1. Which sets of data given below define categories? Warning: some sets of data are incomplete, e.g., do not specify composition. You must reconstruct all missing data (this is a creative process and can have more than one answer). Some items below have negative answers. Some are purposefully nonsensical. If some set of data cannot be made into a category, you have to provide a proof.

- Topological spaces and proper maps. (A map $f\colon X \to Y$ is *proper* if it is continuous and for any compact subset $T \subset Y$ the subset $f^{-1}(T) \subset X$ is also compact.)
- Sets and relations. (More precisely, given two sets X and Y, morphisms $X \to Y$ are *relations* from X to Y, i.e., subsets of $X \times Y$. Two relations $R \subset X \times Y$ and $S \subset Y \times Z$ are composed as follows: $R \circ S = \{(x,z) \mid \exists y \in Y\colon (x,y) \in R \wedge (y,z) \in Z\}$.
- Sets and surjective functions.
- Sets and partially defined functions. (A partially defined function $X \to Y$ is a function $A \to Y$, where $A \subset X$. If $x \in X$, we say that f is *defined* on x (or: $f(x)$ is defined) if $x \in A$. Partially defined functions are composed as follows: if $f\colon X \to Y$ and $g\colon Y \to Z$ are partially defined function, then the composition gf is defined on $x \in X$ if f is defined on x and g is defined on $f(x)$, in which case $(gf)(x) := g(f(x))$.)
- Fix a topological space X and define a category as follows. Objects are continuous functions with codomain X, i.e., $f\colon Y \to X$ (Y is arbitrary). (Here *morphisms* in Top play the role of *objects* in the category that we are constructing.) Morphisms from an object $f\colon Y \to X$ to an object $f'\colon Y' \to X$ are continuous maps $g\colon Y \to Y'$ such that $f'g = f$.
- The category Open. Objects are open subsets of $\mathbf{R}^n$, where n is arbitrary (not fixed). Morphisms $U \to V$ are infinitely differentiable maps $U \to V$.
- The category Open_*. Objects are pairs (U,x), where $U \subset \mathbf{R}^n$ is an open subset and $x \in U$. Morphisms $(U,x) \to (V,y)$ are infinitely differentiable maps $U \to V$ that map x to y.
- $\mathsf{Mat}_{\mathbf{R}}$: objects are natural numbers $n \geq 0$ and morphisms $m \to n$ are matrices of size $n \times m$. Composition is multiplication of matrices.
- $\mathsf{B}\mathbf{R}$: there is only one object $*$. Morphisms $* \to *$ are real numbers. Composition of morphisms is given by multiplication of real numbers.
- There is only one object $*$. Morphisms $* \to *$ are compactly supported continuous functions $\mathbf{R} \to \mathbf{R}$. Composition of morphisms is given by multiplication of functions.
- Poset: objects are partially ordered sets (i.e., a set X with a relation R that is reflexive ($x \leq x$), transitive ($x \leq y$ and $y \leq z$ implies $x \leq z$), and antisymmetric ($x \leq y$ and $y \leq x$ implies $x = y$). Morphisms are functions that preserve the order: if $x \leq y$, then $f(x) \leq f(y)$.

2. Which sets of data below define functors? (Same warning as above.)

- $\mathsf{Open}_* \to \mathsf{Vect}_{\mathbf{R}}$. Send $U \subset \mathbf{R}^m$ to $\mathbf{R}^m$. Send $f\colon (U,x) \to (V,y)$ to the linear map $\mathbf{R}^m \to \mathbf{R}^n$ given by the Jacobian matrix of f at x, i.e., the entry in ith row and jth column is the value of the ith partial derivative of the jth coordinate of f at point x. In symbols: $a_{i,j} = \frac{\partial f_j}{\partial x_i}(x)$. (The jth coordinate of f is the composition $U \to V \subset \mathbf{R}^n \to \mathbf{R}$, where $\mathbf{R}^n \to \mathbf{R}$ is the projection to the jth component.)
- $\mathsf{Mat}_{\mathbf{R}} \to \mathsf{B}\mathbf{R}$: send any object $n \geq 0$ of $\mathsf{Mat}_{\mathbf{R}}$ to the only object of $\mathsf{B}\mathbf{R}$. Send a matrix of size $m \times n$ to its determinant (a morphism in $\mathsf{B}\mathbf{R}$) if $m = n$. Otherwise send it to zero.
- $\mathsf{OpenSet}\colon \mathsf{Top} \to \mathsf{Poset}$: send any topological space X to the poset $\mathsf{OpenSet}(X)$ whose elements are open subsets of X and the ordering is given by inclusion. Send any continuous map $f\colon X \to Y$ of topological spaces to the map of posets $g\colon \mathsf{OpenSet}(X) \to \mathsf{OpenSet}(Y)$ defined as follows: $g(U) = \bigcup_{V \subset f(U)} V$, where V runs over open subsets of Y.

3. Define a functor $\mathsf{Mat}_{\mathbf{R}}^{\mathsf{op}} \to \mathsf{Mat}_{\mathbf{R}}$. Define a functor $\mathsf{Mat}_{\mathbf{R}} \to \mathsf{Vect}_{\mathbf{R}}$. Define a functor $\mathsf{B}\mathbf{R} \to \mathsf{Mat}_{\mathbf{R}}$. Define a functor $\mathsf{Mat}_{\mathbf{R}} \to \mathsf{Open}_*$. Define a functor $\mathsf{Top}^{\mathsf{op}} \to \mathsf{Poset}$.

4. This problem investigates how categories work with measure theory. A novel feature is that measurable maps that differ on a set of measure 0 must be identified.

