SEMINAR QUANTUM HOMOTOPY

Speaker: GRIGORY TAROYAN

FEBRUARY 4TH, 2021

$\mathcal{C} \xrightarrow[\mathrm{Sing}^{Q}_{\bullet}]{} \mathrm{sSet} \qquad X \longmapsto \mathrm{Hom}_{\mathcal{C}}(Q, \cdot)$

$\mathcal{C} = \mathrm{Top} \qquad |\Delta^n| = \{\dots\}$

$\mathcal{D} \in \mathrm{Cat} \qquad \Delta \hookrightarrow \mathrm{Cat}$

$\{0 \to 1 \to 2 \to \dots \to n\} \qquad \mathcal{D} \quad \bullet \to \bullet \to \bullet \to \bullet \to \dots$

$\mathrm{Sing}^{\Delta}_{\bullet}(\mathcal{D}) = N_{\bullet}(\mathcal{D})$

REALIZATION / NERVE adjunction

N (nerve) — $|-|$ (realization)

PROP: Assume $\mathcal{C}$: category, $Q^{\bullet}, \Delta$

$\mathrm{Sing}^{Q}_{\bullet} : \mathcal{C} \to \mathrm{sSet}$.

$\mathcal{C}$ is called complete if it has all small colimits

then there is a left adjoint functor $|-|^{Q}$

PROOF:

$S_{\bullet}$ - sSet is "good" if the functor is corepresentable

$$\mathcal{C} \ni C \longmapsto \mathrm{Hom}_{\mathrm{sSet}}(S_{\bullet}, \mathrm{Sing}^{Q}_{\bullet}(C))$$

$$\| \qquad\qquad$$

$$\mathrm{hom}_{\mathcal{C}}(Q^n, C) \qquad |\Delta^n|^{Q} \rightsquigarrow Q^n$$

$\mathcal{C}$ is cocomplete and the set of all good sSets contains all simplices $\Rightarrow$ ALL sSets ARE "GOOD"! $\quad |\cdot|^{Q} : \mathrm{sSet} \to \mathcal{C}$ ▨

DEF: $S_{\bullet} \in \mathrm{sSet}$ is a KAN COMPLEX iff

(*) $\forall \Lambda^n_r \xrightarrow{f} S_{\bullet}, \ \exists \hat{f} : \Delta^n \to S_{\bullet}$

(**) $\Lambda^n_i \longrightarrow S_{\bullet}$, $\Lambda^n_i \searrow \Delta^n$, $\Delta^n \dashrightarrow S_{\bullet}$ (#)

PROP: $X \in Top$, $Sing_\bullet(X)$ is a **KAN COMPLEX**

PROOF:

Assume we have a representation

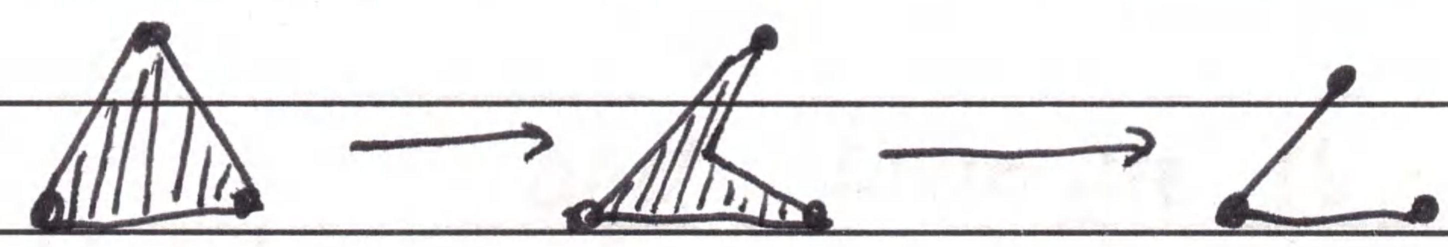
$$\tau_0 : \Lambda^n_r \longrightarrow Sing_\bullet(X)$$

$$\downarrow \quad \# \nearrow$$

$$\Delta^n$$

pass to geometric realization using the adjunction

$|\tau_0|$ retraction map $r(t_0, \dots, t_n) = (t_0 - c, \dots t_n - c)$ where $c = \min\{t_0, \dots t_{i-1}, t_{i+1}, \dots \dots t_n\}$



leave out ith coord.

illustrates extension property

PROP:

If $N_\bullet(\mathcal{C})$: nerve is a **KAN COMPLEX**

Then $\mathcal{C}$ is a **GROUPOID**

PROOF:

