Model structures on $[\mathcal{C}^{op}; \mathrm{Set}^{\Delta^{op}}]$.

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Goal: understand the right side of the Geometric Cobordism Hypothesis (GCH)

mapping simplicial set.

$$\underbrace{\mathbb{R}}_{\text{right derived}} \widehat{\mathrm{Map}}(\underset{\sim}{S}; \underset{\sim}{\mathcal{C}_d^{\times}})$$

simplicial presheaves on the site Emb_d

Def: $\mathcal{C}$ is a category (or simplicial category)

$$\mathrm{sPSh}(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{sSet})$$

(Fun stands for simplicial f-rs if $\mathcal{C}$ is simplicial)

$\mathrm{sPSh}(\mathcal{C})$ is tensored and powered over sSet. I.e. $S : \mathrm{sSet}_0$, $\mathcal{F} : \mathrm{sPSh}(\mathcal{C})_0$

We have

- $S \otimes \mathcal{F} : \mathrm{sPSh}(\mathcal{C})_0$

given by

$$S \otimes \mathcal{F}(T) :\equiv S \times \mathcal{F}(T)$$

$$\triangleright \quad \mathcal{F}^S(T) :\equiv \underbrace{\mathcal{F}(T)^S}_{\text{internal hom in Set}}$$

Construction of a model str-re on $sPSh(\mathcal{T})$

1) "Global" model str-re on $sPSh(\mathcal{T})$

w.e. are objectwise w.e.:

$$\mathcal{F}, \mathcal{G} : sPSh(\mathcal{T}) \qquad \mathcal{F} \xrightarrow{f} \mathcal{G}$$

$$f : W\mathcal{E} \iff f_T : W\mathcal{E}_{sSet} \ \forall T : \mathcal{T}_0$$

$$(f_T : \mathcal{F}(T) \xrightarrow{\sim} \mathcal{G}(T))$$

Projective model structure:

$Fib^{(t.)}$ are defined objectwise

$$\left(\begin{array}{l} \text{and then } CoFib = Proj(Fib^{t.}) \\ \qquad\qquad\quad CoFib^{t.} = Proj(Fib) \end{array} \right)$$

"practical computations"

Injective m.s.

$CoFib^{(t.)}$ are objectwise

CoFib(t.) "theoretical results"

Existence: Smith recognition theorem for left proper combin. m.s. (nlab ⇝ Lurie HTT)

Local model structure on $sPSh(\mathcal{C})$

$\mathcal{C}$ is a site

Weak equivalences are.....

.... 1) stalkwise $\mathcal{F} \xrightarrow{f} G$: $WE_{sPSh(\mathcal{C})}$

if for any p (point) of $\mathcal{C}$

$$Sh(*) = Set \underset{p_*}{\overset{p^*}{\rightleftarrows}} Sh(\mathcal{C})$$

point = geometric morphism

An aside: $\mathcal{C} = Open(X)$, X: Top.

$p : X$ (a point) then $p^* : Sh(X) \to Set$

p_* is the skyscrapper sheaf functor.

$\mathcal{F} \mapsto \mathcal{F}_p$

$\operatorname{colim}_{p \in U} \mathcal{F}(U)$

filtered

colimit ⇒ it commutes w/ finite limits

Ex 2: $\mathcal{C}$ = Cart

$\mathcal{C}$ = Man

$$Sh(Cart) \underset{p_*}{\overset{p^*}{\rightleftarrows}} Set$$

$$p^*\mathcal{F} = \operatorname{colim}_{0 : U \subseteq_{op} \mathbb{R}^n} \mathcal{F}(U)$$

n-dimensional stalk.

$p^* f : p^* \mathcal{F} \to p^* G \quad : WE_{sSet}$.

.... 2) f induces an iso on sheaves of $\pi_\bullet$, $\bullet \geq 0$.

$$\begin{tikzcd} sPSh(\mathcal{E}) \arrow[r, shift left, "\pi_0"] & Sh(\mathcal{E}) \arrow[l, shift left, "\substack{\text{constant} \\ \text{object} \,\circ}"] \end{tikzcd}$$

forgetful f-r

$Sh(\mathcal{E}) \to PSh(\mathcal{E})$

$$\begin{tikzcd} {[X \mapsto \pi_0(\mathcal{F}(X))] : PSh(X)} \arrow[d, squiggly, "a \text{ (assoc. sheaf f-r)}"] \\ (\pi_0 \mathcal{F}) \quad : Sh(X) \end{tikzcd}$$

π_n is defined similarly....

$f : \mathcal{F} \to G \quad : WE_{sPSh(\mathcal{E})}$ if

- $\pi_0 f$ is an iso in $Sh(\mathcal{E})$
- $\pi_{\bullet > 0}(f, b)$ is an iso $\forall b$.

.... 3) Define an ∞-sheaf (∞-stack / homotopy coherent sheaf)

a) $\mathcal{F} : sPSh(\mathcal{E})$ is an ∞-sheaf if $\mathcal{F}$ is projectively (objectwise) fibrant <u>AND</u> $\forall X : \mathcal{C}_0$

$\forall \{ f_i : U_i \to X \}$ (covering families)

$$\mathcal{F}(X) = \underbrace{\mathcal{M}ap}_{\text{sSet (mapping sSet)}}(\underline{X}^{\text{representable}} ; \mathcal{F}) \xrightarrow{\text{sPSh}(\mathcal{C})}$$

$$\xrightarrow{\simeq} \mathcal{M}ap(\check{C}U ; \mathcal{F})$$

Aside: For $Sh(\mathcal{C})$ we have

$$\underbrace{Map}_{\text{Set}}(h_X ; \mathcal{F}) \xrightarrow{\cong} Map(\underbrace{\check{C}U}_{\text{sive generated by } U} ; \mathcal{F})$$

is the set "of matching families".

(b) $\mathcal{F} \xrightarrow{f} \mathcal{G}$ is a local (Čech) WE if $\forall$ injectively fibrant ∞-sheaf $\mathcal{H}$ the map

$$\mathcal{M}ap(\mathcal{G}, \mathcal{H}) \xrightarrow[\text{WE}_{\text{sSet}}]{\simeq} \mathcal{M}ap(\mathcal{F}; \mathcal{H})$$

We have (1) = (2), but in general (2) $\neq$ (3). However (2) = (3) if $\mathcal{C}$ has enough points: $\mathcal{F} \xrightarrow[f]{} \mathcal{G}$, f is an iso $\Leftrightarrow$ $p^* f$ is an iso $\forall p$ point of $\mathcal{C}$.

Ex: Cart has enough points.

"Hypercovers & sPSh" (Dugger)

"Sheaves & homotopy theory" (D.)

The local projective m.s.

WE = local WE

CoFib = Proj (Fib$^{t.}$)

Fib$^{t.}$ are objectwise Fib$^{t.}$

The local inj. m.s.

WE = local WE

CoFib = injective cof = mono.

Fib$^{t.}$ = injective Fib$^{t.}$

Existence: construct the global m.s. & invoke Smith's theorem on existence of left Bousfield localization w.r.t. $\check{C}\mathcal{U} \to \underline{X}$ for $\{\mathcal{U}_i \xrightarrow[f_i]{} X\}$ - cov. families.

In Sh it is an inclusion of a sieve into a rep. PSh.

Invoke Smith's theorem for LPCMC (left proper comb. model categories)