Higher gauge groupoids

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Def. (Lie Groupoids)

$$\mathcal{G} : \quad G_1 \overset{t}{\underset{s}{\rightrightarrows}} G_0$$

C^∞ mflds

$\mathcal{G}$ is a Lie Gpd

Def. A bisection $\sigma : G_0 \to G_1$ is a C^∞-map s.t.

1) $s\sigma = \mathrm{id}_{G_0}$
2) $t\sigma$ is a diffeomorphism on G_0.

This is equivalent to a ho-comm. diagram:

$$\begin{tikzcd} G_0 \arrow[rr, "t\sigma", "\cong"'] \arrow[dr, "\iota_G"'] & \arrow[d, phantom, "\Leftarrow \sigma^{-1}"] & G_0 \arrow[dl, "\iota_G"] \\ & G & \end{tikzcd}$$

We pass to sPSh and we get an honest homotopy.

$$B\,\mathrm{Sect}(G) :\equiv \mathrm{Aut}_{\mathbb{H}}(\iota_G)$$

MC of sPSh(Cat)

Let $f: X \to Y$ in $\mathbb{H}$. Then we get a natural fiber sequence:

Prop 2.3.3

$$\begin{tikzcd}
\mathrm{Aut}_{\mathbb{H}}(f) \arrow[r] \arrow[d] & \mathrm{Aut}(X) \arrow[d] \\
* \arrow[r, "f"'] & {[X,Y]}
\end{tikzcd}$$

$$\simeq$$

$$\begin{tikzcd}
\Omega_f[X,Y] \arrow[r] \arrow[d] & \mathrm{Aut}_{\mathbb{H}}(f) \arrow[d] \\
* \arrow[r] & \mathrm{Aut}(X)
\end{tikzcd}$$

2.9 Denote by $\mathbb{H}_{\leq n}$ an ∞-cat of n-stacks $\mathbb{H}_{\leq n} \hookrightarrow \mathbb{H}$ has a left adjoint: the n-truncation

$$X \longrightarrow X_{\leq n} \longrightarrow *$$

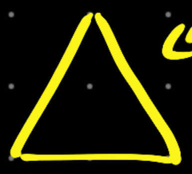
$$\mathbb{H}_{\leq n} \hookrightarrow \mathbb{H}_{\leq n+1} \rightsquigarrow X \to X_{\leq n+1}$$

$\sigma \in (\mathrm{cosk}^n X)_k$ ← is htpy invariant

$\Updownarrow$

$\mathrm{sk}^n \Delta^k \longrightarrow X$

This is equivalent to adding higher simplices killing π_* :

 ← simplicial loop

$\cap$
X

$\Updownarrow$

$\Delta^2 \to \mathrm{cosk}^1 X$

truncation

$$\mathcal{H}_{\le n} \rightleftarrows \mathcal{H}_{\le n+1}$$

$$X \to X_{\le n+1} \longrightarrow X_{\le n} \longrightarrow *$$

$$\vdots$$

$$\begin{tikzcd} & X_{\le 1} \arrow[d] \arrow[dr] & \\ X \arrow[ur] \arrow[r] & X_{\le 0} \arrow[r] & * \end{tikzcd}$$

$f: X \to Y$ we have

$$\begin{tikzcd} & \mathrm{Im}_2(f) \arrow[d, "!"] \arrow[dr, hook, bend left] & \\ X \arrow[ur] \arrow[r, two heads] & \mathrm{Im}_1(f) \arrow[r, hook] & Y \end{tikzcd}$$

$X \twoheadrightarrow \mathrm{Im}_1(f)$ is epi on π_0

<u>Ex 2.9.2</u>:

$\mathcal{G} = G_1 \rightrightarrows G_0$ then canonical

$G_0 \hookrightarrow \mathcal{G}$ is a 1-epi.