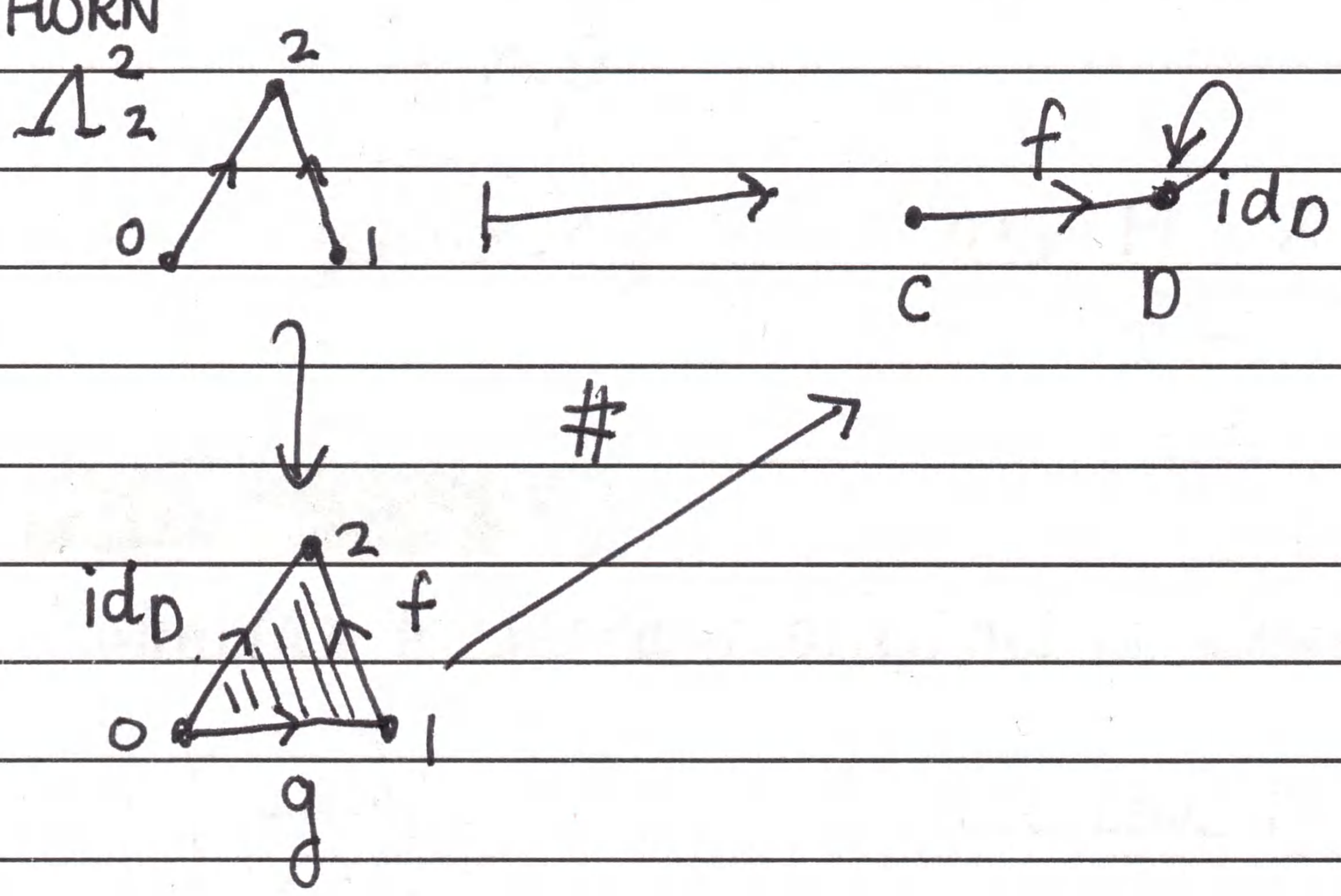
$$Hom_{sset}(\Delta^2, N_\bullet(\mathcal{C})) \xrightarrow{1-1} Hom_{sset}(\Lambda^2_2, N(\mathcal{C}))$$

$\exists\, \tau \in N_2(\mathcal{C})$ such that $d_0(\tau) = f$

$d_1(\tau) = id_D$

$g \equiv d_2(\tau)$

HORN Λ^2_2



$fg = id_D$

$hf = id_C$

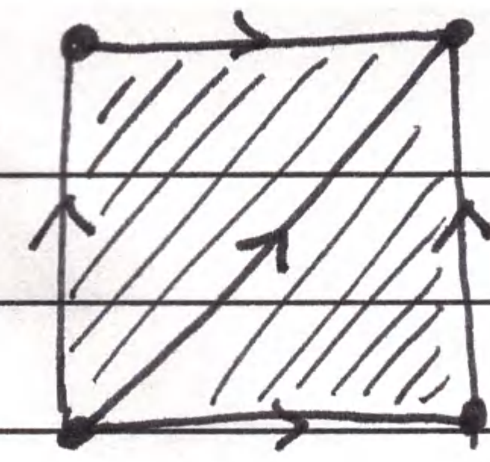
Then $\boxed{h = g}$ ▨

NOTE:

ssets have all limits and colimits (complete/cocomplete)

and $(\lim(\mathcal{F}))_n = \lim(\mathcal{F})_n$, $\mathcal{F} : \mathcal{I} \longrightarrow sset$

EXAMPLE: $\lim(\Delta^1 \to \cdot, \Delta^1 \to \cdot) = \Delta^1 \wedge \Delta^1$



$X_0 = \{([0],[0]), ([0],[1]), ([1],[0]), ([1],[1])\}$

$e, f \in \{[0,0], [0,1], [1,1]\}$, and there are 9 1-simplices in X_0

NOTE: $d_i(x,y) = (d_i x, d_i y)$

$d_0([0,1],[0,1]) = (d_0[0,1], d_0[0,1])$
$= ([1],[1])$

↑ here e projects onto horizontal, f projects onto vertical axis

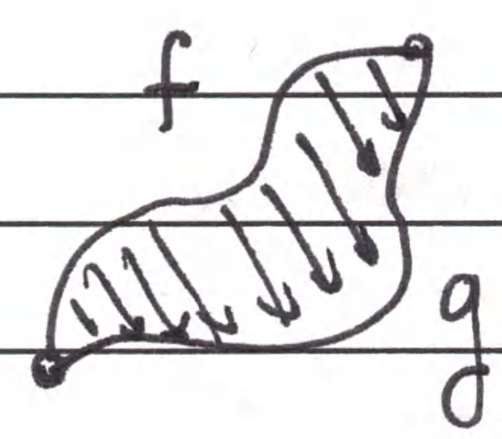
When both projections (horizontal and vertical) are degenerate, the 1-simplex is degenerate

and there are 16 - 2-simplices (14 of which are degenerate)

[and all higher simplices are degenerate.]

