A short story about spans:

Spans

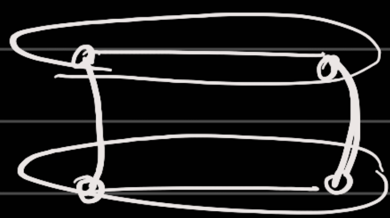
Cospans

1-fold
1-uple

$$\begin{tikzcd} & -\infty \arrow[dl] \arrow[dr] & \\ A_1 & & B_1 \end{tikzcd} \qquad \begin{tikzcd} A_1 \arrow[dr] & & B_1 \arrow[dl] \\ & +\infty & \end{tikzcd}$$

$$\begin{tikzcd} & C^\infty(\bullet\!-\!\!-\!\bullet) \arrow[dl] \arrow[dr, two heads] & \\ C^\infty(\cdot) & & C^\infty(\cdot) \end{tikzcd}$$

2-fold
2-uple



2-fold
cospan

$C^\infty($  $)$

$$\begin{tikzcd} A_1 \arrow[d] \arrow[dr] & & B_1 \arrow[dl] \arrow[d] \\ A_2 \arrow[dr] & & B_2 \arrow[dl] \\ & +\infty & \end{tikzcd}$$

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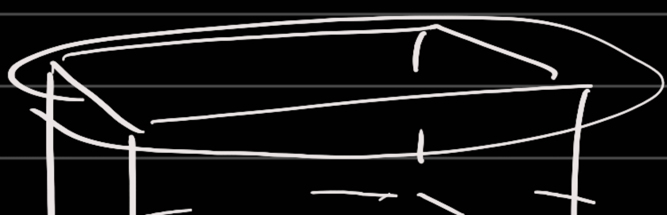
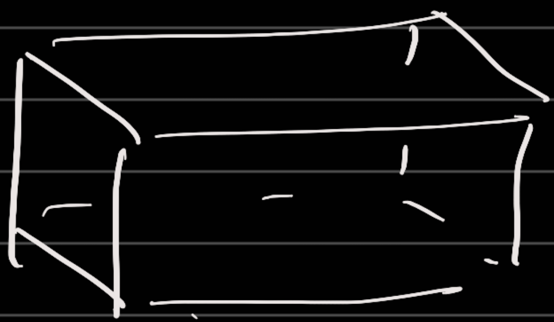


n-fold span

n-uple span $\qquad$ n-uple cospan



$(Sp')^{\times n}$ - n-uple spans

Basic Idea: $\mathrm{Lag}_{S,s}$ - ∞ category

$\mathrm{Or}_{S,s}$ - ∞-category

Higher Lagrange correspondences.

Proposition: $p : K^{\triangleleft} \to \mathrm{PreSymp}_{S,s}$

$-\infty \longmapsto X : dSt^{Art}_{S}$

$\tilde{p} : (K^{\triangleleft})^{op} \to \mathrm{Tw}^r(\mathrm{QCoh}(X))$

If $X_k = p(k)$ and $f_k : X \to X_k$

$p: -\infty \to K$) then $f_K^* \mathbb{T}_{X/S} \to f_K^* \mathbb{L}_{X_u/S}[u]$

induced by 2-form.

Proof: There is a functor $\mathrm{PreSymp}_{S,s}^{op} \to \mathcal{Q}^{\Delta^1}$

$(X,\omega) \rightsquigarrow \mathcal{O}_X \to \otimes^2 \mathbb{L}_{X/S}[s]$

$(K^{\triangleleft})^{op} \to \mathcal{Q}^{\Delta^1}$

$(K^{\triangleleft})^{op} \to (\mathrm{QCoh}(X))^{\Delta^1}$

$k : K \quad k \rightsquigarrow \mathcal{O}_X \simeq f_K^* \mathcal{O}_{X_K} \to$

$\to f_K^* (\otimes^2 \mathbb{L}_{X_K/S}[s]) \simeq \otimes^2 \mathbb{L}_{X/S}[s]$

