String bordism invariants in dimension 3 from U(1)-valued TQFTs

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Outline

M n-dimensional smooth manifold

$$\begin{tikzcd}
 & * \arrow[d] & \\
 & \vdots \arrow[d] & \\
 & \mathbf{B}String \arrow[r, "\frac{1}{6}p_2"] \arrow[d] & \mathbf{B}^8\mathbb{Z} \\
 & \mathbf{B}Spin \arrow[r, "\frac{1}{2}p_1"] \arrow[d] & \mathbf{B}^4\mathbb{Z} \\
 & \mathbf{B}SO \arrow[r, "w_2"] \arrow[d] & \mathbf{B}^2\mathbb{Z}/2\mathbb{Z} \\
M \arrow[r, "TM"] \arrow[ur, bend left] \arrow[uur, bend left] \arrow[uuur, bend left] \arrow[uuuuur, bend left] & \mathbf{B}O \arrow[r, "w_1"] & \mathbf{B}\mathbb{Z}/2\mathbb{Z}
\end{tikzcd}$$

In particular if $n \leq 7$ there are no obstruction to lifting a string structure to a framing.

$$\mathrm{Bord}_3^{String} = \mathrm{Bord}_3^{fr} \underset{\substack{\uparrow \\ (Pontryagin - Thom)}}{=} \pi_3(\mathbb{S}) = \lim_{n \to +\infty} \pi_{n+3}(S^n) = \pi_8(S^5) = \mathbb{Z}/24\mathbb{Z}$$

One may wish to express the isomorphism $\varphi\colon\ : \mathrm{Bord}_3^{String} \xrightarrow{\cong} \mathbb{Z}/24\mathbb{Z}$ as some characteristic number given by integrating some *canonical* differential 3-form on a closed string 3-manifold M

$$\varphi[M] = \int_M \omega_M$$

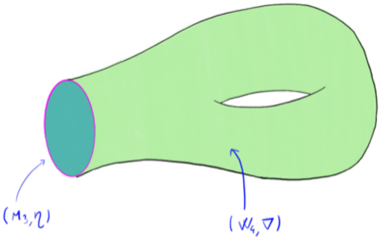
Clearly, there is no hope that this can be true, since the integral takes real values while φ takes values in $\mathbb{Z}/24\mathbb{Z}$, and there is no injective group homomorphism from $\mathbb{Z}/24\mathbb{Z}$ to $\mathbb{R}$.

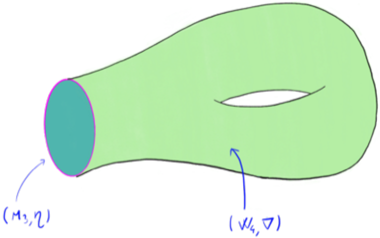
There is however a variant of this construction that may work. Instead of considering just a string 3-manifold M, one considers a string 3-manifold M endowed with some additional structure Υ. This structure should be such that any M admits at least one Υ. To the pair (M, Υ) there could be associated canonical 3-form $\omega_{M,\Upsilon}$ such that $\int_M \omega_{M,\Upsilon}$ takes integral values. Then, if a change in the additional structure Υ results in a change in the value $\int_M \omega_{M,\Upsilon}$ by a multiple of 24 one would have a well defined element

$$\int_M \omega_{M,\Upsilon} \mod 24$$

in $\mathbb{Z}/24\mathbb{Z}$, depending only on the string 3-manifold M; and this could indeed represent the isomorphism φ.