SIMPLICIAL HOMOTOPIES

$f, g: X \longrightarrow Y$ $\quad H: \Delta^1 \times X \longrightarrow Y$ $\quad f = H \circ (d_1 \times X)$
$g = H \circ (d_0 \times X)$

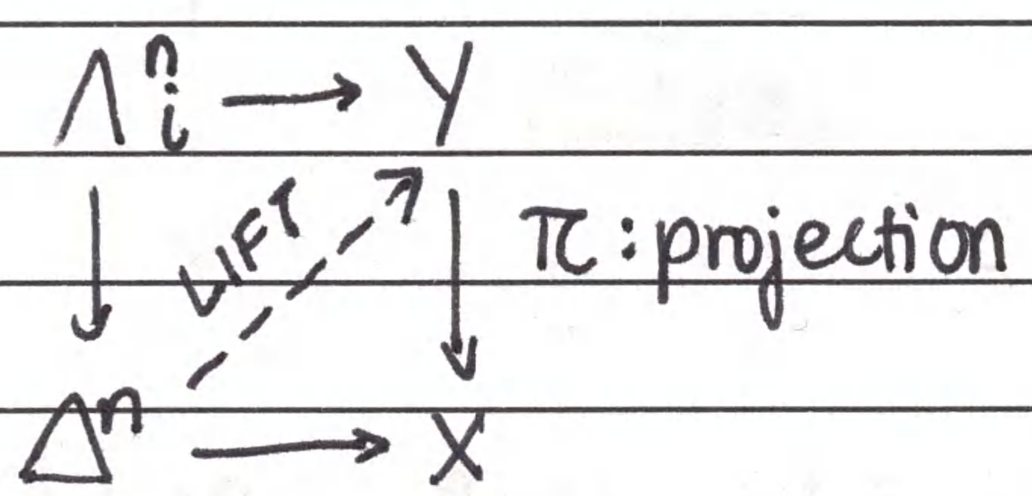


or, $H: X \longrightarrow Y^{\Delta^1}$
$f = Y^{d_1} \circ H$
$g = Y^{d_0} \circ H$

NOTE: Y must be a Kan complex

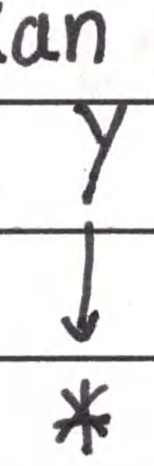
KAN FIBRATION

(similar to Serre fibrations in classical homotopy theory)



"relative Kan complex"

$X = *$



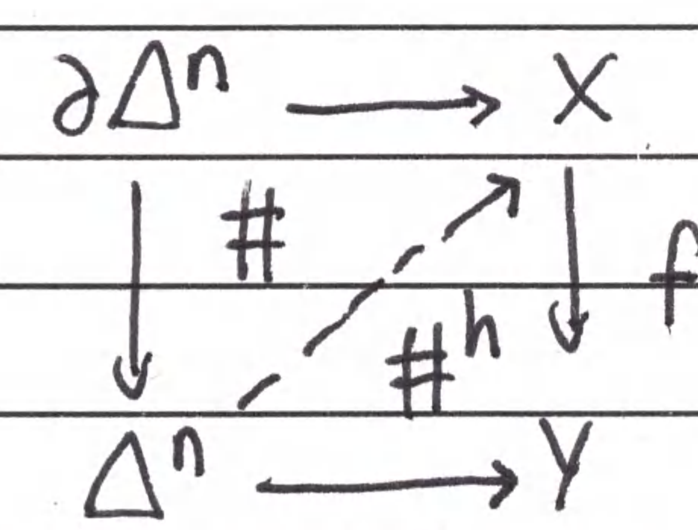
Recall, $\#^h$: commutes up to homotopy

SIMPLICIAL WHITEHEAD THEOREM

$f: X \longrightarrow Y$

↖ ↗
Kan Complexes

f is a simplicial homotopy if we have lifting property in comm. diagram



[PROOF OMITTED]

Another formulation of Simplicial Whitehead Theorem:

$f_* \div \pi_0(X) \xrightarrow{|-|} \pi_0(Y)$

$f_* : \pi_n(X, x) \xrightarrow{\cong} \pi_n(Y, f(x))$

induces isomorphism

NOTE { When $Y = *$ (a point)
X is contractible iff every shphere can be "filled in" by a disk.

HOMOTOPY GROUPS

SPHERE

$$\begin{array}{ccc} \partial\Delta^n & \hookrightarrow & \Delta^n \\ \downarrow & \# & \downarrow \\ * & \longrightarrow & \Delta^n/\partial\Delta^n \\ & & \| \\ & & \mathbb{S}^n \end{array}$$

$s \in S_0$

$\pi_n(S, s) = \hom^{\circledast}_{sSet}(\mathbb{S}^n, S)/\sim$ (pointed Kan complex)

[There is a group structure on $\pi_{n>0}(S,s)$]
[$\pi_{n>1}(S,s)$ are abelian](KERODON 3.2.3)

NOTE:
[Given $(X, x) \xrightarrow{f} (S, s)$ Kan fibration of pointed Kan complexes,
we have a long exact sequence

$\cdots \to \pi_0(X_s, x) \longrightarrow \pi_0(X, x) \longrightarrow \pi_0(S, s)$]