$(K^{\triangleleft})^{op} \to \mathrm{Tw}^r(\mathrm{QCoh}(X))$

$k \rightsquigarrow f_K^* \mathbb{T}_{X_K/S} \to f^* \mathbb{L}_{X/S}$ $\square$

Corollary: Let $p: K^{\triangleleft} \to \mathrm{PreSymp}_{S,s}$

$-\infty \mapsto X$

$\hat{p} : \mathrm{Tw}^r_!(K^{\triangleleft})^{op} \to \mathrm{QCoh}(X)$

A fibrational

naive correspondence

for ∞-cats

Rune Haugseng

$\boxed{\mathrm{Tw}^r_! \dashv \mathrm{Tw}^r}$

Notation:

- Sp^n – n-uple spans
- sp^n – n-fold spans
- $\widetilde{Sp}^n := \mathrm{Tw}_!^{\sim}(Sp^n)$
- $\widetilde{sp}^n := \mathrm{Tw}_!^{\sim}(sp^n)$

$\mathcal{C}$-span

Corollary: $Sp^n \to \mathcal{C}$

(i) n-uple span in $\mathrm{PreSymp}_{s,s}$ with apex X (value on $-\infty$) it induces a diagram $\widetilde{Sp}^n \to \mathrm{QCoh}\,X$

(ii) n-fold span is the same thing, but for sp^n $\widetilde{sp}^n \to \mathrm{QCoh}(X)$

Definition: Let $\mathcal{C}$ be an ∞-category then $\widetilde{Sp}^n \to \mathcal{C}$ (or $\widetilde{sp}^n \to \mathcal{C}$) is non-degenerate if it is a limit diagram.

Definition: n-fold Lagrange correspon-

- dance is a n-fold span

$Sp^n \longrightarrow \mathrm{PreSymp}_{S,S}$ with apex X s.t.

$\widetilde{Sp}^n \longrightarrow QCoh(X)$ $\Rightarrow$ non-degenerate.

Proposition:

(i) TFAE $\Phi : \widetilde{Sp}^n \to \mathcal{C}$

(1) Φ is non-degenerate

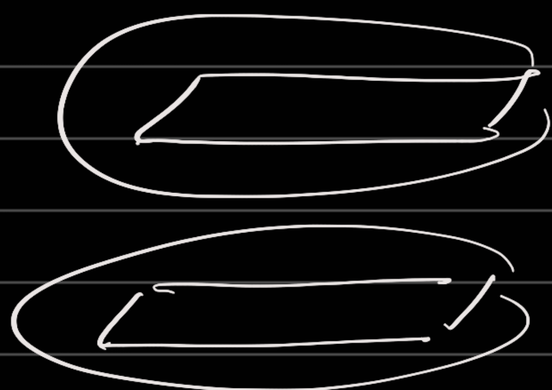
(2) $$\underbrace{Sp^{n,D} \longrightarrow \widetilde{Sp}^n \xrightarrow{\Phi} \mathcal{C}}_{\text{limit diagram}}$$

(ii) TFAE $\Phi : \widetilde{sp}^n \to \mathcal{C}$

(1)
(2) } are the same, but for $\widetilde{sp}^n$

(3) $\widetilde{Sp}^n \xrightarrow{Tw_!^2(j_n)} \widetilde{Sp}^n \xrightarrow{\Phi} \mathcal{C}$ - non-deg.

$j_n : Sp^n \to sp^n$



$Sp^n \quad Sp[D^{-1}]$

$Sp = Sp[K_n]$

Lemma: $\mathcal{C} \to \mathcal{D}$ is coinitial iff $\mathcal{C}^{\triangleright} \to \mathcal{D}^{\triangleright}$ is coinitial.

