Symplectic derived stacks

Notation

① k base field of char $= 0$.

② dSt – ∞-category of derived stacks

étale sheaf valued in category of spaces $\mathcal{S}$

$$cAlg_k^{ⓒ} \xrightarrow{\mathcal{F}} \mathcal{S}$$

connective means $[0, +\infty)$

Conditions of étale sheaf

① $\mathcal{F}$ preserves finite products

② $f : \underbrace{A \to B}_{\text{étale map}} \Rightarrow \mathcal{F}(A) = \lim\left(\mathcal{F}(B) \rightrightarrows \mathcal{F}(B^{\otimes 2}) \rightrightarrows \mathcal{F}(B^{\otimes_A 3}) \ldots\right)$

$$\underbrace{\text{Spaces}^{op}}_{\text{Algebras}} \to \mathcal{S}$$

$$f^* : \operatorname{Spec} B \hookrightarrow \operatorname{Spec} A$$

$$\sqcup U_i$$

$$\mathcal{F} = \operatorname{colim}\left(\mathcal{F}(\sqcup U_i) \leftleftarrows \mathcal{F}(\sqcup U_i \cap U_j) \rightrightarrows \right)$$

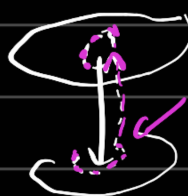
③ dSt^{Art} – full subcategory of Artin stacks in dSt

Long definition:

① Stack is (-1)-geometric if it is affine

② Morphism $Y \to X$ (-1) geometric if for any affine $Z \to X$ $Y \times_X Z$ is affine

③ Morphism of stacks is (-1) smooth if it is (-1)-geometric and $\forall Z : \mathrm{Aff}$ $\underbrace{Y \times_X Z \to Z}_{\text{smooth}}$



affine

④ n-smooth atlas on the stack X is an epimorphism $\bigsqcup_{i:I} U_i \to X$ s.t. $U_i : \mathrm{Aff}$

$U_i \to X$ is $(n-1)$ smooth

⑤ n-geometric stack is X s.t. $X \xrightarrow{\Delta} X \times X$ (n-1)-geometric and X admits n-smooth atlas

(6) $Y \to X$ n-geometric $\forall Z$: Aff $Y \times_X Z$ — n geometric.

(7) $Y \xrightarrow{\varphi} X$ n-smooth if φ is n-geometric and $\forall Z$: Aff $Z \to X$ $\exists$ n-smooth atlas $\sqcup U_i \to Z \times_X Y$ ← n-geometric stack

$$\begin{tikzcd} \sqcup U_i \arrow[r] \arrow[dr, bend right, "\text{smooth for Aff}"'] & Z \times_X Y \arrow[d] \\ & Z \end{tikzcd}$$

$$\underbrace{\boxed{\sqcup U_i \to Z \times_X Y}}_{\text{0-geom.}} \qquad \begin{tikzcd} \sqcup U_i \arrow[d] \\ Z \end{tikzcd}$$

$$Z \times_X Y \xrightarrow{\Delta} (Z \times_X Y)^{\times 2} \quad \text{affine}$$

$U_i \to Z \times_X Y$

(-1)-smooth

(8) A geometric stack is a stack s.t. it is n-geom. for some n

Definition $f : Y \to X$ of finite presentation if it is n-geometric for some n and $\forall Z$: Aff $Z \to Y$ $\exists$ n-smooth atlas

$\sqcup U_i \to X \times_Y Z \to Z$

is finitely presented

Definition: Artin stack$/S$ is geometric and locally of finite presentation.

$$\begin{tikzcd} X\times_S Z \arrow[r] \arrow[d, phantom, "\mathrm{I}"] & X \arrow[d] \\ Z \arrow[r] & S \end{tikzcd}$$

is n-geometric for some n

$Z \in \mathrm{Aff}$ finitely presented

$\simeq$ Lie groupoid

(4) $X : dSt_S \quad s \in \mathbb{Z}, \; p \in \mathbb{Z}_{\geq 0} \quad \mathcal{A}^{p,cl}_S(X, s)$

s-shifted closed p-forms.

__Mixed modules__

$Ch_k^{(\infty)}$ - chain complexes over k

$A : CDGA_k \quad Mod_A(Ch_k)$ - category of DG-modules $/A$.

