X)) \arrow[r, hook] & U(1)\text{-}n\text{-}\mathrm{Conn}(X)
\end{tikzcd}$$

Proof III

A) By uniqueness $?? \simeq \mathrm{Ham\,Symp}(\nabla)$

B) Note that $\Omega_\nabla U(1)\text{-}n\text{-}\mathrm{Conn}(X) \simeq \mathrm{Flat}\text{-}U(1)\text{-}(n-1)\text{-}\mathrm{Conn}(X)$

$$\begin{tikzcd}
\Omega_\nabla U(1)\text{-}n\text{-}\mathrm{Conn}(X) \arrow[r] \arrow[d] \arrow[rr, bend left, "\simeq"] & \ast \arrow[d, "p_\nabla"] & \mathrm{Flat}\ U(1)\text{-}(n-1)\text{-}\mathrm{Conn}(X) \\
\ast \arrow[r, "p_\nabla"'] & U(1)\text{-}n\text{-}\mathrm{Conn}(X) &
\end{tikzcd}$$

This is a version of equiv:

$$\Omega_\nabla \mathbf{H}(X, \mathbf{B}^{n+1} U(1)_{\mathrm{conn}}) \simeq$$

$$\simeq \mathbf{H}(X, \flat \mathbf{B}^n U(1))$$

Proof IV

F) Now flip the diagram of our images:

f: fibr

$$\begin{tikzcd} \Omega \arrow[r] \arrow[d] \arrow[dd, equal] & \mathrm{QuantMorph}(\nabla) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner_h"] & * \arrow[d] \\ * \arrow[r] & \mathrm{HamSymp}(\nabla) \arrow[r] & \mathbf{B}(\mathrm{Flat}\,U(1)\text{-}n\text{-}\mathrm{Conn}(X)) \arrow[d, equal] \\ \mathrm{Flat}\text{-}U(1)\text{-}(n-1)\text{-}\mathrm{Conn}(X) & & \mathbf{B}\Omega_\nabla U(1)\text{-}n\text{-}\mathrm{Conn}(X) \end{tikzcd}$$

$\square$

Corollary for Heisenberg groups

Cor (3.3.2) $G \overset{\rho}{\hookrightarrow} \mathrm{HamSymp}(\nabla)$ - Hamiltonian

Recall:

$$\begin{tikzcd}
\mathrm{Heis}_\rho(\nabla) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{QuantMorph}(\nabla) \arrow[d] \\
G \arrow[r, hook, "\rho"] & \mathrm{HamSymp}(\nabla)
\end{tikzcd}$$

G-action $\Rightarrow$ we have

$$\begin{tikzcd}
\mathrm{Flat}\text{-}U(1)\text{-}(n-1)\text{-}\mathrm{Conn}(X) \arrow[r] \arrow[d] & \mathrm{Heis}_\rho(\nabla) \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] & G \arrow[r, "\nabla_* \rho"'] & \mathbf{B}\,\mathrm{Flat}\,U(1)\text{-}n\text{-}\mathrm{Conn}(X)
\end{tikzcd}$$

Alternative description of $\text{Flat-}\mathcal{U}(1)\text{-}n\text{-Conn}(X)$

Prop (3.3.3) X is $(n-1)$-conn. smooth mfld (i.e. $\pi_i(X)=0$, $i \leq n-1$)

Then $\text{Flat-}\mathcal{U}(1)\text{-}(n-1)\text{-Conn}(X) \simeq B^{n-1}\mathcal{U}(1)$.

Pf: $(n=2)$ (i.e. X is simply connected)

An idea: if X is simply connected $\Rightarrow$ all local systems are trivial $\Rightarrow$ flat bundles = bundles w/ no connection

Proof:

(*) For $(\Pi \dashv \flat)$ - adjunction we have

$\mathbf{H}(X, \flat \mathbf{B}U(1))$ ← space of flat bundles

$\simeq$

$\mathbf{H}(\Pi X, \underbrace{\mathbf{B}U(1)}_{1\text{-stack}}) \simeq \mathbf{H}((\Pi X)_{\leq 1}; \mathbf{B}U(1))$

$(\pi_i(\mathbb{C}P^\infty) = 0 \quad i > 2)$

Proof II

2) $\mathbb{H}(\mathcal{U}; \text{Flat-}U(1)\text{-Conn}(X)) \simeq \mathbb{H}(\mathcal{U} \times \Pi X, \mathbf{B}U(1))$

$\mathcal{U}$ is a 0-stack $\Rightarrow \simeq \mathbb{H}(\mathcal{U} \times (\Pi X)_{\leq 1}; \mathbf{B}U(1))$

3) $(\Pi X)_{\leq 1} \simeq * \Rightarrow \mathbb{H}(\mathcal{U}; \mathbf{B}U(1)) \Rightarrow$

$\text{Flat-}U(1)\text{-Conn}(X) \longrightarrow \mathbf{B}U(1)$

4) For $n \geq 2$ the proof is analogous $\square$

Corollary for (higher) conn. case

Cor (3.3.4) If X is $(n-1)$-conn. we have a fiber seq.:

$$\begin{tikzcd}
B^{n-1}U(1) \arrow[r] \arrow[d] & \mathrm{QuantMorph}(\nabla) \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] & \mathrm{HamSymp}(\nabla) \arrow[r] & B^{n}U(1)
\end{tikzcd}$$

If $G \xrightarrow{\rho} \mathrm{HamSymp}(\nabla)$

$$\begin{tikzcd}
B^{n-1}U(1) \arrow[r] \arrow[d] & \mathrm{Heis}_{\rho}(\nabla) \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] & G \arrow[r] & B^{n}U(1)
\end{tikzcd}$$

$n=1$ Application

If $n=1$ (X is connected preq. symp. mfld)

$$\underbrace{1 \to U(1) \to \mathrm{QuantMorph}(\nabla) \to \mathrm{HamSymp}(\nabla) \to 1}_{\text{Central extension}}$$

Thank you!!!