EX-FUNCTOR

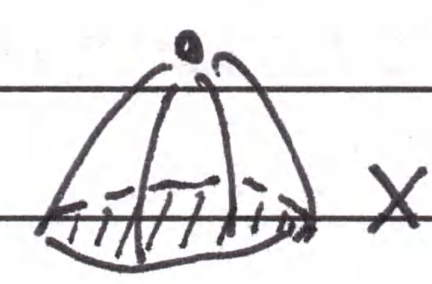
$sd : sSet \longrightarrow sSet$ "subdivision"

where $sd\, \Delta^0 = \Delta^0$

and $sd\, \Delta^n = C(sd(\partial\Delta^n))$

"cone" $C : \Delta \longrightarrow \Delta \quad [m] \longmapsto [m] \cup \{*\}$

$sSet \longrightarrow sSet$



"cone" over X

$Ex(X)_n = \hom_{sSet}(sd\, \Delta^n, X)$

right adjoint functor to baryaentric subdivision

which is $Ex(X)_n =$
$\hom_{sSet}(\Delta^n, Ex(X)_\circ)$
(Ex can be applied in either formulation)

DEF: id sset ⇒ Ex : sset ⟶ sset

$X \longmapsto Ex(X)$ ≪ last vertex representation ≫

$sd(X) \longmapsto X$

$sd\,\Delta^0 = \Delta^0 \xrightarrow{id} \Delta^0$
and
$sd\,\Delta^n = C(sd\,\partial\Delta^n) \to \Delta^n$

Concrete description:
(of simplices in Ex)

n-simplices $\longleftrightarrow$ $sd\,\Delta^n \to X$
simplicial maps

$X \hookrightarrow Ex(X)$

$\tau \in X_n \rightsquigarrow sd\,\Delta^n \to \Delta^n \xrightarrow{\tau} X$
composition

and another way of understanding the Ex-functor

$Ex(X) = Sing_0^{\Gamma}(X)$ $\Gamma([n]) = N_\circ(P([n]))$

DEF: Ex^∞-functor, Ex^∞ : sSet ⟶ sSet

$Ex(Ex(X))\ldots$

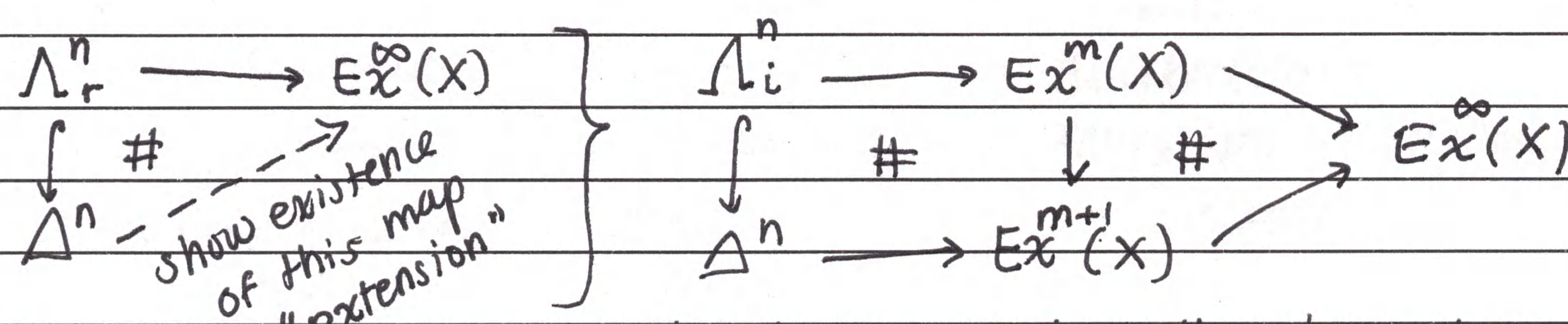
$$Ex^\infty(X) = \operatorname{colim}(X \to Ex(X) \to Ex^2(X) \to \cdots)$$

PROP: Ex^∞ preserves ~~small~~ finite limits, filtered colimits, monomorphisms (of ssets)

(Ex is right adjoint and we take the finite limit as Ex^∞ preserves finite limits)

$sd^k(\Delta^n)$ is a compact simplicial set ("covered" by finitely many nondegenerate simplices)