Definition: Graded mixed module $/A$ is a family $E(p) \in Mod_A(Ch_k) \quad p \in \mathbb{Z}$

$$\varepsilon : E(p) \to E(p+1)[1] \qquad \varepsilon^2 = 0.$$

Now we have a category $GM\,MOD_A(Ch_k)$

symmetrical monoidal model structure

$$(E \otimes E')(p) = \bigoplus_{i+j=p} E(i) \otimes E'(j)$$

$GM\,MOD_A$ − ∞-category assoc. with

Definition: $k(p)[-p-s] \in GM\,MOD_k$ is a graded mixed module with k in weight p and 0 in all other weights.

$$\mathcal{A}^{p,cl}_S(X,s) := \mathrm{Map}_{GM\,MOD_k}(k(p)[-p-s], DR(X/S))$$

$$\mathcal{A}^{p}_S(X,s) := \mathrm{Map}_{GM\,MOD_k}(k(p)[-p-s], U\,DR(X/S))$$

$$U : GM\,CAlg_k \to G\,CAlg_k$$

Definition: $X \to \mathrm{Spec}\, A$ dSt

$DR_A(X) : GM\,CAlg_A$

$$\lim_{p : \mathrm{Aff}^{op}/X} \left(DR_A (p^* \mathcal{O}_X) \right)$$

$$DR_A(p^*\mathcal{O}_X) = \Lambda^* \Omega^1_{p^*\mathcal{O}_X/A}$$

$$DR_A \dashv \mathcal{U} : GMCAlg_A \to CAlg_A$$

(DGA)

⑤ X: dSt $\quad QCoh(X)$ ∞-category

$$\lim_{\substack{\mathrm{Spec} A \to X \\ (\mathrm{Aff}/X)^{op}}} \mathrm{Mod}_A = QCoh(X)$$

⑥ $X : dSt_S \quad \mathbb{L}_{X/S}$

$$\underline{\mathrm{Map}_Q(\mathbb{L}_{X/S}, (Y, \mathcal{F}))} \simeq \mathrm{Map}_{dSt}(Y[\mathcal{F}], X)$$

<u>Square zero extension</u>

$X : dSt_S^{Art} \quad \mathbb{L}_{X/S}$ dualizable

$\mathbb{T}_{X/S} = \mathbb{L}^\vee_{X/S}$ – relative tangent complex X/S

Definition: $X : dSt_S^{Art} \quad \mathcal{A}^p_S(X, s) \cong$

$\cong (\mathcal{O}_X[-s] \longrightarrow \Lambda^p \mathbb{L}_{X/S}) \quad QCoh(X)$

$\shortparallel$

$DR\,(X/\ldots) \quad \boxed{X}$

$DR(X/S)$ $\begin{bmatrix} \downarrow \\ S \end{bmatrix}$

s-shifted 2-form $\omega \leftrightarrow \mathcal{O}_X[-s] \to \Lambda^2 \mathbb{L}_{X/S}$

$\tilde{\omega} : \mathbb{T}_{X/S} \to \mathbb{L}_{X/S}[s]$

ω is **non-degenerate** if the cor. $\tilde{\omega}$ is

a non-degenerate map. (w.E. between)

ω - **closed**, **non-degen.**, s-shifted **2-form**

ω - symplectic

Definition: s-presymplectic stack (X, ω)

$X : dSt_S^{Art}$ $\omega \in A_S^{2,cl}(X, s)$. ω is s-symp.

$\Rightarrow (X, \omega)$ s-symplectic stack

Examples:

(i) $S : dSt_S$ - terminal object, $A_S^{2,cl}(S, s) \simeq *$

$\mathbb{L}_{X/S} = 0$ S carries a unique s-presympl.

str-re $(S, 0)$ which is symplectic.

(ii) A Noetherian k-algebra and G-
- linear algebraic group $\mathfrak{g} = \mathrm{Lie}\, G$.
$\forall c \in (\mathrm{Sym}^2 \mathfrak{g}^\vee)^G$ defines 2-presymplectic
str-re on $\mathbb{B}G$. ($\mathbb{B}G$ is a $\mathrm{Spec}\, A$ stack)
c defines a symplectic str-re iff

$$\boxed{c : \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^\vee} \qquad \mathbb{L}_{\mathbb{B}G/A} = \mathfrak{g}^\vee[1]$$

$$\mathbb{T}_{\mathbb{B}G/A} = \mathfrak{g}[-1]$$

(iii) Perf_A - derived stack of perfect A-modules
2-symplectic st-re given by Chern character's
2-degree component. (Next time)

(iv) $X : dSt_k^{Art}$ $\quad \underbrace{\mathbb{T}^*[n]}\, X$ - stack over X
classifies sections of $\mathbb{L}_X[n]$ $\quad \lambda_X : \mathcal{A}^1_k(n)$

$$\mathcal{A}^1_k(n) \xrightarrow[d_{DR}]{} \mathcal{A}^{2,cl}_k(n)$$

presymplectic str-re $(\mathbb{T}^*[n]X)$ it is
symplectic $\quad d\lambda_X : \mathbb{T}(\mathbb{T}^*[n]X/_X) \to \mathbb{L}(\mathbb{T}^*[n]X/_X)$