Differential cohomology.

$$\begin{tikzcd}
 & H^{k-1}(M;\mathbb{R}/\mathbb{Z}) \arrow[r, "\text{Bock}"'] & H^k(M;\mathbb{Z}) \arrow[dr] & \\
H^{k-1}_{dR}(M) \arrow[ur] \arrow[dr] & & & H^k_{dR}(M) \\
 & \Omega^{k-1}(M)/\mathrm{Im}\, d \arrow[r, "d"] & \Omega^k_{cl}(M) \arrow[ur] &
\end{tikzcd}$$

$$\mathbb{Z} \hookrightarrow \mathbb{R} \to \mathbb{R}/\mathbb{Z}$$

$$H^{k-1}_{dR}(M) \simeq H^{k-1}(M;\mathbb{R})$$

$$\begin{tikzcd}
 & 0 \arrow[dr] & & & 0 & \\
 & & H^{k-1}(M;\mathbb{R}/\mathbb{Z}) \arrow[r] \arrow[dr] & H^k(M;\mathbb{Z}) \arrow[ur] \arrow[dr] & & \\
H^{k-1}_{dR}(M) \arrow[urr] \arrow[drr] & & & \hat{H}^k(M;\mathbb{Z}) \arrow[u, "\text{Bock}"'] \arrow[dr] & H^k_{dR}(M) & \\
 & & \Omega^{k-1}(M)/\mathrm{Im}\, d \arrow[ur] \arrow[rr, "d"] & & \Omega^k_{cl}(M) \arrow[u] \arrow[dr] & \\
 & 0 \arrow[ur] & & & & 0
\end{tikzcd}$$

$$C^\infty(M;\mathbb{R}/\mathbb{Z}) \overset{\simeq}{\hookrightarrow} \mathrm{Map}(M;\mathbb{R}/\mathbb{Z})$$

$$\pi_*(\mathbb{R}/\mathbb{Z}) = 0 \quad * > 1$$

$C^\infty(M;\mathbb{R}/\mathbb{Z})$ is also

1-truncated.

$\mathbb{R}/\mathbb{Z} = K(\mathbb{Z};1) \Rightarrow$

$$\pi_0 C^\infty(M;\mathbb{R}/\mathbb{Z}) = H^1(M;\mathbb{Z})$$

$$C^\infty(M;\mathbb{R}/\mathbb{Z}) \twoheadrightarrow H^1(M;\mathbb{Z})$$

$$\pi_1 C^\infty(M;\mathbb{R}/\mathbb{Z}) \cong \pi_0 \mathrm{Map}_*(S^1, C^\infty(M\ldots))$$

$$\cong \pi_0 \mathrm{Map}_*(S^1, \mathrm{Map}(M;\mathbb{R}/\mathbb{Z})) \cong$$

$$\cong \pi_0 \mathrm{Map}(M, \underbrace{\mathrm{Map}_*(S^1;\mathbb{R}/\mathbb{Z})}_{\substack{= \\ \Omega(\mathbb{R}/\mathbb{Z}) \\ = \\ K(\mathbb{Z};0)}}) \cong$$

$$\cong H^0(M;\mathbb{Z})$$

Construction: vol denotes the standard volume form on $S^1 \cong \mathbb{R}/\mathbb{Z}$. $\mathrm{curv}: C^\infty(M;\mathbb{R}/\mathbb{Z}) \to \Omega^1_{cl}(M)$

$$\mathrm{curv}(f) = f^*(\mathrm{vol})$$

$\ker(\mathrm{curv}) = \{f: M \to \mathbb{R}/\mathbb{Z} \mid f \text{ loc. const}\}$

$= H^0(M; \mathbb{R}/\mathbb{Z})$

$\operatorname{im}(\mathrm{curv}) = \{ \alpha \in \Omega^1_{cl}(M) \mid \int_{S^1} \alpha \in \mathbb{Z} \quad S^1 \hookrightarrow M \}$

subgroup of forms with "integral periods".

Def: Let M: Mfld, $k \in \mathbb{Z}_{\geq 0}$

$\omega : \Omega^k_{cl}(M)$ has integral periods if $\forall\, c \in Z^{sm}_k(M)$

$$\int_c \omega \in \mathbb{Z}$$

$$\Omega^k_{cl}(M)_{\mathbb{Z}} < \Omega^k_{cl}(M)$$

Remark: $\omega \in \Omega^k_{cl}(M)_{\mathbb{Z}} \iff$

$[\omega] \in H^k(M;\mathbb{R})$ is in the image of $H^k(M;\mathbb{Z}) \to H^k(M;\mathbb{R})$

$i: \Omega^0(M) = C^\infty(M;\mathbb{R}) \to C^\infty(M;\mathbb{R}/\mathbb{Z})$ given by $\mathbb{R} \twoheadrightarrow \mathbb{R}/\mathbb{Z}$.