Proof: a functor is coinitial $\Leftrightarrow$

$(\mathcal{C}^{\triangleright})_{/d} \simeq \mathcal{C}_{/d}$ contractible (weakly)

$\forall d \in \mathcal{D}$ if $d = +\infty$ $\mathcal{C}^{\triangleright}_{/\infty} = \underbrace{\mathcal{C}^{\triangleright}}_{\text{contractible}}$

$\square$

Proof (of the proposition)

(1) $Sp^{n,\circ,\triangleright} \longrightarrow \widetilde{Sp}^{n,\circ,\triangleright}$ is coinitial

$Sp^{n,\circ} - \{-\infty\}$

follows from coinitiality of $Tw_!^{\tau}$ + the previous lemma $\square$

Corollary: $\Phi : Sp^n \to PreSymp_{S,s}$ —

n-fold Lagrange $\Leftrightarrow$ $\Phi \circ j_n : Sp^n \to PreSymp_{S,s}$

Proof: $\widetilde{Sp}^n \xrightarrow{Tw_!^{\tau}(j_n)} \widetilde{Sp}^n$

$$\begin{tikzcd} \Phi_j \arrow[dr] & & \hat{\Phi} \arrow[dl] \\ & QCoh(X) & \end{tikzcd}$$

By proposition everything follows

$\square$

Lemma: TFAE

$\Phi : Sp^u \longrightarrow PreSymp_{S,s}$

(i) Φ is Lagrange

(ii) $\forall\ Spec\, A \xrightarrow[p]{} S \quad p^*\Phi$ is Lagrangian

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<u>Low-dimensional example:</u>

$$\begin{tikzcd} & L \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{T}_{L/S} \arrow[r, "g_*"] \arrow[d, "f_*"'] \arrow[dr, phantom, "\ulcorner", very near start] & g^*\mathbb{T}_{Y/S} \arrow[d] \\ f^*\mathbb{T}_{X/S} \arrow[r] & \mathbb{L}_{L/S}[-s] \end{tikzcd}$$

$\Gamma \quad \mathcal{X}_S \qquad\qquad \mathcal{L}_S$

Lagrangian correspondence

Higher oriented cospans:

Proposition: $p : \mathcal{Y}^D \longrightarrow \mathrm{PreOr}_{S,d}$

$X := p(+\infty)$, $X_i := p(i)$ $\quad f_i : X_i \longrightarrow X$

$\mathcal{E} : QCoh(X)$

$$p_{\mathcal{E}} : (\mathcal{Y}^D)^{op} \longrightarrow QCoh(S)_{/\mathcal{O}_S[-d]}$$

$i : \mathcal{Y}$

$$\Gamma_S f_i^* \mathcal{E} \otimes \Gamma_S f_i^*(\mathcal{E}^\vee) \longrightarrow$$

$$\longrightarrow \Gamma_S f_i^*(\mathcal{E} \otimes \mathcal{E}^\vee) \xrightarrow[\Gamma_S f_i^* ev]{} \Gamma_S f_i^* \mathcal{O}_{X_i}[-d] \longrightarrow$$

$$\simeq \Gamma_S \mathcal{O}_{X_i} \xrightarrow[|X_i|]{} \mathcal{O}_S[-d]$$

$\delta: S' \to S$ we have a natural equivalence $\delta^* p_{\mathcal{E}} \simeq (\delta^* p)_{\underset{\sim}{\delta^* \mathcal{E}}}$

Proof: $\mathcal{E} \otimes \mathcal{E}^\vee \to \mathcal{O}_X$ it is a morphism $(\mathcal{E}, \mathcal{E}^\vee) \xrightarrow{\mathcal{E}} \mathcal{O}_X$

$(QCoh(X), \otimes) \longleftrightarrow \langle 2 \rangle \to \langle 1 \rangle$
inert maps
active maps

$\mathcal{Q}^{\otimes}_S \longrightarrow dSt^{op}_S \times \mathbb{F}_*$
finite sets