PROP: $Ex^\infty(X)$ is a Kan complex

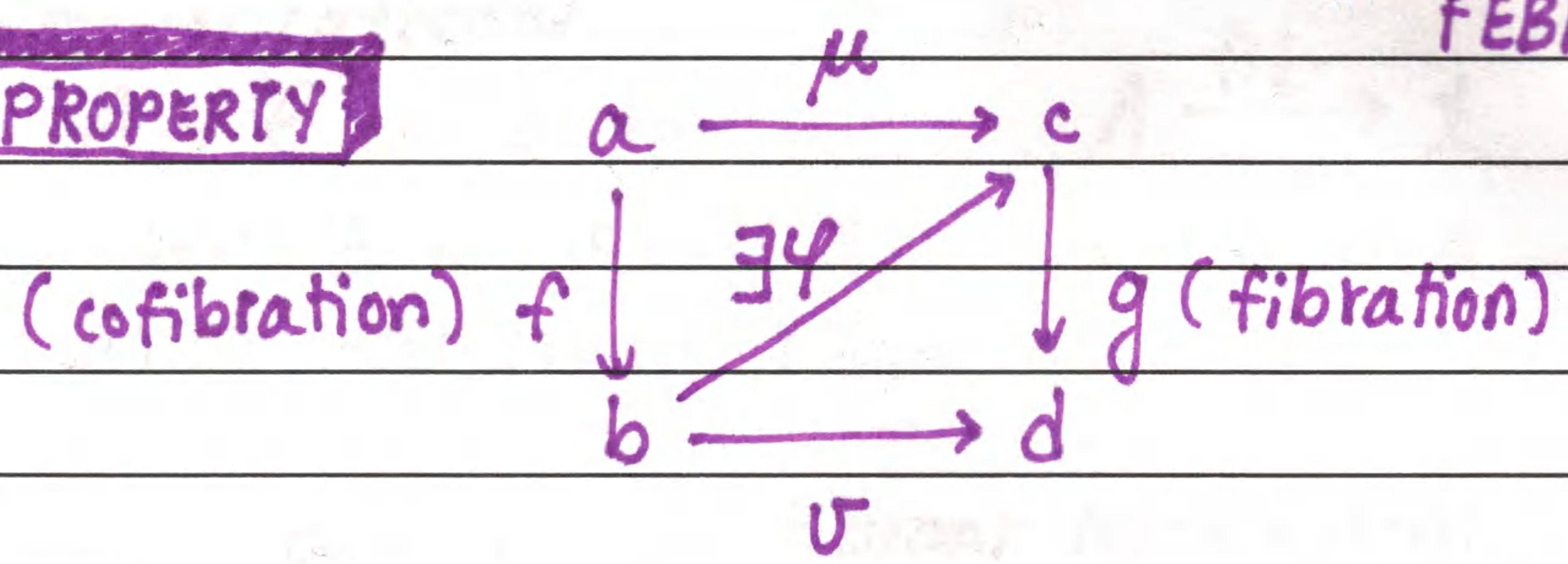


This is acomplished though a stage, m as given by this diagram. ▨

SEMINAR (Ramiro) [I came late due to meeting w/ Razvan]

FEBRUARY 25TH

LIFTING PROPERTY



The lifting property states that $\forall(\mu, \upsilon)$ $\exists \varphi$

Note that f or g is a weak equivalence (only one needed)

MODEL CATEGORY A category equipped with three distinguished classes of morphisms:

① Cofibrations $\subset Mor(\mathcal{C})$ ② Fibrations $\subset Mor(\mathcal{C})$

③ Weak Equivalences $\subset Mor(\mathcal{C})$

When we choose $\mathcal{A} = RMod$, we can define a particular kind of model structures ...

PROJECTIVE MODEL STRUCTURE

Weak equivalences are quasi-isomorphisms, and Fibrations are epimorphisms in all degrees (surjective in RMod) ... and ...

Cofibrations are degreewise monomorphisms with projective cokernels.

INJECTIVE MODEL STRUCTURE

Same weak equivalences, Fibrations are degreewise epimorphisms with injective kernel ... and ...

Cofibrations are monomorphisms in RMod in any degree.

→

NOTES NOT RELEVANT TO SEMINAR:

Add to Resume (AFRL GS-11 due tonight)

Assistant to Statistician (Ph.D.)

5/16 PhD in statistics, keeping record

x19 spreadsheet. Promoted at her recommendation

144, 160 } 304 to Director/Coordinator of ______ (etc.)

August – December 16 weeks (16 × 19 hrs =)

10 hrs / week × 4 weeks = 32 hrs = 40 hrs

5 hrs / week × 4 weeks = 20 hrs = 20 hrs

TO BE CONTINUED... 12 hrs 20 hrs

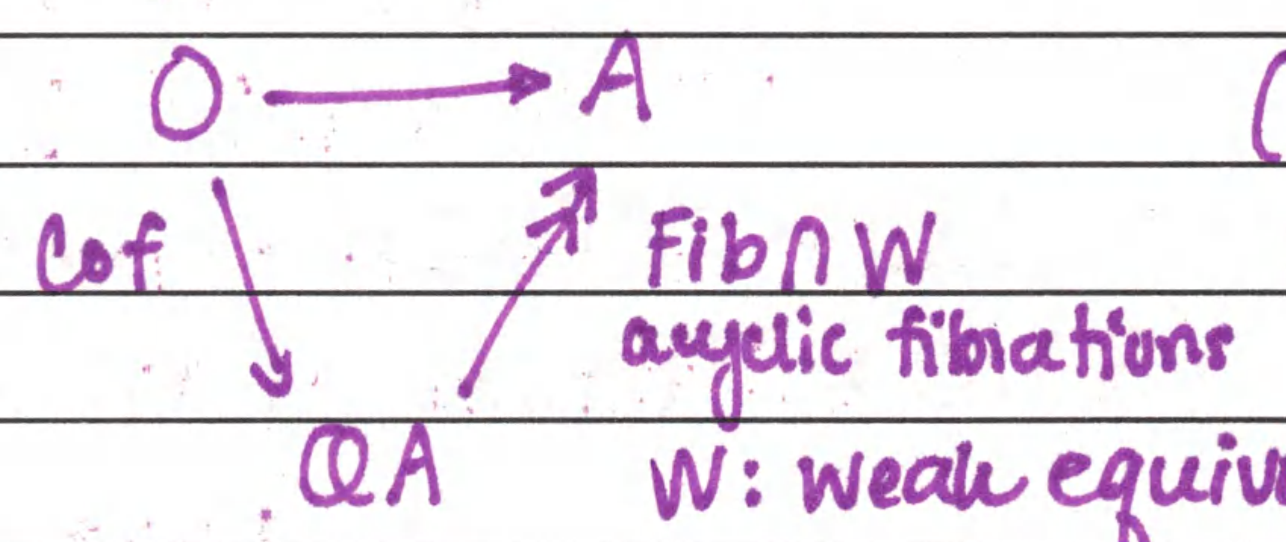
DEF $\mathcal{C}$: model category

The object $A \in \mathcal{C}$ is called a fibrant (cofibrant) object if $\exists!$ fibration (cofibration)