If G is a Lie Group

$* \to \mathbb{B}G$ is a 1-epimorphism

<u>Ex 2.9.7</u> Let $P \to X$ a prin. G-bundle

$\nabla^0 : X \to \mathbb{B}G$

$\mathrm{im}_1 \nabla^0 \xrightarrow{\sim} \mathrm{hocolim}(X \times_{\mathbb{B}G} X \times_{\mathbb{B}G} X \substack{\to\\\to\\\to} X \times_{\mathbb{B}G} X \rightrightarrows X)$

$\pi_0 X \cong X$

$X \longrightarrow \mathbb{B}G$ is <u>not</u> a 1-epi.

Atiyah groupoid of P

$$At(P) = {}^{P\times P}/_{G} \rightrightarrows X$$

$At(\nabla^\circ)$ is the gauge groupoid

$$\Updownarrow$$

$$\left\{ \begin{tikzcd} X \arrow[rr, "\sim"] \arrow[dr, "\nabla^\circ"'] \arrow[rr, phantom, shift right=4ex, "\Downarrow"] & & X \arrow[dl, "\nabla^\circ"] \\ & \mathbb{B}G & \end{tikzcd} \right\} \cong \left\{ \begin{tikzcd} X \arrow[rr, "\cong"] \arrow[dr, "\nabla^\circ"'] \arrow[r, phantom, shift right=3ex, "x_2"] & & X \arrow[dl, "\nabla^\circ"] \arrow[dl, phantom, shift left=2ex] \\ & At(\nabla) \arrow[d] & \\ & \mathbb{B}G & \end{tikzcd} \right\}$$

$$\Updownarrow$$

$$\cong \left\{ \begin{tikzcd} X \arrow[rr, "\sim"] \arrow[dr] \arrow[rr, phantom, shift right=4ex, "\Downarrow"] & & X \arrow[dl] \\ & \mathrm{Aut}\, At(P) & \end{tikzcd} \right\}$$

$\mathbb{B}G$ - groupoid of G-torsors.

3.1 Let $\mathbb{H}$ an ∞-topos

$G : \mathrm{Grp}(\mathbb{H})$ (an ∞-grp)

Let $P \to X$ be a G-prin.

∞ - bundle

Def: Let $\nabla^n : X \to \mathbb{B}G$

The higher (Atiyah) gauge grpd

$At(P)$ is the groupoid obj. in $\mathbb{H}$ given by $\mathrm{Im}_1(\nabla^0)$

$At(\nabla^0)$

$$\mathbb{B}\,\mathrm{Sect}(At(\nabla^0)) \cong \mathrm{Aut}_{\mathbb{H}}(\nabla^0)$$

$$\begin{tikzcd}
\Omega_{\nabla^0}[X;\mathbb{B}G] \arrow[r] \arrow[d] & \mathrm{Aut}_{\mathbb{H}}(\nabla^0) \arrow[d] \\
* \arrow[r] & \mathrm{Aut}(X)
\end{tikzcd}$$

Thm 3.1.9 We have a pasting

$$\begin{tikzcd}
\Omega_{\nabla^n}[X;\mathbb{B}G] \arrow[r] \arrow[d] & \mathrm{Aut}_{\mathbb{H}}(\nabla^0) \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] & \mathrm{Aut}_p(X)^{\nabla} \arrow[r] & \mathbb{B}(\Omega_{\nabla^0}[X;\mathbb{B}G])
\end{tikzcd}$$

$$\mathbb{B}(\nabla^\circ) : \mathbb{B}\mathrm{Aut}_P(X) \to \mathbb{B}^2(\Omega_{\nabla^\circ}[X;\mathbb{B}G])$$

$$\begin{tikzcd}
\mathrm{Aut}_{\mathbb{H}}(\nabla^\circ) \arrow[r, two heads] \arrow[rr, bend right] & \mathrm{Aut}_P(X) \arrow[r, hook] & \mathrm{Aut}(X)
\end{tikzcd}$$