2-shifted symplectic structure on Perf_A

$\exists\, A : DGA^{\leq 0}$

$\mathbb{R}\mathrm{Perf} : A \to \mathrm{Perf}_A$ homologically finitely presented (dualizable objects in DGM_A)

$\mathbb{R}\mathrm{Perf} = \bigcup$ Artin stacks classifying stack

$$BGL = \bigcup_{n=1}^{\infty} BGL(n)$$

$\mathcal{E}$ is a universal perfect complex

if $\dim \mathcal{E} = n$

$\mathcal{A} := \mathbb{R}\mathrm{Hom}(\mathcal{E},\mathcal{E}) = \mathcal{E}^\vee \otimes \mathcal{E}$

$\mathbb{R}\mathrm{Hom}(\mathcal{E},\mathcal{E}) = \mathrm{Mat}_n(\mathcal{A})$

$\mathcal{L}\,\mathbb{R}\mathrm{Perf} := \mathrm{Map}(S^1, \mathbb{R}\mathrm{Perf})$

S^1 based loops

$\mathcal{A}^\times = GL_1(\mathcal{A})$

$\mathbb{T}_{\mathbb{R}\mathrm{Perf}} \simeq \mathcal{A}[1]$ has a structure of dg Lie algebra

$$\mathbb{T}_{\mathbb{R}\mathrm{Perf}} \wedge \mathbb{T}_{\mathbb{R}\mathrm{Perf}} \simeq \mathrm{Sym}^2(\mathcal{A})[2] \xrightarrow{\omega} \mathcal{O}_{\mathbb{R}\mathrm{Perf}}[2]$$

$$\mathcal{A} \otimes \mathcal{A} \xrightarrow[\mu]{} \mathcal{A} \xrightarrow[T_r]{} \mathcal{O}_{\mathbb{R}\mathrm{Perf}}$$

$$T_2 : \mathcal{E}^\vee \otimes \mathcal{E} \longrightarrow \mathcal{O}_{\mathbb{R}\mathrm{Perf}}$$

$$\mathbb{T}_{\mathbb{R}\mathrm{Perf}} \xrightarrow{\tilde{\omega}} \mathbb{L}_{\mathbb{R}\mathrm{Perf}} = \mathbb{T}^\vee_{\mathbb{R}\mathrm{Perf}}$$

$\mathrm{Tr}(- \cdot -) : \mathrm{Mat}_n(A)^{\otimes 2} \to A$

Oriented & preoriented stacks

Notation:

① $f : X \to Y$ in dSt_S **cocontinuous** if $f_* : QCoh(X) \to QCoh(Y)$

f_* **preserves colimits**. universally cocontinuous if

for every $g : S' \to S$ corresponding $g^* f$ is

cocontinuous

dSt_S^{ucc} , $\mathrm{Fun}(\Delta^1, dSt)^{ucc}$

$$\begin{array}{ccc} X & \xrightarrow{f} & * \\ QCoh(X) & \xrightarrow[\Gamma]{} & QCoh(S) \end{array}$$

Definition: d-preorientation

on $X : dSt_S$ is a morphism $[X] : \underbrace{\Gamma_S \mathcal{O}_X \to \mathcal{O}_S[-d]}_{QCoh(S)}$

$(X, [X])$ d-preoriented stack over S if

$X \to S$ is a UCC-morphism and $[X]$ is

a d-preorientation of X.

Definition: Let $(X, [X])$ · d preoriented stack $/S$

$\mathcal{E} : QCoh(X)$ $\quad \mathcal{E} \otimes \mathcal{E}^\vee \longrightarrow \mathcal{O}_X$

$\Sigma^\vee = \underline{\mathrm{hom}}(\Sigma, \mathcal{O}_X)$ Γ_S is lax monoidal functor

$$\Gamma_S(\Sigma) \otimes \Gamma_S(\Sigma^\vee) \to \Gamma_S(\Sigma \otimes \Sigma^\vee) \to \Gamma_S(\mathcal{O}_X) \xrightarrow{[X]} \mathcal{O}_S[-d]$$

$$\Gamma_S(\Sigma^\vee) \to \Gamma_S(\Sigma)^\vee[-d]$$

$(X, [X])$ is weakly d-oriented if is an equiv. $\forall\, \Sigma$ dualizable in $\mathrm{QCoh}(X)$.