In this form the statement is indeed almost true. The correct version of it has been found by Bunke–Naumann and Redden. Their additional datum Υ consists of a triple (η, W, ∇), where η is a geometric string structure on M in the sense of Waldorf, W is a spin 4-manifold with $\partial W = M$ and ∇ is a spin connection on W such that the restriction $\nabla\big|_M$ coincides with the spin connection datum of the geometric string structure η.





$$\psi(M, \eta, W, \nabla) := \frac{1}{2} \int_W \mathbf{p}_1^{CW}(\nabla) - \int_M \omega_\eta,$$

From the interplay between geometric string structures and differential cohomology it follows that $\psi(M, \eta, W, \nabla) \in \mathbb{Z}$. Keeping (M, η) fixed and letting (W, ∇) vary, one finds

$$\psi(M, \eta, W_1, \nabla_1) - \psi(M, \eta, W_0, \nabla_0) = \frac{1}{2}\int_W p_1(W) \underset{\substack{\Big\uparrow \\ (Atiyah - Singer)}}{=} -12\hat{A}(W),$$

where $W = W_1 \cup_M W_0^{\mathrm{opp}}$ denotes the closed spin 4-manifold obtained gluing together W_0 and W_1 along M.
Therefore the function

$$\psi(M, \eta) := \psi(M, \eta, W, \nabla) \mod 24$$

is well defined.

One concludes by showing that $\psi(M, \eta)$ is actually independent of the geometric string structure η, and only depending on the string cobordism class of M. Additivity is manifest from the definition, so the above integral formula defines a group homomorphism $\psi\colon \mathrm{Bord}_3^{String} \xrightarrow{\cong} \mathbb{Z}/24\mathbb{Z}$.

A direct computation with the canonical generator of Bord_3^{String}, i.e., with S^3 endowed with the trivialization of its tangent bundle coming from $S^3 \cong SU(2)$, then shows that ψ is indeed an isomorphism.

The aim of this talk is to show how the above integral formula for ψ, as well as its main properties, naturally emerge in the context of topological field theories with values in the symmetric monoidal categories associated with morphisms of abelian groups.

By $\mathrm{Bord}^{\xi}_{d,d-1}(X)$ we will denote the symmetric monoidal category of $(d, d-1)$-bordism with tangential structure ξ and background fields X.

The monoidal structure on $\mathrm{Bord}^{\xi}_{d,d-1}(X)$ is given by disjoint union.

The only tangential structures we will be concerned with will be orientations, spin, and string structures; we will denote them by or, Spin, and String, respectively.

Example

Let $X = \Omega_{cl}^{d-1}$ be the smooth stack of closed $(d-1)$-forms. Then an object of $\mathrm{Bord}^{\mathrm{or}}_{d,d-1}(\Omega^{d-1}_{cl})$ is given by a closed oriented $(d-1)$-manifold M equipped with an (automatically closed) $(d-1)$-form $\omega_{d-1;M}$. A morphism $W\colon M_0 \to M_1$ in $\mathrm{Bord}^{or}_{d,d-1}(\Omega^{d-1}_{cl})$ is the datum of an oriented d-manifold W with $\partial W = M_1 \coprod M_0^{opp}$, where "opp" denotes the opposite orientation, equipped with a closed $(d-1)$-form $\omega_{d-1;W}$ such that

$$\omega_{d-1;W}\Big|_{M_i} = \omega_{d-1;M_i}$$

for $i = 0, 1$.

Definition

Let $\mathcal{C}$ be a symmetric monoidal category. A $(d, d-1)$-dimensional $\mathcal{C}$-valued topological quantum field theory (TQFT for short) with tangential structure ξ and background fields X is a symmetric monoidal functor

$$Z\colon \mathrm{Bord}^{\xi}_{d,d-1}(X) \to \mathcal{C}.$$

A typical target is $\mathcal{C} = \mathrm{Vect}$, the category of vector spaces (over some fixed field $\mathbb{K}$). Yet there are plenty of interesting targets other than Vect. Here we will be concerned with the symmetric monoidal categories naturally associated with abelian groups and with morphisms of abelian groups.

Definition

Let $(A,+)$ be an abelian group. By $A^{\otimes}$ we will denote the symmetric monoidal category with

$$\mathrm{Ob}(A^{\otimes}) = A;$$

$$\mathrm{Hom}_{A^{\otimes}}(a,b) = \begin{cases} \mathrm{id}_a & \text{if } a=b \\ \emptyset & \text{otherwise} \end{cases}$$

The tensor product is given by the sum (or multiplication) in A and the unit object is the zero (or the unit) of A. Associators, unitors and braidings are the trivial ones.