$\ker i = \{ f \in C^\infty(M;\mathbb{R}) \mid$ locally constant and have integral values $\}$

$\ker i = \Omega^0_{cl}(M)_{\mathbb{Z}}$

$$\begin{tikzcd}
0 \arrow[dr, dashed] & & & & 0 \\
& H^0(M;\mathbb{R}/\mathbb{Z}) \arrow[rr, "\mathrm{Bock}"] \arrow[dr] & & H^1(M;\mathbb{Z}) \arrow[ur, dashed] \arrow[dr] & \\
H^0_{dR}(M) \arrow[ur] \arrow[dr] & & C^\infty(M;\mathbb{R}/\mathbb{Z}) \arrow[ur, "\pi_0"] \arrow[dr, "\mathrm{curv}"] & & H^1_{dR}(M) \\
& \Omega^0(M)/\Omega^0_{cl}(M)_{\mathbb{Z}} \arrow[ur, "i"] \arrow[rr, "d"] & & \Omega^1_{cl}(M)_{\mathbb{Z}} \arrow[ur] \arrow[dr, dashed] & \\
0 \arrow[ur, dashed] & & & & 0
\end{tikzcd}$$

Moral: $\Omega^{k-1}(M)/d\Omega^{k-2}(M)$ should be replaced by $\Omega^{k-1}(M)/\Omega^{k-1}_{cl}(M)_{\mathbb{Z}}$

Differential characters

Notation: $C_i^{sm}(M;\mathbb{Z})$ group of smooth singular chains.

$Z_i^{sm}(M;\mathbb{Z})$ – the group of smooth singular cycles

Def: Let $k \in \mathbb{Z}_{\geq 1}$, M: Mfld

differential character of degree k on M is a homomorphism

$\chi : Z_{k-1}^{sm}(M;\mathbb{Z}) \to \mathbb{R}/\mathbb{Z}$ s.t.

$\exists\, \omega(\chi) : \Omega^k(M)$ s.t. $\forall c : C_k^{sm}(M;\mathbb{Z})$

$$\chi(\partial c) = \int_c \omega(\chi) \mod \mathbb{Z}$$

$$\hat{H}^k(M;\mathbb{Z}) \subset \mathrm{Hom}_{\mathbb{Z}}(Z_{k-1}^{sm}(M;\mathbb{Z}); \mathbb{R}/\mathbb{Z})$$

$\omega(\chi)$ is uniquely determined by χ

$\omega(\chi)$ is called curvature of χ

$\mathrm{curv}: \hat{H}^k(M;\mathbb{Z}) \twoheadrightarrow \Omega^k_{cl}(M)_{\mathbb{Z}}$

$$\chi \longmapsto \omega(\chi)$$

$k=0 \quad \hat{H}^0(M;\mathbb{Z}) = H^0(M;\mathbb{Z})$

Construction: We have a map

$cc: \hat{H}^k(M;\mathbb{Z}) \longrightarrow H^k(M;\mathbb{Z})$

$Z^{sm}_{k-1}(M;\mathbb{Z})$ is a free abelian group $\Rightarrow \chi: Z^{sm}_{k-1}(M;\mathbb{Z}) \to \mathbb{R}/\mathbb{Z}$

$\bar{\chi}: Z^{sm}_{k-1}(M;\mathbb{Z}) \longrightarrow \mathbb{R}$

$I(\bar{\chi}): C^{sm}_k(M;\mathbb{Z}) \to \mathbb{Z}$

$c \mapsto -\bar{\chi}(\partial c) + \int_c \mathrm{curv}(\chi)$

$\mathrm{curv}(\chi): \Omega^k_{cl}(M)_{\mathbb{Z}} \Rightarrow$

$I(\bar{\chi})$ - cocycle, $I(\bar{\chi}) \in$

$\in C^k_{sm}(M;\mathbb{Z})$

$[I(\bar{x})] \in H^k(M;\mathbb{Z})$ is independent of the choice of $\bar{x}$.

$$cc : \hat{H}^k(M;\mathbb{Z}) \longrightarrow H^k(M;\mathbb{Z})$$
$$\check{x} \longmapsto [I(\bar{x})]$$

Construction:

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(H_{i-1}(M;\mathbb{Z});\mathbb{R}/\mathbb{Z}) \to H^i(M;\mathbb{R}/\mathbb{Z}) \xrightarrow{\langle -,-\rangle} \mathrm{Hom}_{\mathbb{Z}}(H_i(M;\mathbb{Z});\mathbb{R}/\mathbb{Z}) \to 0$$

$\langle -,-\rangle$ u - cocycle

$$\langle u,-\rangle : H_i(M;\mathbb{Z}) \to \mathbb{R}/\mathbb{Z}$$
$$[z] \longmapsto u(z)$$

$\mathbb{R}/\mathbb{Z}$ injective $\Rightarrow$ $\mathrm{Ext}^j_{\mathbb{Z}}(A;\mathbb{R}/\mathbb{Z}) = 0$ $\forall j>0$ $\forall A$ $\Rightarrow$ $\langle -,-\rangle$ is an iso.

