Examples of TFT via GCH

$R : \mathcal{FEmb}_1^{op} \to Set$ $\qquad \mathcal{M} \xrightarrow[p]{} \mathcal{U}$ dim fib(p) = 1 is sent by R to the set of fiberwise Riem. metrices w/ orientations on $\mathcal{M}$.

R/k: $[0,\ell] \simeq [0;1]$

Consider field theories of the form:

$$Z : Bord_1^R \longrightarrow Vect$$

We evaluate this TFT on $* = U : Cart$.

$Ob(Vect) = N(Vect^{\simeq})$ $\quad Mor(Vect) = N((Vect^{2})^{\simeq})$

$$\begin{tikzcd}
V \arrow[r, "L"] \arrow[d, "A", "\simeq"'] & W \arrow[d, "B", "\simeq"'] \\
Y \arrow[r, "K"'] & Z
\end{tikzcd} \longleftrightarrow (1,1) \text{ simplex in Vect.}$$

Z sends $+ : Bord_1^R$ to V (f.d. vector space)

$- \longmapsto V^{\vee}$. Consider the interval of length ℓ $\underbrace{[0,\ell]}$ oriented so that it is compatible w/ the "+" orientation on germs of endpoints. A bordism $[0;\ell] \mapsto [L_\ell : V \to V]$

$\exists$ (1,1)-simplex h in $Bord_1^R$ s.t. $[0;\ell] \simeq_h [0;0]$

$$\begin{tikzcd}
+ \arrow[r, "{[0;\ell]}", "\simeq"'] \arrow[d, "{([0,\ell],h)}"', "\simeq"] & + \arrow[d, "id"] \\
+ \arrow[r, "id"'] & +
\end{tikzcd} \quad \overset{Z}{\longmapsto} \quad \begin{tikzcd}
V \arrow[r, "L_\ell", "\simeq"'] \arrow[d, "\simeq"', "L_\ell"] & V \arrow[d, "id_V"] \\
V \arrow[r, "id_V"'] & V
\end{tikzcd}$$

$\forall\, \varepsilon > 0 \quad \exists\, w : Ob(Bord_1^R)_2 \quad d_1(w) = ([0,\varepsilon],h)$

$d_0(w) = id \quad , d_2(w) = ([0,\ell], h)$

$L_\ell = L_\varepsilon$

$L_\ell \neq L_\varepsilon$.

Def: $C_d : sPSh(\mathcal{F}Emb_d)_{\check{C}, fle} \to PSh(\underline{Cat}, \underline{sSet}^{GL(d)})_{\check{C}}$

The idea is that for various $\varepsilon > 0$ we have equivalences

of categories $Bord^{\delta}_{B^d(\varepsilon)} \xrightarrow{\simeq} Bord^{\delta}_{B^d(\varepsilon')}$ $\quad \varepsilon' > \varepsilon$

$GL(d) \leftrightarrow S(GL(d))$

Lemma: We have an equivalence of $GL(1)$-equiv. sPSh:

$$\mathbb{C}_1(R) \simeq \mathbb{Z}/2 \times BR$$

$$GL(1) \xrightarrow[sgn]{} \pi_0(GL_1(1)) \cong \mathbb{Z}/2$$

Proof: R we replace it w/ $R' \subset R$ it sends $M \to U$

to a set of fiberwise metrics on M of fin. length

$$R' \simeq \underbrace{\operatorname{hcolim}_{(\mathbb{R}\times U \to U) \to R'}}_{j/R} (\mathbb{R}\times U \to U)$$

$j : \mathcal{F}Emb_1 \hookrightarrow sPSh(\mathcal{F}Emb_1)$.

