All definitions used in the problems can be found in the notes, and also in the recommended textbooks.

1. Recall the category of enhanced measurable spaces (Homework 1, Problem 4). More precisely, we constructed two different categories: PreEMS and its quotient $\mathsf{StrictEMS}$ modulo the equivalence relation of equality almost everywhere. Consider the category $\mathsf{CAlg}^*_{\mathbf{C}}$ of commutative *complex *-algebras*, defined as follows. Objects are complex algebras A equipped with a complex-antilinear operation $*\colon A \to A$ (the involution) such that $(ab)^* = b^*a^*$, $1^* = 1$, and $a^{**} = a$. Morphisms are morphisms of complex algebras $f\colon A \to B$ such that $f(a^*) = f(a)^*$.

- Extend the following construction to a functor $\mathsf{L}^\infty\colon \mathsf{PreEMS}^{\mathsf{op}} \to \mathsf{CAlg}^*_{\mathbf{C}}$. Send an enhanced measurable space (X, Σ, N) to the complex *-algebra of *bounded* morphisms $(X, \Sigma, N) \to (\mathbf{C}, \Sigma_{\mathbf{C}}, \{\emptyset\})$, where $\Sigma_{\mathbf{C}}$ denotes the Borel σ-algebra of $\mathbf{C}$. All operations are pointwise, with f^* being the pointwise complex conjugate of f. Here a morphism is *bounded* if it factors (when restricted to a conegligible subset of X) through some bounded subset of $\mathbf{C}$.
- Show that if C is a category with an equivalence relation R on its sets of morphisms, then precomposing a functor $\mathsf{C}/R \to \mathsf{D}$ with the quotient functor $\mathsf{C} \to \mathsf{C}/R$ establishes a bijection between functors $\mathsf{C}/R \to \mathsf{D}$ and functors $\mathsf{C} \to \mathsf{D}$ that send equivalent morphisms in C to equal morphisms in D. Apply this observation to construct a functor $\mathsf{L}^\infty\colon \mathsf{StrictEMS}^{\mathsf{op}} \to \mathsf{CAlg}^*_{\mathbf{C}}$
- [***] (Very optional.) Prove that the functor L^∞ is *not* faithful: there is an object (X, Σ, N) in the quotient $\mathsf{StrictEMS}$ and two different morphisms $f, g\colon (X, \Sigma, N) \to (\mathbf{C}, \Sigma_{\mathbf{C}}, \{\emptyset\})$ such that $\mathsf{L}^\infty(f) = \mathsf{L}^\infty(g)$.
- Consider again the category PreEMS. Show that the following relation of *weak equality almost everywhere* gives rise to another quotient category of PreEMS, denoted simply by EMS: $f \approx g\colon (X, \Sigma, N) \to (X', \Sigma', N')$ if for any $m \in \Sigma'$ the symmetric difference $f^*m \oplus g^*m$ is a negligible subset of X.
- Construct a functor $\mathsf{L}^\infty\colon \mathsf{EMS}^{\mathsf{op}} \to \mathsf{CAlg}^*_{\mathbf{C}}$ and prove that it is faithful.
- [***] (Very optional.) A functor $\mathsf{F}\colon \mathsf{C} \to \mathsf{D}$ is *full* if for any objects $X, Y \in \mathsf{C}$ the map of sets $\mathsf{C}(X, Y) \to \mathsf{D}(X, Y)$ is surjective. Show that the functor L^∞ is full.

2. Recall the category Open of open subsets of $\mathbf{R}^n$ (for any $n \geq 0$) and infinitely differentiable maps (Homework 1, Problem 1). (Attain extra bonus points by working with arbitrary second countable Hausdorff smooth manifolds instead.)

