Model structures on simplicial presheaves II

Examples & applications

Last time: we had a site $\mathcal{C}$

e.g. Cart_{C^∞} Ob: $\mathbb{R}^n$, $n \geq 0$ Mor: C^∞-maps

$$sPSh(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{op}, sSet)$$

Relative category str-re:

- global (∞-presheaves) W.E are objectwise.
- local (∞-sheaves) w.e. are stalkwise.

(ho) $\underset{\varepsilon > 0}{\mathrm{colim}}\, \mathcal{F}(B^n_\varepsilon)$

For Set it corresponds:

- global $= PSh(\mathcal{C})$ W.E $=$ Isos.
- local $= PSh(\mathcal{C})$ [Stalkwise Isos^{-1}]

Model structures on $sPSh(\mathcal{C})$

a choice of resolution

projective   injective

global: Fib, Fib^t objectwise

global: CoFib objectwise ($CoFib^t$

$CoFib = {}^{\boxslash}Fib^t$

$CoFib^t = {}^{\boxslash}Fib$

local: Fib^t-objectwise

local: CoFib objectwise

Today: give examples for the local projectwise model structure.

General recipe:

Ingridients: $\mathcal{F} : sPSh(\mathcal{C})$

"the moduli stack of a dif. geom str-re"

Ex: a) $n : \mathbb{Z}$ $\Omega^n : sPSh(Cart)$

$[S \mapsto \Omega^n S]^c$

b) G : Lie Grp $\mathbb{B}_\nabla G : sPSh(Cart)$

$\mathbb{B}_\nabla G : S \longmapsto N$ (Grpd of prin. G-bun. w/ ∇ on S and ∇-preserving homs)

c) Riemannian metrics are sPSh but on another site. Many other things too: foliations, conformal str-res etc.

d) A is an Abelian Lie group

(e.g. $\mathcal{A} = U(1)$)

Deligne complex

$$\left(S \mapsto \amalg\,\Gamma\big(\underset{0}{\Omega^n S} \leftarrow \dots \leftarrow \underset{n}{\Omega^0 S} \hookleftarrow \underset{n+1}{\mathbb{Z}}\big)\right)$$

$\Big\|$ $\mathcal{A}$-banded bundle $(n-1)$-gerbes w/ ∇

$\mathbf{B}^n_\nabla \mathcal{A}$

$G \to \mathrm{Aut}(V)$ or $O(V)$

or $Sp(V)$

or something of the sort.

$$\mathbf{B}_\nabla G \to \underbrace{\mathbf{B}_\nabla O(V)}_{}$$

vector bundles w/ metric, ∇, typical fibers V

$M \to \mathbf{B}_\nabla G$: sections = gauge fields

sections = = matter fields

For Fermions take $Spin(V)$-reps.

(then take Λ^*)

Want to compute the space of geo. structs on M

$$\mathbb{R}\mathrm{Map}(M, \mathcal{F}) \in sPSh(\mathcal{C})$$

$$S \mapsto C^\infty(S, M)$$

$$\mathrm{Map}(\mathcal{E}, \mathcal{F})_n = sPSh(\mathcal{C})(\mathcal{E}\otimes\Delta^n, \mathcal{F})$$

$\mathcal{F}$ typically is locally fibrant (on Cart) — locally fibrant = ∞-sheaf + glob. fibrant

$$\mathbb{R}\mathrm{Map}(\mathcal{E},\mathcal{F}) = \mathrm{Map}(Q\mathcal{E}, R\mathcal{F})$$

$Q\mathcal{E}$: cofib. rep. $R\mathcal{F}$: fib. rep.