Let $i = k-1$ then the composition

with $Z^{sm}_{k-1}(M;\mathbb{Z}) \twoheadrightarrow H_{k-1}(M;\mathbb{Z})$

$$H^{k-1}(M;\mathbb{R}/\mathbb{Z}) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Z}}(H_{k-1}(M;\mathbb{Z}); \mathbb{R}/\mathbb{Z}) \hookrightarrow$$

$$\hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(Z^{sm}_{k-1}(M;\mathbb{Z}); \mathbb{R}/\mathbb{Z})$$

$$\begin{tikzcd} H^{k-1}(M;\mathbb{R}/\mathbb{Z}) \arrow[r, "\langle -,- \rangle"] & \mathrm{Hom}_{\mathbb{Z}}(Z^{sm}_{k-1}(M;\mathbb{Z}); \mathbb{R}/\mathbb{Z}) \\ & \hat{H}^{k}(M;\mathbb{Z}) \arrow[u, hook] \end{tikzcd}$$

<u>Construction:</u> $\iota : \Omega^{k-1}(M) \to \hat{H}^{k}(M;\mathbb{Z})$

$$\iota(\omega)(z) := \exp\left(2\pi i \int_z \omega\right)$$

$\forall z \in Z^{sm}_{k-1}(M)$

$$\mathrm{curv}(\iota(\omega)) = d\omega$$

$$\bar{\iota}(\omega)(z) := \int_z \omega \quad \forall z \in Z^{sm}_{k-1}(M;\mathbb{Z})$$

$$I(\bar{\iota}(\omega))(c) = -\bar{\iota}(\omega)(\partial c) +$$

$$+ \int_c \mathrm{curv}(\iota(\omega)) = -\int_{\partial c} \omega + \int_c d\omega = 0$$

$\forall c \in C^{sm}_{k}(M) \Rightarrow c c \circ \iota = 0$

ker $\imath$ consists of closed forms s.t. $\int_z \omega \in \mathbb{Z}$ $\forall z \in Z^{sm}_{k-1}(M;\mathbb{Z})$

$\Rightarrow$ ker$(\imath) = \Omega^{k-1}_{cl}(M)_{\mathbb{Z}}$ $\Rightarrow$

$\imath$ descends to a monomorphism

$$\imath : \Omega^{k-1}(M) \big/ \Omega^{k-1}_{cl}(M)_{\mathbb{Z}} \rightarrowtail \hat{H}^k(M;\mathbb{Z})$$

Dif. coh. $\Diamond$

$\mathcal{M}fld$ - category of manifolds

GrAb - category of graded ab groups

Theorem: There is an essentially unique functor

$$\hat{H}^*(-;\mathbb{Z}) : \mathcal{M}fld^{op} \longrightarrow GrAb$$

with natural transf:

- $\langle -,- \rangle : H^{k-1}(-;\mathbb{R}/\mathbb{Z}) \Rightarrow \hat{H}^k(-;\mathbb{Z})$
- $\imath : \Omega^{*-1}(-) \big/ \Omega^{*-1}_{cl}(-)_{\mathbb{Z}} \Rightarrow \hat{H}^*(-;\mathbb{Z})$
- $cc : \hat{H}^*(-;\mathbb{Z}) \Rightarrow H^*(-;\mathbb{Z})$
- curv $: \hat{H}^*(-;\mathbb{Z}) \Rightarrow \Omega^*_{cl}(-)_{\mathbb{Z}}$

s.t. the hexagon has exact diagonals

$$\begin{tikzcd}
 & H^{k-1}(M;\mathbb{R}/\mathbb{Z}) \arrow[rr, "\text{Bock}"] \arrow[dr, "\langle -,- \rangle"] & & H^k(M;\mathbb{Z}) \arrow[dr] & \\
H^{k-1}_{dR}(M) \arrow[ur] \arrow[dr] & & \hat{H}^k(M;\mathbb{Z}) \arrow[ur, "cc"] \arrow[dr, "\text{curv}"] & & H^k_{dR}(M) \\
 & \Omega^{k-1}(M)/\Omega^{k-1}_{cl}(M)_{\mathbb{Z}} \arrow[ur, "\iota"] \arrow[rr, "d"] & & \Omega^k_{cl}(M)_{\mathbb{Z}} \arrow[ur] &
\end{tikzcd}$$

Any functor with the properties above is called ordinary dif. cohomology

Remark: Let $\mathbb{Z}(k)$ be a complex of sheaves on M of the form

$$0 \to \mathbb{Z} \hookrightarrow \Omega^0 \xrightarrow{d} \Omega^1 \to \dots \to \Omega^{k-1} \to 0$$

$$H^k(M;\mathbb{Z}(k)) \cong \hat{H}^k(M;\mathbb{Z})$$