- Construct a functor $\mathsf{C}^\infty\colon \mathsf{Open}^{\mathsf{op}} \to \mathsf{CAlg}_{\mathbf{R}}$ to the category of real commutative algebras. Send an object $U \in \mathsf{Open}$ to its algebra of smooth functions $U \to \mathbf{R}$.
- Prove that the functor C^∞ is faithful (see Problem 1).
- Consider the map of sets $U \to \mathsf{Hom}(\mathsf{C}^\infty(U), \mathbf{R})$ that sends a point $u \in U$ to the homomorphism $\mathsf{C}^\infty(U) \to R$ $(f \mapsto f(u))$. Prove that this maps of sets is bijective. Hint: consider evaluating homomorphisms on coordinate functions $x_i \in \mathsf{C}^\infty(U)$. Hint: (re)familiarize yourself with Hadamard's lemma from elementary analysis.
- Prove that a subset $W \subset \mathsf{Hom}(\mathsf{C}^\infty(U), \mathbf{R}) \cong U$ is closed in U if and only if there is $f \in \mathsf{C}^\infty(U)$ such that $W = \{h\colon \mathsf{C}^\infty(U) \to \mathbf{R} \mid h(f) = 0\}$. Hint: (re)familiarize yourself with the smooth Tietze extension theorem, also known as the Whitney extension theorem.
- [**] (Optional.) Use the previous items to show that the functor C^∞ is full (see Problem 1).

3. Suppose R is a commutative ring. A *radical ideal* if R is an ideal I of R such that $x^n \in I$ for some $n \geq 0$ implies $x \in I$. Consider the poset $\mathsf{Spec}R$ of radical ideals of R ordered by inclusion.

- Prove that the poset $\mathsf{Spec}R$ admits suprema and infima for all subsets.
- Prove that in the poset $\mathsf{Spec}R$ we have $x \wedge \bigvee_{i \in I} y_i = \bigvee_{i \in I}(x \wedge y_i)$ for any arbitrary set I. Here $\wedge$ denotes infimum and $\bigvee$ denotes the supremum of a family.
- Promote the above construction to a functor $\mathsf{Spec}\colon \mathsf{CRing} \to \mathsf{Frame}$. Here Frame has as objects posets that admit arbitrary suprema, finite infima, and $x \wedge \bigvee_{i \in I} y_i = \bigvee_{i \in I}(x \wedge y_i)$ holds. Morphisms in Frame are order-preserving maps that preserve arbitrary suprema and finite infima.
- Construct a functor $\mathsf{Top} \to \mathsf{Frame}$. Prove that this functor is not faithful.
- Prove that it is faithful when restricted to Hausdorff spaces. Prove that $\mathsf{Spec}(\mathbf{Z})$ is not in the image of the restricted functor.

4. Review the definition of a Banach space. Consider the category Ban of Banach spaces and *contractive maps*. These are linear maps f such that $\|f(x)\| \leq \|x\|$ for all x. Consider the category Ball of *compact unit* *balls*. It has *compact unit balls* as objects, defined as pairs (V, B) consisting of a Hausdorff locally convex topological vector space V and a compact Hausdorff topological subspace $B \subset V$ such that B is *balanced* (i.e., $0 \in B$ and for any $x \in B$ and number t such that $|t| \le 1$ we have $tx \in B$), and B is *convex* (i.e., for any $x, y \in B$ and real numbers $r \ge 0$ and $s \ge 0$ such that $r + s \le 1$ we have $rx + sy \in B$). Morphisms $(V, B) \to (V', B')$ are continuous linear maps $V \to V'$ that send B to B'.

- Construct a functor $\mathsf{Ban}^{\mathsf{op}} \to \mathsf{Ball}$ that sends a Banach space X to the unit ball $(X^*, X^*_{\le 1})$, where X^* denotes the space of linear functionals on X equipped with the weak-* topology and $X^*_{\le 1}$ denotes its subspace of functionals of norm at most 1. (Look up the Banach–Alaoglu theorem.)
- Construct a functor $\mathsf{Ball} \to \mathsf{Ban}^{\mathsf{op}}$ that sends a unit ball (V, B) to the Banach space of continuous linear functionals on V, with the norm given by the supremum over B.
- Show that monomorphisms in Ban are precisely injective maps.
- Show that epimorphisms in Ball are precisely those morphisms $(V, B) \to (V', B')$ for which the maps $V \to V'$ and $B \to B'$ are surjective.
- Assuming the functors defined above form an equivalence of categories, prove that given an inclusion $A \to B$ of Banach spaces, any linear functional on A can be extended to a linear functional on B that has the same norm. You may use the fact that monomorphisms are precisely epimorphisms in the opposite category.