M : Man $\Rightarrow$ $\underline{M}$: sPSh(C) is cofibrant $\Leftrightarrow$ M : Cart

$$\check{C}(\mathcal{U}, M) = \check{C}\mathcal{U} = QM$$

($\mathcal{U}$: covering of M)

$\mathcal{U}$ is a good open cover of M.

$$(\check{C}\mathcal{U})_n = \coprod_{i_0 \dots i_n} \underline{\mathcal{U}_{i_0} \cap \dots \cap \mathcal{U}_{i_n}}$$

$\mathcal{U}_{i_0} \cap \dots \mathcal{U}_{i_n}$

projectively cofibrant "in each degree = coproduct of reps + split deg. simplices"

Thus $\mathbb{R}\mathrm{Map}(M, \mathcal{F}) = \mathrm{Map}(\check{C}\mathcal{U}, \mathcal{F})$

w/ $\mathcal{F} = \Omega^n$

$$\check{C}\mathcal{U} \longrightarrow \Omega^n$$

(Ω^n in degree 0)

$$\coprod_i \underline{\mathcal{U}}_i \longrightarrow \Omega^n$$

$$\{\underline{\mathcal{U}}_i \to \Omega^n\}_i$$

$$\{\omega_i : \Omega^n(\mathcal{U}_i)\}_i$$

$\forall j,k \quad \omega_j|_{U_j\cap U_k} = \omega_k|_{U_j\cap U_k}$

b) $\mathcal{F} = \mathbb{B}_\nabla G$

$$\check{C}\mathcal{U} \longrightarrow \mathbb{B}_\nabla G = \Omega^1(-;\mathfrak{g}) /\!/ C^\infty(-,G)$$

$$\coprod_i \underline{U_i} \longrightarrow \Omega^1(-;\mathfrak{g})$$

$$\{\omega_i \in \Omega^1(U_i;\mathfrak{g})\}$$

$$\coprod_{j,k} \underline{U_j\cap U_k} \to C^\infty(-;G)$$

$$t_{j,k} \in C^\infty(U_j\cap U_k, G)$$

$$\omega_k|_{U_j\cap U_k} - \omega_j|_{U_j\cap U_k} = dt_{j,k} + t^*_{j,k}\theta$$

MC-form

$$t_{jk}t_{ij} = t_{ik}$$

$$\check{C}\mathcal{U} \to \mathcal{N}(\mathfrak{o}\check{\imath} \otimes \Gamma(\Omega^n \leftarrow \dots \leftarrow \Omega^0 \leftarrow \mathbb{Z}))$$

LieA

normalised chains

$$N\,\mathbb{Z}[\check{C}\mathcal{U}] \to (\Omega^n \leftarrow \dots \leftarrow \Omega^0 \leftarrow \mathbb{Z})$$

simplicial chains

On an sSet

$\omega_i : \Omega^n(U_i)$

$\omega_{jk} : \Omega^{n-1}(U_j \cap U_k)$

$\omega_{lmn} : \Omega^{n-2}(U_l \cap U_m \cap U_n)$

$\omega_{pqrs} : \Omega^{n-3}(U_p \cap U_q \cap U_r \cap U_s)$

$d\omega_{jk} = \omega_k - \omega_j$

$d\omega_{lmn} = \omega_{mn} - \omega_{ln} + \omega_{lm}$

$d\omega_{pqrs} = \omega_{qrs} - \omega_{prs} + \omega_{pqs} - \omega_{pqr}$

$\omega_{a\ldots z} : \Omega^0(U_a \cap \ldots \cap U_z) \quad d\omega_{a\ldots z} = \ldots$

$\omega_{a\ldots z\ddot{o}} : \mathbb{Z}$

the last map is <u>not</u> a diff.

$\omega_{abc\ldots z\ddot{o}} \ominus \omega_{bc\ldots z\ddot{o}} - \omega_{ac\ldots z\ddot{o}} + \ldots \pm \omega_{ab\ldots z}$

<u>Geometric Cobordism Hypothesis</u>

$\mathbb{R}\mathrm{Map}(\mathcal{G}, \partial_d^{\times}) = \mathrm{sPSh}(\mathrm{FEmb}_d)$

($\partial_d^{\times}$ ← sym. mon. cat. w/ duals)

$$= FFT_{d,g,\omega}$$