A TQFT with tangential structure ξ and background fields X with values in $A^{\otimes}$ consists into a rule that associates with any closed $(d-1)$-manifold M_{d-1} (with tangential structure and background fields) an element $Z(M_{d-1}) \in A$ in such a way that:

- $Z(M_{d-1} \sqcup M'_{d-1}) = Z(M_{d-1}) + Z(M'_{d-1})$ (monoidality);
- if $M_{d-1} = \partial W_d$ then $Z(M_{d-1}) = 0$ (functoriality).

Example (Stokes' theorem for closed forms)

A paradigmatic example of a TQFT with values in an abelian group is provided by Stokes' theorem. Take the stack X of background fields to be the smooth stack Ω_{cl}^{d-1} of closed $(d-1)$-forms and let $\mathbb{R}^{\otimes}$ be the symmetric monoidal category associated with the abelian group $(\mathbb{R},+)$. Then

$$\begin{aligned} Z\colon \mathrm{Bord}^{\mathrm{or}}_{d,d-1}(\Omega_{cl}^{d-1}) &\to \mathbb{R}^{\otimes} \\ (M_{d-1},\omega_{d-1}) &\mapsto \int_{M_{d-1}} \omega_{d-1} \end{aligned}$$

is a TQFT.

Remark

Chern-Weil theory provides differential form representatives for Pontryagin classes

$$\mathbf{p}_k^{\mathrm{CW}} \colon \mathbf{B}\mathrm{SO}_\nabla \to \Omega^{4k}_{cl},$$

We have an induced symmetric monoidal morphism

$$Bord^{\mathrm{or}}_{4k+1,4k}(\mathbf{B}\mathrm{SO}_\nabla) \to Bord^{\mathrm{or}}_{4k+1,4k}(\Omega^{4k}_{cl})$$

and so a TQFT

$$\begin{aligned} Z \colon Bord^{\mathrm{or}}_{4k+1,4k}(\mathbf{B}\mathrm{SO}_\nabla) &\to \mathbb{R}^{\otimes} \\ (M_{4k}, P, \nabla) &\mapsto \int_{M_{4k}} \mathbf{p}_k^{\mathrm{CW}}(\nabla). \end{aligned}$$

Remark

This TQFT descends to a TQFT with background fields $\mathbf{B}\mathrm{SO}$, i.e., we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Bord}^{\mathit{or}}_{4k+1,4k}(\mathbf{B}\mathrm{SO}_{\nabla}) \arrow[r] \arrow[d] & \mathbb{R}^{\otimes} \\
\mathrm{Bord}^{\mathit{or}}_{4k+1,4k}(\mathbf{B}\mathrm{SO}) \arrow[ur] &
\end{tikzcd}$$

The tangent bundle provides a symmetric monoidal section to the forgetful morphism $\mathrm{Bord}^{\mathrm{or}}_{4k+1,4k}(\mathbf{B}\mathrm{SO}) \to \mathrm{Bord}^{\mathrm{or}}_{4k+1,4k}$, so we get an oriented TQFT

$$\begin{aligned} Z\colon \mathrm{Bord}^{\mathrm{or}}_{4k+1,4k} &\to \mathbb{R}^{\otimes} \\ M_{4k} &\mapsto \int_{M_{4k}} \mathbf{p}_k(TM). \end{aligned}$$

Remark

The same argument applies replacing the single Pontryagin class p_k with a polynomial $\Phi = \Phi(p_1, p_2, \dots)$ in the Pontryagin classes. This way one obtains plenty of $\mathbb{R}$-valued oriented TQFTs. These are in particular $\mathbb{R}$-valued oriented cobordism invariants, and Thom's isomorphism

$$\boldsymbol{\Omega}_\bullet^{\mathrm{SO}} \otimes \mathbb{R} \cong \mathbb{R}[p_1, p_2, \dots]$$

implies that indeed every $\mathbb{R}$-valued oriented cobordism invariant is of this form.