5. Given a topological space X, consider the category Cov_X of open covers of X. Objects are open covers of X, defined as maps of sets $f : I \to \mathrm{Open}(X)$ such that $\bigcup_{i \in I} f(i) = X$, where $\mathrm{Open}(X)$ denotes the collection of open subsets of X. Morphisms $(f : I \to \mathrm{Open}(X)) \to (g : J \to \mathrm{Open}(X))$ are maps of sets $p : I \to J$ such that for every $i \in I$ we have $f(i) \subset g(p(i))$.

- Extend the following construction to a functor $\mathsf{Cov}_X \to \mathsf{Set}$. Send an open cover $f : I \to \mathrm{Open}(X)$ to the set $\pi_0(f)$, defined as the quotient of I by the equivalence relation $\sim$, where $i \sim i'$ if $f(i) \cap f(i') \ne \emptyset$.
- If X is a locally connected topological space (look it up), explain how the functor π_0 relates to the traditional set of connected components.
- [**] (Optional.) Suppose $X = \{0, 1\}^{\mathbf{N}}$ (an infinite product of two-point topological spaces; also known as the Cantor space). Is there an open cover f of X such that the canonical map from the set of connected components of X to $\pi_0(X)(f)$ is an isomorphism?
- Look up the definition of a groupoid and the category Grpd of groupoids and functors. Extend the following construction to a functor $\mathsf{Cov}_X \to \mathsf{Grpd}$. Send an open cover $f : I \to \mathrm{Open}(X)$ to the groupoid $\pi_{\le 1}(f)$. Objects are elements of I. Morphisms $i \to i'$ are equivalence classes of chains of elements of I: $i = i_0, i_1, \ldots, i_k = i'$ such that $f(i_j) \cap f(i_{j+1}) \ne \emptyset$. Two chains are equivalent if they can be connected by a sequence of elementary transformations or their inverses. An elementary transformation takes two consecutive elements i_j, i_{j+1} and replaces them with a single i'_j such that $f(i_j) \cap f(i_{j+1}) \cap f(i'_j) \ne \emptyset$.
- Compute $\pi_{\le 1}(S^1)(f)$, where S^1 is a circle and f is a sufficiently fine cover, for example, three overlapping intervals.

6. Look up the definition of a *covering space* in topology.

- Prove that for a topological space X, covering spaces $A \to X$ over X form a category. Morphisms $(A \to X) \to (B \to X)$ are continuous maps $A \to B$ that commute with the map to X.
- Prove that for a topological space X, the following construction defines a category $\pi_{\le 1}(X)$. Objects are points in X. Morphisms $x \to x'$ are equivalences classes of continuous maps $p : [0, 1] \to X$ such that $p(0) = x$, $p(1) = x'$. Two paths $p, q : [0, 1] \to X$ are equivalent if they are homotopic. Composition is given by concatenating paths (define exactly how).
- Given a covering space $p : A \to X$, prove that the following construction defines a functor $\pi_{\le 1}(X) \to \mathsf{Set}$. Send a point $x \in X$ to the fiber $A_x = \{a \in A \mid p(a) = x\}$. Send a path $p : [0, 1] \to X$ to the map of sets $A_p : A_{p(0)} \to A_{p(1)}$ defined as follows. Given $a \in A_{p(0)}$, denote by $q : [0, 1] \to A$ the unique lift of p such that $q(0) = a$. (Look up the lifting property for covering spaces.) Then A_p sends a to the point $q(1)$.