Denote $\mathbb{C}$ a cat. w/ $Ob\, \mathbb{C} = \{\alpha : U \to \mathbb{R}_{>0}\}$

$(h : U \to U', \beta : U \to \mathbb{R}_{\geq 0}) : \alpha \to \alpha'$ s.t. $\alpha' h \geq \alpha + \beta$

$(h' : U' \to U'', \beta' : U' \to \mathbb{R}_{\geq 0}) \cdot (h : U \to U', \beta : U \to \mathbb{R}_{\geq 0}) =$

$= (h'h : U \to U'', \beta' h + \beta : U \to \mathbb{R}_{\geq 0})$

$\imath : \mathbb{C} \hookrightarrow j/R$

$\underbrace{\alpha : \mathcal{U} \to \mathbb{R}_{>0}}$ $\quad p : \underbrace{(0,\alpha) = \underbrace{\{(t,u) \in \mathbb{R}\times\mathcal{U} \mid 0 < t < \alpha(u)\}}\to \mathcal{U}}$

$\boxed{\mathbb{R}\times\mathcal{U} \to \mathcal{U}}$

$(h : \mathcal{U} \to \mathcal{U}', \beta : \mathcal{U} \to \mathbb{R}_{\geq 0}) : \alpha \to \alpha'$ in $\mathbb{C}_1$

$(0,\alpha) \to (0,\alpha') \quad (t,u) \mapsto (t+\beta(u), h(u))$

$2 : \mathbb{C} \xrightarrow{\simeq} j/_R$

$C_1(R) \simeq C_1(R') \simeq C_1(\operatorname{hocolim}_{j/R}(\mathbb{R}\times U \to \mathcal{U})) \simeq$

$\simeq C_1(\operatorname{hocolim}_{\mathbb{C}}((0,\alpha) \to U)) \simeq \operatorname{hocolim}_{\mathbb{C}} \underbrace{C_1((0,\alpha)\to U)}$

$\simeq \operatorname{hocolim}_{\mathbb{C}}(\mathbb{Z}/2 \times U)$

W: Cat_0 $\quad D_W$ a cat w/ $GL(1)$-action

$(\alpha : U \to \mathbb{R}_{>0}, f : W \to U, \sigma : \mathbb{Z}/2) : (D_W)_0$

$(h : \mathcal{U} \to \mathcal{U}', \beta : U \to \mathbb{R}_{\geq 0}) : (\alpha : U \to \mathbb{R}_{>0}, f : W \to \mathcal{U}, \sigma : \mathbb{Z}/2) \to$

$\to (\alpha' : \mathcal{U}' \to \mathbb{R}_{>0}, f' : W \to \mathcal{U}', \sigma' : \mathbb{Z}/2)$

$hf = f'$, $\sigma' = \sigma$ $\quad \alpha' h \geq \alpha + \beta$

$\varphi : GL(1)$ acts on $(\alpha, f, \sigma) \mapsto (\alpha, f, \operatorname{sgn}(\varphi)\sigma)$

$D'_W \hookrightarrow D_W \quad \boxed{\sigma = 0} \quad D_W = \mathbb{Z}/2 \times D'_W$

E_W w/ objects $A : W \to \mathbb{R}_{>0}$ $\quad A \xrightarrow{B} A'$ $\quad B : W \to \mathbb{R}_{\geq 0}$

s.t. $A + B \leq A'$ $\quad D'_W \to E_W$ sending $(\alpha, f) \mapsto \alpha f : W \to \mathbb{R}_{>0}$

$(h, \beta) : (\alpha, f) \to (\alpha', f') \mapsto \beta f : W \to \mathbb{R}_{\geq 0}$

$E_W \to D'_W$ $\quad A: W \to \mathbb{R}_{>0} \mapsto (A, id_W)$

$B: W \to \mathbb{R}_{\geq 0} \mapsto (id_W, B)$

$$\begin{tikzcd} E_W \arrow[r] \arrow[rr, bend right, "1_{E_W}"'] & D'_W \arrow[r] & E_W \end{tikzcd} \qquad \begin{tikzcd} D'_W \arrow[r] \arrow[rr, bend right, "\mathcal{F}"'] & E_W \arrow[r] & D'_W \end{tikzcd}$$

$\mathcal{F} \Rightarrow 1_{D'_W}$ $\quad \mathcal{F}_{(\alpha,f)} = (f,0) : (\alpha f, id_W) \to (\alpha, f)$

$N(D'_W) \simeq N(E_W)$

$B(C^\infty(W;\mathbb{R}),+)$ a groupoid with a single object and pointwise add. on morphisms.