More generally, one can associate a symmetric monoidal category with a morphism of abelian groups, as follows.

Definition

Let $\varphi_A \colon A_{\mathrm{mor}} \to A_{\mathrm{ob}}$ be a morphism of abelian groups. By $\varphi_A^{\otimes}$ we will denote the symmetric monoidal category with

$$\mathrm{Ob}(\varphi_A^{\otimes}) = A_{\mathrm{ob}};$$

$$\mathrm{Hom}_{\varphi_A^{\otimes}}(a,b) = \{x \in A_{\mathrm{mor}} \ : \ a + \varphi_A(x) = b\}.$$

The composition of morphism is given by the sum in A_{mor}. The tensor product of objects and morphisms is given by the sum in A_{ob} and in A_{mor}, respectively. The unit object is the zero in A_{ob}. Associators, unitors and braidings are the trivial ones, i.e., they are given by the zero in A_{mor}.

A TQFT with values in $\varphi_A^{\otimes}$. It consists into a rule that associates with any closed $(d-1)$-manifold M_{d-1} (with tangential structure and background fields) an element $Z(M_{d-1}) \in A_{\mathrm{ob}}$, and with any d-manifold W_d (with tangential structure and background fields) an element $Z(W_d) \in A_{\mathrm{mor}}$ in such a way that:

- $Z(M_{d-1} \sqcup M'_{d-1}) = Z(M_{d-1}) + Z(M'_{d-1})$ and $Z(W_d \sqcup W'_d) = Z(W_d) + Z(W'_d)$ (monoidality);
- if $M_{d-1} = \partial W_d$ then $Z(M_{d-1}) = \varphi_A(Z(W_d))$ (functoriality).

Example (Stokes' theorem)

Take as stack of background fields the smooth stack Ω^{d-1} of smooth $(d-1)$-forms. Then we have a TQFT

$$\begin{aligned} Z\colon \mathit{Bord}^{\mathrm{or}}_{d,d-1}(\Omega^{d-1}) &\to \mathrm{id}^{\otimes}_{\mathbb{R}} \\ (M_{d-1}, \omega_{d-1}) &\mapsto \int_{M_{d-1}} \omega_{d-1} \\ (W_d, \omega_{d-1}) &\mapsto \int_{W_d} \mathrm{d}\omega_{d-1}. \end{aligned}$$

Example (Holonomy and curvature)

A generalization of the above Example for $d = 2$ is obtained by taking $X = \mathbf{B}\mathrm{U}(1)_\nabla$ and $\exp(2\pi i -)^{\otimes}$ as target category.

$$\begin{aligned} Z \colon \mathit{Bord}_{2,1}^{\mathrm{or}}(\mathbf{B}\mathrm{U}(1)_\nabla) &\to \exp(2\pi i -)^{\otimes} \\ (M_1, P, \nabla) &\mapsto \mathrm{hol}_{M_1}(\nabla) \\ (W_2, P, \nabla) &\mapsto \frac{1}{2\pi i}\int_{W_2} F_\nabla , \end{aligned}$$

The fact that Z is a TQFT is encoded in the fundamental integral identity relating holonomy along the boundary and curvature in the interior:

$$\mathrm{hol}_{\partial W_2}(\nabla) = \exp\left(\int_{W_2} F_\nabla\right).$$

More generally one has an $(n+1, n)$-dimensional TQFT as

$$\begin{aligned} Z\colon \mathrm{Bord}^{\mathrm{or}}_{n+1,n}(\mathbf{B}^n\mathrm{U}(1)_\nabla) &\to \exp(2\pi i -)^{\otimes} \\ (M_n, P, \nabla) &\mapsto \mathrm{hol}_{M_n}(\nabla) \\ (W_{n+1}, P, \nabla) &\mapsto \frac{1}{2\pi i}\int_{W_{n+1}} F_\nabla, \end{aligned}$$

Example (TQFTs from spin connections)