$A \xrightarrow{\text{fib}} 1$

$0 \xrightarrow{\text{cofib}} A$

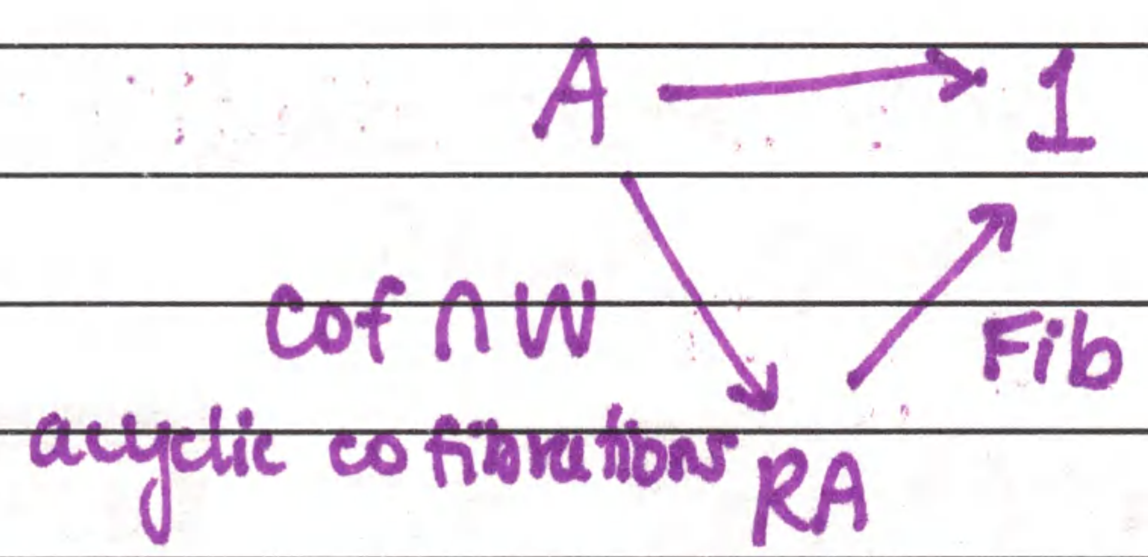
Every object A admits a cofibrant replacement $A \longmapsto QA$



(In Projective Model structure)
PROJECTIVE RESOLUTION

and similarly,

INJECTIVE RESOLUTION
(In Injective Model structure)



DEF A left Quillen functor is a functor that preserves cofibrations and acyclic fibrations, $Cof \cap W$ (acyclic cofibration).

A right Quillen functor is a functor that preserves fibrations and acyclic fibrations, $fib \cap W$ (acyclic fibrations)

We can "derive" the functor, that is, modify it so as to preserve weak equivalences.

DEF A left derived functor $\mathbb{L}F: \mathcal{C} \longrightarrow \mathcal{D}$ of a left quillen functor $F: \mathcal{C} \longrightarrow \mathcal{D}$ is obtained by applying F on cofibrant objects (cofibrant replacement)

$$\mathbb{L}F(A) := F(\underbrace{QA}_{\text{cofibrant replacement of } A})$$

DEF A right-derived functor $\mathbb{R}G : \mathcal{D} \to \mathcal{C}$ of a right Quillen functor $G : \mathcal{D} \to \mathcal{C}$ is obtained by applying G on fibrant objects

$$\mathbb{R}G(A) := G(RA)$$

HOM AND TENSOR

recall,

$$- \otimes - : Ch(A)^{op} \times Ch(A) \longrightarrow Ch(A)$$
$$Hom(-,-) : Ch(A)^{op} \times Ch(A) \longrightarrow Ch(A)$$

DEF $Ext^1(A,-) := \mathbb{R}Hom(A,-)$

$Tor_1(A,-) := A \overset{\mathbb{L}}{\otimes} -$

Obtained from Hom and $\otimes$ via cofibrant and fibrant replacement.

EXAMPLE $Tor_1(\mathbb{Z}/p, A) = Tor_1(\mathbb{Z} \xleftarrow{p} \mathbb{Z}, A)$

cofibrantly replace w/ projective resolution

$$A \otimes \mathbb{Z} \xleftarrow{p} A \otimes \mathbb{Z}$$
$$\cong \qquad\qquad \cong$$
$$A \xleftarrow{p} A \xleftarrow{0} 0$$
$$(\deg 0) \qquad (\deg 1)$$

so, $Tor_0(\mathbb{Z}/p, A) = A/pA$

$Tor_1(\mathbb{Z}/p, A) = \{a \in A \mid pa = 0\}$

p-torsion elements of A

$= Tor(A, \mathbb{Z}/p)$

Seminar MARCH 11, 2021

"A LOOK @ RATIONAL HOMOTOPY THEORY"
Sullivan's Approach to a version of homotopy over $\mathbb{Q}$
the study $\pi_*(X) \otimes \mathbb{Q}$ ("torsion-free")

Associate to $X \in Top$
its polynomial deRham complex $A_{PL}(X)$ → study homotopy type using sullivan forms!
(simplicial commDGA)
commutative differential graded algebras

functor

In other words... we have a functor $\mathcal{A}: sSet \to CDGA_{\mathbb{Q}}^{op}$
(MR. MEADOR!)
$K \mapsto sSet(K, (A^*_{poly}).)$