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encodes monoidal structure on $\mathcal{Q}_S$

$$\begin{tikzcd}
(\mathcal{J}^{\triangleright})^{op} \times \Delta^1 \arrow[r] \arrow[d] & \mathcal{Q}^{\otimes}_S \arrow[d] \\
PreO_2{}^{op}_{S,d} \times \Delta^1 \arrow[r, "\sim"] & dSt^{op}_S \times \mathbb{F}_*
\end{tikzcd}$$

$i: \mathcal{J}$ $\quad (f_i^* \mathcal{E}, f_i^* \mathcal{E}^\vee) \xrightarrow[f_i^* \mathcal{E}]{} f_i^* \mathcal{O}_X \simeq \mathcal{O}_{X_i}$

$(\mathcal{J}^{\triangleright})^{op} \times \Delta^1 \longrightarrow QCoh^{\otimes}$

$$\mathrm{PreOr}_{S,d}^{op} \times \Delta^1 \longrightarrow \mathrm{Fin}_*$$

$$\mathrm{QCoh}(S)^{\otimes} \to \mathrm{Fin}_*$$ – cocartesian fibration

$\langle 2 \rangle \to \langle 1 \rangle$

$$(\mathcal{J}^{\triangleright})^{op} \times \Delta^1 \longrightarrow \mathrm{QCoh}(S)$$

$i : \mathcal{J}$

$$\Gamma_S f_i^* \mathcal{E} \otimes \Gamma_S f_i^* \mathcal{E}^\vee \to \Gamma_S(f_i^* \mathcal{E} \otimes f_i^* \mathcal{E}^\vee)$$

$$\longrightarrow \Gamma_S f_i^* \mathcal{O}_X \simeq \Gamma_S \mathcal{O}_X$$

$\Gamma_S f_i^* \mathcal{E}$

$$\mathrm{PreOr}_{S,d}^{op} \to \mathrm{QCoh}(S)/\mathcal{O}_S[-d]$$

$$(\mathcal{J}^{\triangleright})^{op} \times \Delta^1 \longrightarrow \mathrm{QCoh}(S)$$

$$(\mathcal{J}^{\triangleright})^{op} \times \Delta^2 \longrightarrow \mathrm{QCoh}(S)$$

$\Delta^{\{0,2\}}$ we get ...

we get Φ_Σ.

$\square$

Corollary: n-uple (n-fold) cospan Φ in $\mathrm{PreOr}_{S,d}$ with bottom vertex X induces $\forall \Sigma : \mathrm{QCoh}(X)$ ad-m

$$\Phi_\Sigma : \widetilde{Sp}^n \to \mathrm{QCoh}(S)$$

Definition: Φ - cospan $\forall \Sigma : \mathrm{QCoh}(X)$ dualizable

Φ_Σ non-degenerate.

Definition: $\forall \sigma : \mathrm{Spec}\, A \to S$

$\sigma^* \Phi$ is weakly oriented $\Rightarrow$ Φ is oriented

Example of equiv. condition of non-degeneracy

(1) $\Phi : \widetilde{Sp}^n \to \mathcal{C}$ has non-degenerate boundary if $\Phi|_{\widetilde{Sp}^{n,0,\partial}}$ is a right Kan extension of $\Phi|_{Sp^{n,op}}$

② Let Φ be a d-n with non-deg. Boundary then TFAE:

① Φ is non-degenerate

② Φ is a right Kan extension of $Sp^{n,op} \hookrightarrow \widetilde{Sp}^n$

③ $\Phi(-\infty)$ is the limit of Φ restricted to $Sp^{n,op}$.

Note: condition ① is equivalent to saying that

$\forall S \subsetneq \{0 \dots n\}$ $\qquad \underline{\widetilde{Sp}^{|S|} \hookrightarrow \widetilde{Sp}^n \to \mathcal{C}}$

$\qquad\qquad\qquad\qquad\qquad$ <u>non-degenerate</u>

Higher categories of Symplectic & Lagrangian Derived stacks.

Lag_n^{SS} $\qquad\qquad\qquad\qquad$ Oz_n^{Scl}

i-morphisms:

Lagrangian i-spaces　　　　Oriented i-cospans