By Brylinksi–McLaughlin and F.–Schreiber–Stasheff the characteristic class $\frac{1}{2}p_1 \in H^4(BSpin;\mathbb{Z})$ refines to a commutative diagram of morphisms of smooth stacks

$$\begin{tikzcd}
\mathbf{B}\mathrm{Spin}_\nabla \arrow[r, "\frac{1}{2}\hat{\mathbf{p}}_1"'] \arrow[rr, bend left, "\frac{1}{2}\mathbf{p}_1^{CW}"] \arrow[d] & \mathbf{B}^3\mathrm{U}(1)_\nabla \arrow[r, "\frac{1}{2\pi i}F"'] \arrow[d] & \Omega^4_{cl} \\
\mathbf{B}\mathrm{Spin} \arrow[r, "\frac{1}{2}\mathbf{p}_1"'] & \mathbf{B}^3\mathrm{U}(1), &
\end{tikzcd}$$

Example (TQFTs from spin connections)

This induces $(4,3)$-dimensional TQFT with background fields given by spin connections and target $\exp(2\pi i -)^{\otimes}$ given by

$$\begin{aligned} Z_{\mathrm{Spin}} \colon \mathit{Bord}^{\mathrm{or}}_{4,3}(\mathbf{B}\mathrm{Spin}_{\nabla}) &\to \exp(2\pi i -)^{\otimes} \\ (M_3, P, \nabla) &\mapsto \mathrm{hol}_{M_3}\left(\frac{1}{2}\hat{\mathbf{p}}_1(\nabla)\right) \\ (W_4, P, \nabla) &\mapsto \int_{W_4} \frac{1}{2}\mathbf{p}_1^{CW}(\nabla). \end{aligned}$$

We have a sequence of morphisms of smooth stacks

$$\mathbf{B}^3\mathrm{U}(1)_\nabla \to \mathbf{B}\mathbf{B}^2\mathrm{U}(1)_\nabla \to \mathbf{B}^2\mathbf{B}\mathrm{U}(1)_\nabla \to \mathbf{B}^3\mathrm{U}(1).$$

Composing this on the left with $\frac{1}{2}\hat{\mathbf{p}}_1 : \mathbf{B}\mathrm{Spin}_\nabla \to \mathbf{B}^3\mathrm{U}(1)_\nabla$ we obtain maps

$$\frac{1}{2}\hat{\mathbf{p}}_1^{(i)} \colon \mathbf{B}\mathrm{Spin}_\nabla \to \mathbf{B}^{3-i}\mathbf{B}^i\mathrm{U}(1)_\nabla,$$

for $i = 0, \ldots, 3$.

Definition

For $i = 0, \ldots, 3$, the smooth stack $\mathbf{B}\mathrm{String}_{\nabla}^{(i)}$ is defined as the homotopy pullback

$$\begin{tikzcd}
\mathbf{B}\mathrm{String}_{\nabla}^{(i)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d] \\
\mathbf{B}\mathrm{Spin}_{\nabla} \arrow[r, "\frac{1}{2}\hat{\mathbf{p}}_1^{(i)}"] & \mathbf{B}^{3-i}\mathbf{B}^{i}\mathrm{U}(1)_{\nabla}
\end{tikzcd} .$$

The stack $\mathbf{B}\mathrm{String}_{\nabla}^{(2)}$ will be called the stack of *geometric string structures* (Waldorf).

By the pasting law for homotopy pullbacks, the defining diagram for the stack of geometric string structures can be factored as

$$\begin{tikzcd}
\mathbf{B}\mathrm{String}^{(2)}_{\nabla} \arrow[r, "\omega_3"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Omega^3 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d] \\
\mathbf{B}\mathrm{Spin}_{\nabla} \arrow[r, "\frac{1}{2}\hat{\mathbf{p}}_1"] & \mathbf{B}^3\mathrm{U}(1)_{\nabla} \arrow[r] & \mathbf{B}\mathbf{B}^2\mathrm{U}(1)_{\nabla}
\end{tikzcd} .$$

where both squares are homotopy pullbacks. This in particular shows that a geometric string structure comes equipped with a canonical 3-form.

