Geometric structures & geom. cobordism hypothesis.

GCH: $\mathrm{FFT}_{d,S,V} = \underset{\mathbb{R}\mathrm{Map}}{\overset{}{\mathrm{Fun}^{\otimes}}}(\mathrm{Bord}_d^S, V)^{\simeq}$

$\simeq \mathbb{R}\mathrm{Map}(S, V_d^{\times})$

simplicial presheaves

What is S?

(our site

$S: \mathrm{sPSh}(\mathrm{FEmb}_d)$

Def: $d \geq 0$. FEmb_d is a site w/ objects given by submersions:

Ob: $T \twoheadrightarrow B$ $\quad \dim T - \dim B = d$

$\in \mathrm{MFLD}_0$ (or even Cart then we obtain a dense subsite)

Mor:
$$\begin{tikzcd} T_1 \arrow[r] \arrow[d, "p_1"'] \arrow[dr, phantom, "\#"] & T_2 \arrow[d, "p_2"] \\ B_1 \arrow[r] & B_2 \end{tikzcd}$$
s.t. fiberwise open embeddings:

$\forall x \in B_1$ $\quad p_1^{-1}\{x\} \xrightarrow{t_*} p_2^{-1}\{b(x)\}$

is an open embedding (or a local diffeomorphism)

$$\left\{ \begin{array}{ccc} T_i & \xrightarrow{t_i} & T \\ \downarrow p_i & \# & \downarrow p \\ B_i & \xrightarrow[b_i]{} & B \end{array} \right\}$$ is a covering family

in $\mathcal{F}\mathcal{E}mb_d$ if $\{T_i \to T\}$ is a cov. family

in Man.

Def: A d-dimensional geometric str-re

is $S : sPSh(\mathcal{F}\mathcal{E}mb_d)$

Ex: $S : sPSh(\mathcal{F}\mathcal{E}mb_d)$

(1) M : Man, $M : sPSh(Cart)$

$$S\left(\begin{array}{c} T \\ \downarrow \\ B \end{array}\right) = C^\infty(T, M)$$

$$S\left(\begin{array}{ccc} T & \to & T' \\ \downarrow & \# & \downarrow \\ B & \to & B' \end{array}\right) = C^\infty(T \to T', M)$$

(2) $\begin{array}{c} T \\ p \downarrow \\ B \end{array} \mapsto$ The set of Riem. metrics on $T\left(\begin{array}{c} T \\ \downarrow \\ B \end{array}\right) = \ker(T_p : TT \to TB)$

Generalizes to any diff. geom. def. w/

tangent / cotangent bundles.

For example: $\begin{array}{c} T \\ \downarrow p \\ B \end{array} \mapsto$ Groupoid of principal G-bundles w/ ∇

(Chern - Simons) theory

$\mathcal{M}$ - orbifold $\quad \mathcal{M} : \mathbb{B}_\nabla G.$

③ What about foliations?

Prequantization:

Diff. cohomology

$$\begin{tikzcd}
\mathcal{D} = (\Omega^n \leftarrow \Omega^{n-1} \leftarrow \dots) \arrow[r] \arrow[d] & \mathbb{Z}[n+1] \arrow[d, bend left] \\
\Omega^{n+1}_{cl} \arrow[r] & (\Omega^{n+1}_{cl} \leftarrow \Omega^n \leftarrow \dots \leftarrow \Omega^0) \simeq \mathbb{R}[n+1]
\end{tikzcd}$$

Lagrangian density $S \xrightarrow{\omega} \Omega^{n+1}$

top. sector $S \hookrightarrow \mathbb{Z}[n+1] : H^{n+1}(S;\mathbb{Z})$

$[\omega] = c \;;\; d\Psi = \omega - c$

$S \to P = \mathbb{B}^n_\nabla U(1) =$ "bundle $(n-1)$-gerbes w/ ∇"

Example: (Chern–Simons theory)

$n=3 \quad \omega : \Omega^n_{cl} \;,\; S = \mathbb{B}_\nabla G$

ⓐ $S \xrightarrow{\omega} \Omega^4_{cl}$

$(P,\nabla) \mapsto \underbrace{F}_{\text{curvature}} \mapsto \mathrm{tr}(F \wedge F)$

(b) $S \mapsto \mathbb{Z}[4]$ $\quad c \in H^4(\mathbb{B}G;\mathbb{Z})$

level

(c) $\psi : d\psi = \omega - c$ $\quad \psi$ - Chern-Simons form $=$

$$= tr(F \wedge A - \tfrac{1}{3}A^{\wedge 3}) = tr(dA \wedge A + \tfrac{2}{3}A^{\wedge 3})$$

$$\underbrace{\mathbb{B}_\nabla G}_{S} \xrightarrow{\quad} \overbrace{\mathbb{B}^3_\nabla U(1)}^{e^x_3}$$ the Chern-Simons bundle 2-gerbe, $\mathcal{C} = \mathbb{B}^3 U(1)$

$s\mathrm{PSh}(\mathrm{Cart})$

$s\mathrm{PSh}(\mathcal{F}\mathrm{Emb}_3)_1$

$$\mathrm{FFT}_{3, \mathbb{B}_\nabla G, \mathbb{B}^3 U(1)} = \mathrm{Fun}^{\otimes}(\mathrm{Bord}^3_{\mathbb{B}_\nabla G},$$

$\mathbb{B}^3 U(1)$) - the prequantum C-S. theory

Foliations:

$T \to B$ $\mapsto$ The set of fiberwise foliations on $T \to B$ (every leaf is contained in one fiber)

Supermanifolds:

$$\begin{array}{c} T \\ \downarrow \\ B \end{array} \mapsto$$ the set of fiberwise supermanifold structure.