Bundle $(d-1)$-gerbes w/ conn. as fully ext. nontop. FFT '12

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Theorem: If $\mathcal{A}$ - abelian Lie group (like $U(1)$) $d \geq 0$ $M : \mathrm{Man}_0$, then

$$\mathrm{FFT}_{d, M, B^d \mathcal{A}} \simeq$$

$$\Gamma\left(\begin{array}{l}\mathcal{A}\text{-banded } (d-1)\text{-gerbes w/ conn.} \\ \text{over } M\end{array}\right)$$

$\uparrow$ $(N \dashv \Gamma)$ - Dold-Kan adj.

What is a bundle $(d-1)$-gerbe?

Consider $D_{\mathcal{A},d} : \mathrm{Man}^{op} \to \mathrm{Ch}_{\geq 0}$

$$D_{\mathcal{A},d}(M) = \Omega^d(M; \mathfrak{a}) \leftarrow \Omega^{d-1}(M; \mathfrak{a}) \leftarrow \cdots \leftarrow \Omega^0(M; \mathfrak{a}) \leftarrow \mathbb{Z}_{\mathcal{A}}$$

$$\mathfrak{a} \cong \mathrm{Lie}(\mathcal{A})$$

$$0 \to \mathbb{Z}_{\mathcal{A}} \xrightarrow{\mathrm{tr}} \mathfrak{a} \to \mathcal{A} \to 0$$

$$0 \to \underbrace{2\pi i\mathbb{Z}}_{\text{quanta}} \xrightarrow{\iota} i\mathbb{R} \longrightarrow U(1) \to 0$$

$$0 \to \overbrace{(2\pi i\mathbb{Z})^n} \to (i\mathbb{R})^n \longrightarrow U(1)^n \to 0$$

de Rham cohomology = $\mathbb{R}$-banded gerbes

There is an equiv descr.:

$$\Omega^d(M;\mathfrak{a}) \leftarrow \dots \leftarrow \Omega^1(M;\mathfrak{a}) \xleftarrow{d\log} C^\infty(M;A) \leftarrow 0$$

<u>Deligne complex</u>

The ch. complex of A-banded bundle $(d-1)$-gerbes w/ conn. on M is the value of ∞-sheafification of $\mathcal{D}_{A,d}$ on M

<u>In practice</u>: pick a good cover of M

Then $\mathrm{Map}(\check{C}\mathcal{U}, \mathcal{D}_{A,d})$ computes the value of the ∞-sheafification of $\mathcal{D}_{A,d}$ on M.

Even more concretely:

$$\mathrm{Map}(\check{C}\mathcal{U}; D_{A,d}) = \underline{\mathrm{Tot}(\Omega^d(\sqcup \mathcal{U}_i;\mathfrak{a}))}$$

$$\begin{tikzcd}
\Omega^d(\sqcup \mathcal{U}_i;\mathfrak{a}) \arrow[d] & \Omega^{d-1}(\sqcup_i \mathcal{U}_i;\mathfrak{a}) \arrow[l] \arrow[d] & \cdots \arrow[l] & C^\infty(\sqcup \mathcal{U}_i;A) \arrow[l] \arrow[d] \\
\Omega^d(\sqcup_{jk} \mathcal{U}_{jk};\mathfrak{a}) \arrow[d] & \Omega^{d-1}(\sqcup_{jk} \mathcal{U}_{jk};\mathfrak{a}) \arrow[l] \arrow[d] & \cdots \arrow[l] & C^\infty(\sqcup \mathcal{U}_{jk};A) \arrow[l] \arrow[d] \\
\vdots & \vdots & & \vdots \\
\Omega^d(\sqcup \mathcal{U}_I;\mathfrak{a}) \arrow[d] & \Omega^{d-1}(\sqcup \mathcal{U}_I;\mathfrak{a}) \arrow[l] \arrow[d] & \cdots \arrow[l] & C^\infty(\sqcup \mathcal{U}_I;A) \arrow[l] \arrow[d] \\
\vdots & \vdots & & \vdots
\end{tikzcd}$$

$$\mathcal{U}_I := \bigcap_{i:I} \mathcal{U}_i$$

$\underline{d=1}$:

$$\begin{tikzcd}
\Omega^1(\sqcup \mathcal{U}_i;\mathfrak{a}) \arrow[d] & C^\infty(\sqcup \mathcal{U}_i;A) \arrow[l, "d\log"'] \arrow[d] \\
\Omega^1(\sqcup \mathcal{U}_{jk};\mathfrak{a}) & C^\infty(\sqcup \mathcal{U}_{jk};A) \arrow[l, "d\log"'] \arrow[d] \\
 & C^\infty(\sqcup \mathcal{U}_{lmn};A)
\end{tikzcd}$$

The chain complex of $\mathcal{A}$-banded bundle gerbes

$$Z_0 \leftarrow C^\infty(\sqcup U_i ; \mathcal{A})$$

$$Z_0 = \{ \omega_i : \Omega^1(U_i ; \mathfrak{a}), \quad t_{jk} : C^\infty(\sqcup U_{jk} ; \mathcal{A}) \mid$$

$$\forall j,k \quad \omega_k - \omega_j = d\log(t_{j,k})$$

$$\forall l,m,n : t_{mn} - t_{ln} + t_{lm} = 0 \}$$

R/k: all other properties of t_{ij} follow from the cocycle cond.:

$$t_{ll} - t_{ll} + t_{ll} = 0 \Rightarrow t_{ll} = 0$$

$$t_{mn} - t_{ll} + t_{nm} = 0 \Rightarrow t_{mn} = -t_{nm}$$

$$\mathcal{D}_{\mathcal{A},d} = \Omega^{d+1}_{cl}(-;\mathfrak{a})[0] \times^h C^\infty(-;\mathbb{Z}_A)[d+1] \atop C^\infty_{cl}(-;\mathfrak{a})[d+1]$$

$$\mathcal{FFT}_{d,M,B^d A} = \mathcal{F}un^{\otimes}(\mathcal{B}ord^M_d ; B^d A) = \mathcal{M}ap(M ; (B^d A)^\times_d)$$

Thm: $(\mathbb{B}^d\mathcal{A})^\times_d \simeq \mathcal{D}^{\mathrm{FEmb}\hat{\hat{d}}}_{d,\mathcal{A}}$

Thm: $\mathbb{R}\mathrm{Map}(\mathcal{M};\mathcal{D}^{\mathrm{FEmb}\hat{\hat{d}}}_{d,\mathcal{A}}) \simeq$

$\simeq \mathbb{R}\mathrm{Map}_{\mathrm{Cart}}(\mathcal{M},\mathcal{D}_{d,\mathcal{A}})$

Def: $\mathcal{D}^{\mathrm{FEmb}\hat{\hat{d}}}_{d,\mathcal{A}} \longrightarrow (\mathbb{B}^d\mathcal{A})^\times_d$

$\omega \in \Omega^d(\mathbb{R}^d;\mathfrak{a}) \quad B \hookrightarrow \mathbb{R}^d$

$\exp \int_B \omega \in \mathcal{A}$