From this we obtain the commutative diagram

$$\begin{tikzcd}
\mathbf{B}\mathrm{String}^{(2)}_{\nabla} \arrow[r, "\omega_3"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Omega^3 \arrow[d] \arrow[dr, bend left, "\mathrm{d}"] & \\
\mathbf{B}\mathrm{Spin}_{\nabla} \arrow[r, "\frac{1}{2}\hat{\mathbf{p}}_1"] \arrow[rr, bend right, "\frac{1}{2}\mathbf{p}_1^{CW}"'] & \mathbf{B}^3\mathrm{U}(1)_{\nabla} \arrow[r, "\frac{1}{2\pi i}F"] & \Omega^4_{cl}
\end{tikzcd}$$

showing that, if $\omega_{3,M}$ is the canonical 3-form on a smooth manifold M equipped with a geometric string structure and ∇ is the underlying spin connection, then one has

$$\mathrm{d}\omega_{3,M} = \frac{1}{2}\mathbf{p}_1^{CW}(\nabla).$$

The obstructions to lifting a topological string structure on M to a geometric string structure lie in $H^{3-i}(M;\Omega^i)$ for $i = 1, 2$. Both these groups vanish.

Proposition

Let M be a smooth manifold, and let $P\colon M \to \mathbf{B}\mathrm{Spin}$ be a principal spin bundle on M. Then P can be enhanced to a geometric string structure on M if and only if $\frac{1}{2}p_1(P) = 0$.

Morphisms of abelian groups are the objects of a category whose morphisms are commutative diagrams: if $\varphi_H \colon H_{\mathrm{mor}} \to H_{\mathrm{ob}}$ and $\varphi_G \colon G_{\mathrm{mor}} \to G_{\mathrm{ob}}$ are morphisms of abelian groups, then a morphism from φ_H to φ_G is a commutative diagram of the form

$$\begin{tikzcd} H_{\mathrm{mor}} \arrow[r, "\varphi_H"] \arrow[d, "f_{\mathrm{mor}}"'] & H_{\mathrm{ob}} \arrow[d, "f_{\mathrm{ob}}"] \\ G_{\mathrm{mor}} \arrow[r, "\varphi_G"] & G_{\mathrm{ob}} \end{tikzcd}$$

The pair $(f_{\mathrm{ob}}, f_{\mathrm{mor}})$ defines a symmetric monoidal functor $f \colon \varphi_H^{\otimes} \to \varphi_G^{\otimes}$.

Definition

Given a functor $p : \mathcal{D} \to \mathcal{C}$ and an object c of $\mathcal{C}$, the *homotopy fiber* (or *essential fiber*) of p over c is the category $\mathrm{hofib}\,(p; c)$ with objects the pairs (x, b) with x an object in $\mathcal{D}$ and $b \in \hom_{\mathcal{C}}(c, p(x))$ an isomorphism; morphisms from (x, b) to (x', b') in $\mathrm{hofib}\,(p; c)$ are those morphisms $a : x \to x'$ in $\mathcal{D}$ such that the diagram

$$\begin{tikzcd}
p(x) \arrow[rr, "p(a)"] & & p(x') \\
& c \arrow[ul, "b"] \arrow[ur, "b'"'] &
\end{tikzcd}$$

commutes.

By relaxing the condition that b is an isomorphism, and allowing it to be an arbitrary morphism, we obtain the notion of *lax homotopy fiber* and denote it by $\mathrm{hofib}_{\mathrm{lax}}(p; c)$.

When $p : \mathcal{D} \to \mathcal{C}$ is a monoidal functor between monoidal categories, we will always take c to be the monoidal unit $\mathbf{1}_{\mathcal{C}}$ of $\mathcal{C}$, and simply write $\mathrm{hofib}(p)$ and $\mathrm{hofib}_{\mathrm{lax}}(p)$.