- Define a *measurable space* as a pair (X, Σ), where X is a set and Σ is a σ-algebra on X, i.e., a collection of subsets of X that is closed under complements and countable unions. Define a *measurable map* $(X,\Sigma) \to (X',\Sigma')$ as a map of sets $f\colon X \to X'$ such that the f-preimage of any element of Σ' is an element of Σ. Do measurable sets and measurable maps form a category?
- Define an *enhanced measurable space* as a triple (X,Σ,N), where (X,Σ) is a measurable space and N is a σ-ideal of Σ, i.e., a collection of elements of Σ that is closed under passage to subsets (if $A \in \Sigma$,

$B \in N$, and $A \subset B$, then $A \in N$) and countable unions. A *negligible set* is defined as a subset of some element of N. Define a *measurable map* $(X, \Sigma, N) \to (X', \Sigma', N')$ as a map of sets $f\colon X_f \to X'$ (where $X_f \subset X$ is some subset) with the following properties: (1) The set $X \setminus X_f$ is negligible. (2) For any $m' \in \Sigma'$ there is $m \in \Sigma$ such that the set $f^* m' \oplus m$ is negligible. (3) For any $n' \in \Sigma'$ the set $f^* n'$ is negligible. Do enhanced measurable sets and measurable maps form a category?

- Two mesurable maps $f, g\colon (X, \Sigma, N) \to (X', \Sigma', N')$ are *equal almost everywhere* ($f \sim g$) if $\{x \in X_f \cap X_g \mid f(x) \neq g(x)\}$ is negligible. Show that equality almost everywhere defines an equivalence relation that is compatible with composition: if $f \sim f'$ and $g \sim g'$, then $fg \sim f'g'$.
- Suppose C is a category and for every pair of objects $X, Y \in \mathsf{C}$ we are given an equivalence relation $R_{X,Y}$ on $\mathsf{C}(X, Y)$ that is compatible with composition: if $f \sim f'$ and $g \sim g'$, then $fg \sim f'g'$. Show that taking quotients of $\mathsf{C}(X, Y)$ with respect to these equivalence relations produces a category.

5. Fix a category C. A *section* of a morphism $f\colon X \to Y$ in C is a morphism $g\colon Y \to X$ such that $fg = \mathrm{id}_Y$. Give an example of a category C such that all epimorphisms have sections. Give an example of a category C and an epimorphism f in C that does not have a section. Hint: it suffices to use the examples that we studied in class.

6. Fix a category C. A *bimorphism* in C is a morphism f that is simultaneously a monomorphism and an epimorphism. Is any isomorphism a bimorphism? Give an example of a category C and a bimorphism f in C that is not an isomorphism.

7. Construct two functors $D\colon \mathsf{Set} \to \mathsf{Set}$ and $I\colon \mathsf{Set}^{\mathsf{op}} \to \mathsf{Set}$ such that for any set X we have $D(X) = I(X) = 2^X$, where 2^X denotes the set of all subsets of X. (In other words, you must define D and I on morphisms and prove that composition and identity maps are respected.)

8. Construct a functor $\mathsf{L}^1\colon \mathsf{Set} \to \mathsf{Ban}_1$ such that for any set S the Banach space $\mathsf{L}^1(S)$ is the space of functions $f\colon S \to \mathbf{R}$ such that the sum $\sum_{s \in S} f(s)$ exists (and is finite). Construct a functor $\mathsf{L}^\infty\colon \mathsf{Set}^{\mathsf{op}} \to \mathsf{Ban}_1$ such that $\mathsf{L}^\infty(S)$ is the space of all bounded functions $S \to \mathbf{R}$.

9. Show that the class of epimorphisms in the category of Hausdorff topological spaces coincides with the class of continuous maps whose image is dense.

10. Describe concretely all monomorphisms and epimorphisms in $\mathsf{B}\mathbf{R}$. Same question for Open and Open_*. Same question for Poset.

11. A *idempotent ring* is a ring R such that $x^2 = x$ for any $x \in R$. (Rings are assumed to be associative and unital, homomorphisms of rings preserve units.)

- Show that any idempotent ring is commutative: $xy = yx$ for all x and y.
- Show that the relation $x \le y \colon= (x = xy)$ defines a partial order on R.
- Show that given a set X, equipping 2^X (the set of subsets of X) with the following operations: $0 \colon= \emptyset$, $x + y \colon= (x \setminus y) \cup (y \setminus x)$, $-x \colon= X \setminus x$, $1 \colon= X$, $xy \colon= x \cap y$ produces an idempotent ring.
- Recall that the supremum of a subset $A \subset R$, if it exists, is the unique element $s \in R$ such that for all $a \in A$ we have $a \le s$ and if s' is another element with the same property, then $s \le s'$. Show that in the idempotent ring 2^X every subset has a supremum.
- An *atom* in an idempotent ring is an element $a \in R$ such that $a \neq 0$ and if $0 \le b \le a$ for some $b \in R$, then $b = 0$ or $b = a$. Show that in the idempotent ring 2^X every element can be represented as the supremum of a set of atoms.
- Show that the assignment $X \mapsto 2^X$ can be extended to a contravariant functor $2^{(-)}$ from the category of sets to the category whose objects are idempotent rings in which every subset has a supremum and every element is the supremum of a set of atoms, and morphisms are homomorphisms of rings that preserve suprema ($f\colon R \to R'$ preserves suprema if for any $S \subset R$ we have $\sup f(S) = f(\sup S)$).
- Construct a contravariant functor going in the opposite direction. (Hint: it is useful to keep the example of the idempotent ring 2^X when constructing this functor.)
- Prove that the two functors together form an equivalence of categories.