Definition: Assume we are given a pullback:

$$\begin{tikzcd} \sigma^* X \arrow[r, "\xi"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "\text{in } dSt"] \\ S' \arrow[r, "\sigma"'] & S \end{tikzcd}$$

$[X]$ on X/S we get $\sigma^*[X]$ on $\sigma^* X$

$$\Gamma_{S'}(\mathcal{O}_{\sigma^* X}) \xrightarrow{\simeq} \Gamma_S\, \xi^* \mathcal{O}_X \xrightarrow{\simeq} \sigma^* \Gamma_S \mathcal{O}_X \xrightarrow{[\sigma^* X]} \sigma^* \mathcal{O}_S[-d] \simeq \mathcal{O}_{S'}[-d]$$

$$\sigma^*[X]$$

$(X, [X])$ d-oriented if $\forall\, \sigma : S' \to S$

$(\sigma^* X, \sigma^*[X])$ is weakly d-oriented over S'.

Propositi

Proposition:

Let $(X, [X])$ be a d-preoriented stack$/S$

Then $(X,[X])$ d-or. $\Leftrightarrow \forall p: \operatorname{Spec} A \to S$

$(p^* X, p^*[X])$ weakly d-oriented.

Examples:

① $\emptyset$ in dSt_S has unique d-preorientation $\forall d$

$\mathcal{O}_\emptyset = 0$ d-orient. $QCoh(\emptyset) = 0$

② M: $MFLD_d^{comp.}$. M_B - Betti stack over k

$\operatorname{colim}_{\Delta \operatorname{Sing}(M)} \operatorname{Spec} k = M_B : dSt^{ucc}_{\operatorname{Spec} k}$

M is closed and oriented

$$\int_{[M]} : \underbrace{\Gamma \mathcal{O}_{M_B}}_{C^*(M;k)} \to k[-d]$$

③ M_B - Betti stack for Poincaré duality space $(M,[M])$ $\underline{[M] \cap -}: \Gamma\mathcal{O}_{M_B} \to k[-d]$

④ $Y_{\underline{DR}}$ stack for smooth proper Deligne-Mumford stack $\mathcal{Y}$ over A (A - Noetherian algebra/k)

$\mathcal{Y}$ should have connected geometric fibers.

Any choice of fundamental class of $\mathcal{Y}$ leads to

$2d$-orientation on $\mathcal{Y}_{DR}$. $2d$ orientation for

$\mathcal{Y}_{Dolb}$ but the fund. class in $H^{*,*}_{\bar\partial}(\mathcal{Y})$.

∞-category of presymplectic and preoriented stacks

Notation

$\mathcal{Q} \to dSt^{op}$ cocartesian fibration assoc. to $QCoh$

$QCoh$

$\mathcal{O}: dSt^{op} \to \mathcal{Q}$ $\quad X \rightsquigarrow \mathcal{O}_X$

$CIDF^{2,s} \to dSt^{\Delta'}$ fibration

$\mathcal{A}^{2,cl}_{(-)}(-, s) : dSt^{\Delta', op} \to \mathcal{S}$

$(X \to S) \rightsquigarrow \mathcal{A}^{2,cl}_{S}(X, s)$

$$\begin{tikzcd} CIDF^{2,s} \arrow[r] \arrow[rr, bend left, "\text{cart. fib.}"] & dSt^{\Delta'} \arrow[r, "ev_1"'] & dSt \end{tikzcd}$$

$S \longmapsto dSt_{S/\mathcal{A}^{2,cl}_{S}(s)}$ $\qquad S \overset{\varsigma}{\to} S'$

$$\delta^* : dSt_{S'/\mathcal{A}^{2,cl}_{S'}(s)} \longrightarrow dSt_{S/\mathcal{A}^{2,cl}_{S}(s)}$$