The monoidal structures of $\mathcal{C}$ and $\mathcal{D}$ and the monoidality of p induce a natural monoidal category structure on $\mathrm{hofib}(p)$ and on $\mathrm{hofib}_{\mathrm{lax}}(p)$.

The homotopy fiber of $f\colon \varphi_H^{\otimes} \to \varphi_G^{\otimes}$ admits a simple explicit description.

Lemma

Let

$$\begin{tikzcd} H_{\mathrm{mor}} \arrow[r, "\varphi_H"] \arrow[d, "f_{\mathrm{mor}}"'] & H_{\mathrm{ob}} \arrow[d, "f_{\mathrm{ob}}"] \\ G_{\mathrm{mor}} \arrow[r, "\varphi_G"] & G_{\mathrm{ob}} \end{tikzcd}$$

be a commutative diagram of abelian groups, and let $f\colon \varphi_H^{\otimes} \to \varphi_G^{\otimes}$ *be the associated monoidal functor. Then we have*

$$\mathrm{Ob}(\mathrm{hofib}\,(f)) = G_{\mathrm{mor}} \times_{G_{\mathrm{ob}}} H_{\mathrm{ob}}$$

$$\mathrm{Mor}\,((g,h),(g',h')) = \left\{ x \in H_{\mathrm{mor}} \;\; s.t. \; \begin{cases} f_{\mathrm{mor}}(x) = g' - g \\ \varphi_H(x) = h' - h \end{cases} \right\}.$$

Lemma

A commutative diagram of abelian groups of the form

$$\begin{tikzcd}
H_{\mathrm{mor}} \arrow[r, "\varphi_H"] \arrow[d, "f_{\mathrm{mor}}"'] & H_{\mathrm{ob}} \arrow[d, "f_{\mathrm{ob}}"] \arrow[dl, "\lambda"'] \\
G_{\mathrm{mor}} \arrow[r, "\varphi_G"] & G_{\mathrm{ob}}
\end{tikzcd}$$

induces a symmetric monoidal functor

$$\Xi \colon \mathrm{hofib}\,(f) \to \mathsf{ker}(\varphi_G)^{\otimes}$$

acting on the objects as $(g, h) \mapsto g - \lambda(h)$.

Moreover, Ξ *is an equivalence iff* φ_H *is an isomorphism.*

We have $\mathbf{B}\mathrm{String}_{\nabla}^{(2)} \xrightarrow{\omega_3} \Omega^3$ and so a symmetric monoidal functor

$$Z_{\mathrm{String}} \colon \mathrm{Bord}^{\mathrm{or}}_{4,3}(\mathbf{B}\mathrm{String}_{\nabla}^{(2)}) \to \mathrm{Bord}^{\mathrm{or}}_{4,3}(\Omega^3) \to \mathrm{id}_{\mathbb{R}}^{\otimes}.$$

$$(M_3, \eta) \mapsto \int_{M_3} \omega_{3;M}$$

$$(W_4, \nabla) \mapsto \int_{W_4} d\omega_{3;M} = \frac{1}{2}\int_{W_4} \mathbf{p}_1^{CW}(\nabla)$$

We also have the projection $\mathbf{B}\mathrm{String}_{\nabla}^{(2)} \to \mathbf{B}\mathrm{Spin}_{\nabla}$ inducing the symmetric monoidal functor

$$\mathrm{Bord}_{4,3}^{\mathrm{or}}(\mathbf{B}\mathrm{String}_{\nabla}^{(2)}) \to \mathrm{Bord}_{4,3}^{\mathrm{or}}(\mathbf{B}\mathrm{Spin}_{\nabla})$$

and the commutative diagram of abelian groups

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "\mathrm{id}_{\mathbb{R}}"] \arrow[d, "\mathrm{id}_{\mathbb{R}}"'] & \mathbb{R} \arrow[d, "\exp(2\pi i -)"] \\
\mathbb{R} \arrow[r, "\exp(2\pi i -)"] & \mathrm{U}(1)
\end{tikzcd}$$

inducing the symmetric monoidal functor

$$(\mathrm{id}_{\mathbb{R}}, \exp(2\pi i -)) \colon \mathrm{id}_{\mathbb{R}}^{\otimes} \to \exp(2\pi i -)^{\otimes}.$$