An "improvement" of the cochain algebra $C^\bullet(K;\mathbb{Q})$ which is only commutative at the level of cohomology (via cup product); this correction only takes place over $\mathbb{Q}$ (2002 Mandell)
["chain Multiplication"]

(continuous extension of $A^*_{PL}(\Delta^n)$)

Theorem: Rational types are in 1-1 correspondence with minimal Sullivan models

Definition: A topological space X (nilpotent) is called "rational" if any of the following equivalent conditions hold:

(i) $\pi_* X$ are $\mathbb{Q}$-vector spaces
(ii) $H_*(X, \mathbb{Z})$ are $\mathbb{Q}$-vector spaces
(iii) $H^*(\Omega X, \mathbb{Z})$ are $\mathbb{Q}$-vector spaces

(i) ↔ (ii): Serre s.s. induction (spectral sequence)

- Two simply connected spaces X, Y have the same rational homotopy type if there is a sequence

$$X \leftarrow Z(0) \leftarrow Z(1) \leftarrow \cdots \leftarrow Z(n) \rightarrow Y$$

of rational homotopy maps (maps inducing isos in rational homotopy groups, with cohomology defined previously)

Goal: Assign the "simplest possible" cochain algebra (A,d) to X together with a weak equivalence of DGA's:

$$(A,d) \longrightarrow \cdots \longleftarrow A^*_{PL}(\mathrm{Sing}(X))$$

Definition: A "Sulliven model" for a cochain algebra (A,d) is a quiso

$$m: (\Lambda V, d) \longrightarrow (A,d)$$

($(\Lambda V, d)$: Sullivan algebra)

and this correction only takes place over $\mathbb{Q}$.

The model is minimal

$\alpha < \beta \Rightarrow \deg v_\alpha \leq \deg v_\beta$

where $\{v_\iota \in V : \iota \in I\}$ is a well ordered basis for V

Example: Sphere, S^n in singular cohomology, there is a fundamental class $[\omega] \in H_n(S^n; \mathbb{Z})$ induced by $I^n/\partial I^n \xrightarrow{\sim} S^n$

$\partial I^{n+1} \xrightarrow{\sim} S^n$

$\exists! [\omega] \in H^n(A_{PL}(S^n))$ ~~[illegible]~~

and this serves as a basis for $H^n(A_{PL}(S^n))$

Consider the exterior algebra on one generator with trivial differential:

for odd n:
$$\varphi: (\Lambda(x), 0) \longrightarrow A_{PL}(S^n)$$
$$x \longmapsto \omega$$
(such that degree x is n)

This is a quasi-isomorphism (quiso)

for even n:
φ extends to a quasi-isomorphism φ', with additional generator in degree $2n-1$

y: additional generator, $\deg y = 2n-1$

$$\varphi' : (\Lambda(x,y), d) \longrightarrow A_{PL}(S^n) \quad \rightarrow$$

$$\varphi' : (\Lambda(x,y), d) \longrightarrow A_{PL}(S^n)$$

$x \longmapsto \omega$

$y \longmapsto y$

and

$\omega^2 = dy$ and $x \wedge x = dy$

$dx = 0$

$dy = x \wedge x$

and so the cohomology ring of this Sullivan model is the

$$H^\bullet(\Lambda(x,y), d) \simeq \mathbb{Q}[x]/x^2$$

with $\{1, x\}$ a basis

This establishes φ' as a quasi-isomorphism

So it's Sullivan minimal model

Serre's Theorem

ODD $\pi_k(S^{2n+1}) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & k = 2n+1 \\ 0 & \text{otherwise} \end{cases}$

EVEN $\pi_k(S^{2n}) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & k = 2n, 4n-1 \\ 0 & \text{otherwise} \end{cases}$

The Sullivan's Realization functor

$\varnothing : CDGA_{\mathbb{Q}} \longrightarrow sSet$

$(A, d) \longmapsto \langle A, d \rangle$ } notation due to Felix, Halperin, Thomas "Rational Homotopy Theory I & II"

The n-simplices of $\langle A, d \rangle$ are just DGA-morphisms $\sigma : (A, d) \longrightarrow (A_{PL})_n$

The face maps $\partial_i \circ \sigma$

The degeneracies $s_j \circ \sigma$

$\varphi : (A, d) \longrightarrow (B, d)$

$\langle \varphi \rangle : \langle B, d \rangle \longrightarrow \langle A, d \rangle$

$\sigma \longmapsto \sigma \circ \varphi$

Hence, CDGA's, we have

$$CDGA_{\mathbb{Q}}\big((A,d),(A_{PL})_n\big) = \langle A,d\rangle_n$$

Now, the geometric realization (ordinary geometric realization)

$|\langle A,d\rangle| = |A,d|$ is what the literature calls "spatial realization"

We have an adjoint pair

$$a^* : sSet \underset{R}{\overset{L}{\rightleftarrows}} CDGA^{op}_{\mathbb{Q}} : \varnothing$$

a^*: PL deRham Functor; $\varnothing$: "sullivan realization [Nerve Realization Adjunction]

So we have adjoint of the identity:

$r : id \longrightarrow a^* \circ \varnothing$ (the unit)

This yields a canonical DGA morphism $\eta_A : (A,d) \longrightarrow A^*_{PL}(\langle A,d\rangle)$

$\varepsilon : Sing \circ |-| \longrightarrow id$ (the counit)

The counit has a lift (unique up to homotopy) By the lifting lemma for arbitrary model categories.

