Quantomorphism & Heisenberg groupoids II

(pre-) quantum geometry

Q: What is correct notion of a class. symmetry?

A: Hamiltonian G-action.

<u>Def</u>: Let $G \to \mathrm{Diff}(M)$ be a smooth Lie group action $\Rightarrow$ $\forall v \in \mathfrak{g}$ we have $X_v \in \mathfrak{X}(M)$ given by $X_v(m) = \frac{d}{dt}\Big|_{t=0} e^{tv} \cdot m$

(i) The G-action is called symplectic if

$$\begin{tikzcd} G \arrow[rr] \arrow[dr] & & \mathrm{Diff}(M) \\ & \mathrm{Symp}(M,\omega) \arrow[ur] & \end{tikzcd}$$

$$\mathcal{L}_{X_v} \omega = 0 \quad \forall v \in \mathfrak{g}$$

(ii) Weakly Hamiltonian if each X_v is Hamiltonian w.r.t. μ^v which vary <u>linearly</u> i.e. we have a lin.

map $\mathfrak{g} \xrightarrow[\mu^{(-)}]{} C^\infty(M)$

(iii) Hamiltonian if μ is a map of <u>Lie</u> algebras:

$$\begin{tikzcd} & C^\infty \arrow[d, "!"] \\ \mathfrak{g} \arrow[ur, "\mu^{(-)}"] \arrow[r, "\hat{\mu}"'] & \mathfrak{X}_{Ham} \end{tikzcd}$$

$\leftarrow$ The usual Poisson str-re

"moment map"

$$\mu : \mathfrak{g} \longrightarrow C^\infty(M, \omega)$$

linear dual

$$\mu^* : M \longrightarrow \mathfrak{g}^*$$

is called the moment map.

E. Noether's theorem

Let $G \to \mathrm{Diff}(M, \omega)$ be a smooth action

&) There $\exists$ a lift

$$G \to \mathrm{Diff}(T^*M, \omega)$$

of the action and the lift is <u>Hamiltonian</u>

<u>non-trivial</u>

(⇐) Let $G \to \mathrm{Symp}(M,\omega)$ be a <u>Hamiltonian</u> action w/ a comoment map $\mu : \mathfrak{g} \to C^\infty(M)$

And G-invariant $H : C^\infty(M)$

$\Rightarrow \forall v : \mathfrak{g}$

$$\{H, \mu^v\} = 0$$

<u>Ex:</u>

Action by translations

$$a : \mathbb{R}^n \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$$

$$(\vec{v}, (\vec{q}, \vec{p})) = (\vec{q} + \vec{v}, \vec{p})$$

$$a_{inf} : T_0 \mathbb{R}^n = \mathbb{R}^n \longrightarrow \mathfrak{X}(\mathbb{R}^{2n})$$

$$\vec{e}_j \longmapsto \frac{\partial}{\partial q^j}$$

We have a linear map

$$\mu : T_0 \mathbb{R}^n = \mathbb{R}^n \longrightarrow C^\infty(\mathbb{R}^{2n})$$

$$e_j \longmapsto \underbrace{p^j : \mathbb{R}^{2n} \to \mathbb{R}}_{\text{as a f-n}}$$

$$\iota_{X_{p^j}} \omega = d p^j$$

$$\iota_{X_{p_j}} = \frac{\partial}{\partial q^j} = a_{inf}(\vec{e}_j)$$

$$X_{p_j} = X^{q^i}\frac{\partial}{\partial q^i} + X^{p_i}\frac{\partial}{\partial p_i}$$

$$\omega = dq^i \wedge dp_i$$

$$\iota_{X_{p_j}}\omega = X^{q^i} dp_i - X^{p_i} dq^i =$$

$$= dp_j.$$

$$\ominus \frac{\partial}{\partial q^j} = a_{inf}(\vec{e}_j)$$

By E. Noether:

For $H : C^\infty(M)^G$

$$\{H, p_j\} = 0$$

$$\frac{\partial H}{\partial p_j}\frac{\partial p^j}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial p^j}{\partial p^i} =$$

$$0$$

$$= -\frac{\partial H}{\partial q^j}.$$

Def: The Heisenberg Lie algebra on $2n$ vars is a free $\mathbb{R}$-Lie algebra with presentation

$$\langle q^i, p_i, z \rangle_{i=1}^{n} \Big/ \begin{matrix} [q^i, q^j] = \\ = [p_i, p^j] = [z, z] = 0 \\ [q^i, z] = [p_i, z] = 0 \\ [q^i, p_j] = z\,\delta_{ij} \end{matrix}$$

$$\mathfrak{Heis}_{2n}$$

Rk:

$$\begin{tikzcd} \mathfrak{Heis}_{2n} \arrow[r, hook] \arrow[d] & (C^\infty(\mathbb{R}^{2n}), \{-,-\}) \arrow[d] \\ \mathbb{R}^{2n} \arrow[r] & (\mathfrak{X}_{Ham}(\mathbb{R}^{2n}), [-,-]) \end{tikzcd}$$

Ansatz quantization by Dirac

(M, ω, H) a Quant. is an $\mathbb{R}$-linear map

$Q : C^\infty(M) \supseteq \mathcal{O} \to L \subseteq \mathrm{End}(X)$

X : Hilb space

(D1) $Q(1) = \lambda \, Id_X$

(D2) $\langle Q(f), Q(g) \rangle = c\, Q(\{f, g\})$

For suitable $\lambda, c : \mathbb{C}^\times$

$H : \mathcal{O}$, All $Q(f)$ should be self-adjoint

(D2) Can hold only on a dense sub set.

A key ingredient is missing:

What is X, Q etc?

Geom. quantization:

provides a framework to construct Q & X suitably for (M, ω, H)

Def: (M,ω) : Symp is prequant. if $[\omega] \in H^2_{dR}(M)$ is in the image of $\check{H}^1(M; U(1)) \to$

$\to H^2(M; \mathbb{R}) = H^2_{dR}(M)$

Def: A prequantization of (M,ω) is VBun

$$\begin{array}{c} (E, \nabla) \\ \downarrow p \\ (M,\omega) \end{array} \quad \longleftarrow \text{Hermitian } \underline{\text{line Bundle}}$$

s.t. $\frac{1}{i\hbar} F_\nabla = \omega$

$X \simeq \Gamma(E)$

Prin. Bun. p.o.v.

$$\begin{array}{c} (Y, \alpha) \\ p \downarrow \\ (M,\omega) \end{array}$$

p: Prin $U(1)$-Bundle $/M$ s.t.

$\frac{1}{i\hbar} d\alpha = \omega$

$X \cong C^\infty(Y; \mathbb{C})^{U(1)}$

<u>VBun:</u>

$Q : C^\infty(M) \to X$

$f \mapsto i\hbar \nabla_{X_f} + f$

<u>Prin</u>

$Q : C^\infty(M) \to X$

$f \mapsto i\hbar \tilde{X}_f + \frac{1}{2\pi i} p^* f \, \partial_{2\pi i}$

$\tilde{X}_f$ - hor. lift. of X_f to Y

$\partial_{2\pi i}$: v. field on Y generated by inf. ~~action~~ inf. action of $2\pi i : U(1)$.