Lemma

The diagram of symmetric monoidal functors

$$\begin{tikzcd}
\mathrm{Bord}^{\mathrm{or}}_{4,3}(\mathbf{B}\mathrm{String}^{(2)}_{\nabla}) \arrow[r, "Z_{\mathrm{String}}"] \arrow[d] & \mathrm{id}^{\otimes}_{\mathbb{R}} \arrow[d, "(\mathrm{id}_{\mathbb{R}}{,}\exp(2\pi i -))"] \\
\mathrm{Bord}^{\mathrm{or}}_{4,3}(\mathbf{B}\mathrm{Spin}_{\nabla}) \arrow[r, "Z_{\mathrm{Spin}}"] & \exp(2\pi i -)^{\otimes}
\end{tikzcd}$$

commutes, with identity 2-cell.

We have therefore an induced monoidal functor

$$\mathrm{hofib}_{\mathrm{lax}}\left(\mathit{Bord}^{\mathrm{or}}_{4,3}\left(\mathbf{B}\mathrm{String}^{(2)}_{\nabla}\right) \to \mathit{Bord}^{\mathrm{or}}_{4,3}\left(\mathbf{B}\mathrm{Spin}_{\nabla}\right)\right) \\ \to \mathrm{hofib}\left(\mathrm{id}_{\mathbb{R}}, \exp(2\pi i -)\right).$$

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "\mathrm{id}_{\mathbb{R}}"] \arrow[d, "\mathrm{id}_{\mathbb{R}}"'] & \mathbb{R} \arrow[d, "\mathsf{exp}(2\pi i-)"] \arrow[dl, "\mathrm{id}_{\mathbb{R}}"'] \\
\mathbb{R} \arrow[r, "\mathsf{exp}(2\pi i-)"'] & \mathrm{U}(1)
\end{tikzcd} ,$$

So we have a symmetric monoidal equivalence

$$\Xi\colon \operatorname{hofib}\left((\mathrm{id}_{\mathbb{R}}, \mathsf{exp}(2\pi i-))\right) \to \mathsf{ker}(\mathsf{exp}(2\pi i-)^{\otimes} = \mathbb{Z}^{\otimes},$$

acting on objects as

$$\begin{aligned} \mathbb{R} \times_{\mathrm{U}(1)} \mathbb{R} &\mapsto \mathbb{Z} \\ (g, h) &\mapsto g - h \end{aligned}$$

Putting everything together we obtain a symmetric monoidal functor

$$Z^{\mathrm{Spin}}_{\mathrm{String}} \colon \mathrm{hofib}_{\mathrm{lax}}\left(\mathrm{Bord}^{\mathrm{or}}_{4,3}(\mathbf{B}\mathrm{String}^{(2)}_{\nabla}) \to \mathrm{Bord}^{\mathrm{or}}_{4,3}(\mathbf{B}\mathrm{Spin}_{\nabla})\right) \to \mathbb{Z}^{\otimes}$$

Proposition

The symmetric monoidal functor $Z^{\mathrm{Spin}}_{\mathrm{String}}$ is the Bunke–Naumann–Redden map ψ from the Introduction.

By replacing the first fractional Pontryagin class $\frac{1}{2}p_1$ with the first Chern class c_1 one obtains an integral formula realizing the isomorphism $\mathrm{Bord}_1^{SU} \cong \mathbb{Z}/2\mathbb{Z}$.

Question

Can one obtain an integral formula realizing the isomorphism $Bord_7^{Fivebrane} \cong \mathbb{Z}/240\mathbb{Z}$ by replacing $\frac{1}{2}p_1$ with $\frac{1}{6}p_2$?

Yes, if every string bundle can be endowed with a string connection. The answer to this last question is presently not clear (at least not to me).

Thanks!