$\mathcal{F}: E_W \longrightarrow B(C^\infty(W;\mathbb{R}),+)$

$\mathcal{F}(B) = B$

We want to show $N(E_W) \xrightarrow{\simeq} N(B(C^\infty(W;\mathbb{R}),+))$

Use Quillen's theorem A i.e. show $* \downarrow \mathcal{F} \simeq pt$

$P: \quad (A: W \to \mathbb{R}_{>0}, g: W \to \mathbb{R})$

$(A: W \to \mathbb{R}_{>0}, g: W \to \mathbb{R}) \to (A': W' \to \mathbb{R}_{>0}, g: W' \to \mathbb{R})$

$B: W \to \mathbb{R}_{\geq 0} \quad A' \geq A + B, \; g = g' + B$

$B = g - g'$

$P \neq \emptyset \quad (1,0) : P_0$

$(A,g), (A',g') : P_0$ we want (Ψ, G)

$$\begin{tikzcd} (A,g) \arrow[r, "B"] & (\Psi, G) \\ (A',g') \arrow[ur, "B'"'] & \end{tikzcd}$$

$\Psi \geq A+B$, $g = G+B$, $\Psi \geq A'+B'$

$g' = G+B'$. $\underline{g-g' = B-B'}$ we choose $B, B' : U \to \mathbb{R}_{>0}$

s.t. $G = g - B = g' - B'$ $\quad \Psi : W \to \mathbb{R}_{>0}$ s.t.

$\Psi \geq A+B, A'+B'$.

$$\operatorname{hocolim}_{\alpha: U \to \mathbb{R}_{>0}} (\mathbb{Z}/2 \times U) \simeq \underline{\mathbb{Z}/2 \times B\mathbb{R}} \qquad \square$$

Thm: Let $\mathcal{C}$ be a smooth sym. monoidal $(\infty,1)$ cat. with duals. We have an equiv. of $(\infty,1)$ cats:

$$FFT_{1,R,\mathcal{C}} = Fun^{\otimes}(Bord_1^{R}, \mathcal{C}) \simeq Map(B\mathbb{R}, \mathcal{C}^{\simeq})$$

$$FFT_{1,R^{un},\mathcal{C}} = Fun^{\otimes}(Bord_1^{R^{un}}, \mathcal{C}) \simeq Map^{\mathbb{Z}/2}(B\mathbb{R}, \mathcal{C}^{\simeq})$$

$\mathbb{Z}/2 \curvearrowright B\mathbb{R}$ trivially $\quad \mathbb{Z}/2 \curvearrowright \mathcal{C}^{\simeq}$ $\quad V \mapsto V^{\vee}$

Proof: $\mathcal{C}^{\simeq}$ can be promoted to f.l.c. presheaf $\mathcal{C}_1^{\simeq}$ on $FEmb_1$. $\mathcal{C}_1^{\simeq}$ can be identified w/ $C_1(\mathcal{C}_1^{\times})$ on Cart

$C_1(R) \simeq \mathbb{Z}/2 \times B\mathbb{R} \Rightarrow$

$$Map(R, \mathcal{C}_1^{\simeq}) \simeq Map_{GL(1)}(C_1(R), C_1(\mathcal{C}_1^{\simeq})) \simeq$$

$$\simeq Map_{GL(1)}(\mathbb{Z}/2 \times B\mathbb{R}, C_1(\mathcal{C}_1^{\simeq})) \simeq Map(B\mathbb{R}, \mathcal{C}^{\simeq})$$

$$\mathbb{Z}/2 \times B\mathbb{R} \to \underset{\mathcal{C}^{\simeq}}{\underset{\wr\!\wr}{C_1(\mathcal{C}_1^{\simeq})}} \quad \mathbb{Z}/2\text{-equiv} \leftrightarrow B\mathbb{R} \to \mathcal{C}^{\simeq}$$

$\square$