Definition: s-presymplectic stack over S is equivalent to a morphism $X \to \mathcal{A}^{2,cl}_S(s)$ in dSt_S. Thus we may define an ∞-category $PreSymp_{S,s}$ s-presymplectic stacks $/S$ as

$$PreSymp_{S,s} := \left(dSt^{Art}_S\right) \times_{dSt_S} \left(dSt_{S/\mathcal{A}^{2,cl}_S(s)}\right)$$

$$\begin{tikzcd} X \arrow[r] \arrow[rr, bend right] & \mathcal{A}^{2,cl}_S(s) \arrow[r] & S \end{tikzcd}$$

$$\widehat{dSt}^{Art}_s$$

Definition: $PreSymp_s := dSt^{\Delta', Art} \times_{dSt^{\Delta'}} \underline{CIDF^{2S}}$

Remark: $(X,\omega) : PreSymp_{S,s}$ $\quad \underline{\tilde{\omega} : \mathbb{T}_{X/S} \to \mathbb{L}_{X/S}[s]}$

$$X' \xrightarrow{\exists} X$$

$$\downarrow \qquad \downarrow \qquad \delta^*\omega \text{ corresponds to}$$

$S' \xrightarrow{\xi} S$

$$\mathbb{T}_{X'/S'} \simeq \xi^* \mathbb{T}_{X/S} \xrightarrow{\xi^*\omega^\flat} \xi^* \mathbb{L}_{X/S}[s] \simeq \mathbb{L}_{X'/S'}[s]$$

Lemma:

TFAE

① X symplectic stack

② $\forall\, S' \xrightarrow{\xi} S$ s-presymplectic stack $(\xi^* X, \xi^*_\omega)$ is symplectic over S'.

③ —"— but for $S' = \operatorname{Spec} A$.

Lemma: ∞-category $\mathrm{PreSymp}_{S,s}$ has pullbacks (and actually all weakly contractible finite l-ts) limits are computed in dSt_S.

Proof: ∞-category $dSt_{S/\mathcal{A}^{2,cl}_S(s)}$ has all limits and a forgetful functor

$$\mathcal{U}: dSt_{S/\mathcal{A}^{2,cl}_S(s)} \longrightarrow dSt_S$$

$\mathcal{U}$ preserves weakly contractible limits.

On the other hand dSt_S^{Art} is closed under finite limits.

$$\begin{tikzcd}
\mathrm{PreSymp}_{S,s} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & dSt_S/\mathcal{A}_S^{2,cl}(s) \arrow[d, "\mathcal{U}"] \\
\underbrace{dSt_S^{Art}}_{\text{finite limits}} \arrow[r, "\mathcal{U}"'] & dSt_S
\end{tikzcd}$$

preserve weakly contr. limits

$\mathrm{PreSymp}_{S,s}$ has weakly contractible finite limits. $\square$

Definition: $\Gamma_S \mathcal{O} : dSt_S^{op} \to QCoh(S)$

$f : X \to S \rightsquigarrow f_* \mathcal{O}_X$

$\mathrm{PreOr}_{S,d}$ is a pullback:

$$\begin{tikzcd}
\mathrm{PreOr}_{S,d}^{op} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & QCoh(S)/\mathcal{O}_S[-d] \arrow[d] \\
{} & {}
\end{tikzcd}$$

$$\begin{tikzcd} {} \arrow[d] & & {} \arrow[d, "\mathcal{V}"] \\ dSt_S^{ucc, op} \arrow[rr, "\Gamma_S \mathcal{O}"'] & & QCoh(S) \end{tikzcd}$$

Lemma $\mathrm{PreOr}_{S,d}^{op} \longrightarrow dSt_S^{ucc, op}$ detects weak equivalences.

Proof: $\mathrm{PreOr}_{S,d}^{op}$ - equivalence $\Leftrightarrow$ $QCoh(S) / \mathcal{O}_S[-d]$

and in dSt_S are equivalences.

$$QCoh(S)^{op} / \mathcal{O}_S[-d] \xrightarrow{\mathcal{V}} QCoh(S)^{op}$$

$\mathcal{V}$ detects equivalences $\Rightarrow$ equiv. are detectable at the level of $dSt_S^{ucc, op}$ $\square$

Proposition: ∞-cat. $\mathrm{PreOr}_{S,d}$ contains pushouts and all weakly contractible colimits and they are computed in dSt_S.