Recall, generic lifting square

$$\begin{array}{ccc} (A,d) & \longrightarrow & (B,d) \\ {\scriptstyle i}\downarrow & \overset{\exists! h}{\nearrow} & \downarrow \\ (A\otimes \Lambda V,d) & \longrightarrow & (C,d) \end{array}$$

Generally,

(recall, relative sullivan algebra is cofibration in $CDGA_{\mathbb{Q}}$ if $i \vee p$ is a quiso)

Thus, here,

$$\begin{array}{ccc} \mathbb{Q} & \longrightarrow & A^*_{PL}(\langle \Lambda V,d\rangle) \\ {\scriptstyle \text{sullivan algebra}}\downarrow & \overset{\exists! m_{\Lambda V,d}}{\nearrow} & \downarrow \simeq \; a^*(\varepsilon\varnothing(\Lambda V,d)) \\ (\Lambda V,d) & \underset{\eta_{(\Lambda V,d)}}{\longrightarrow} & a^*(\varnothing(\Lambda V,d)) \end{array}$$

We get a map $m_{\Lambda V,d}$ where

$m_{\Lambda V,d} : (\Lambda V,d) \to A_{PL}(\langle \Lambda V,d\rangle)$

and there is an isomorphism

$\pi_*(|\Lambda V,d|) \cong hom_{\mathbb{Q}}(V,\mathbb{Q})$ ∎

DEF: A commutative real algebra is

→ ① a set, A

→ ② an operation, $Af: A^n \longrightarrow A$

for any real polynomial $f(x_1, \dots, x_n)$

where $Af(a_1, \dots, a_n) = f(a_1, \dots, a_n)$ (with $A \in$ above)

satisfying axioms:

apply g_i's individually to argument

$$f(g_1(a_{1,1}, \dots a_{1n}), \dots, g_m(a_{m1}, \dots, a_{mn}))$$

$$\|$$

$$f(g_1, \dots, g_m)(a_{11}, \dots, a_{mn})$$

vs. apply g to argument

a real polynomial in variables $x_{11}, \dots, x_{mn}$

(mn-arguments)

$$\begin{cases} x_1 \in \mathbb{R}[x_1, \dots, x_n] \\ x_i(a_1, \dots, a_n) = a_i \end{cases}$$

TRADITIONAL DEF:

and here...

A

$x_1 + x_2 : A^2 \xrightarrow{+} A$ addition

$-x_1 \quad A \xrightarrow{-} A$ inverse

0

$x_1 \cdot x_2 \quad A^2 \xrightarrow{\cdot} A$ multiplication

1

$\forall r \in \mathbb{R}, \quad r.$

$$A^n \xrightarrow[Af]{} A$$

$$a_{11}, \dots, a_{1n} \longmapsto f(a_1, \dots, a_n)$$

Seminar (a little late!)

Generating (acyclic) cofibrations

any cofibration is a retract of transfinite composition of cobase changes of generating (acyclic) cofibrations

g. cofibrations : $S^{n-1} \longrightarrow D^n$

g.a. cofibrations: $0 \longrightarrow D^n$

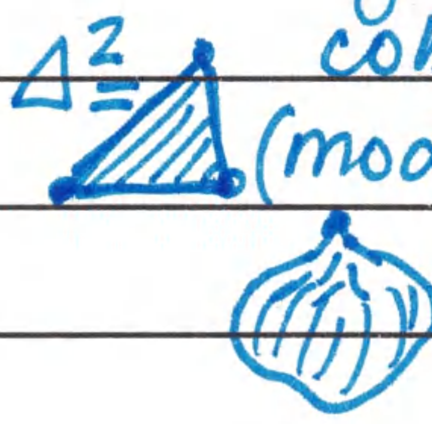
$S^{n-1} = \mathbb{R}[S^{n-1}]$

$D^n = \mathbb{R}[D^n]$

where S^{n-1} is a point $*$ and some nondegenerate $(n-1)$-simplex, i.e., $\Delta^{n-1}/\partial\Delta^{n-1}$

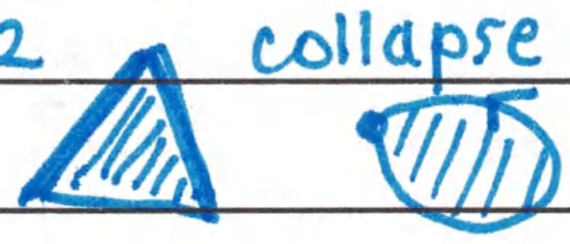
when $n=3$, $S^2 = \Delta^2/\partial\Delta^2$

and $\mathbb{R}[S^{n-1}] = \mathbb{R}[n-1]$

Δ^2  collapse boundary (mod boundary) "dumpling"

and D^n nondegenerate n-simplex and nondegenerate $(n-1)$-simplex

$D^n = \Delta^n/\Lambda^n_k$ $\qquad \mathbb{R}[D^n] = \begin{array}{c}\mathbb{R} \;\; n \\ \downarrow 1 \\ \mathbb{R} \;\; n-1\end{array}$

$n=2$  collapse horn (mod horn)

$$Ch_{\mathbb{R}} \underset{R=\mathcal{U}}{\overset{L}{\rightleftarrows}} DGC^{\infty}\mathbb{R}$$

TRANSFERRING MODEL STRUCTURE

If I, J is a set of generating (acyclic) (relations) cofibrations for $Ch_{\mathbb{R}}$, then $L(I), L(J)$ is a set of generating (acyclic) cofibrations for $DGC^{\infty}\mathbb{R}$.
(Differential Graded C^{∞}-Rings)

The consequence is that it is very easy to say what cofibrations are!

Suppose $I = \left\{ \mathbb{R}[n-1] \hookrightarrow \begin{array}{c}\mathbb{R}\,(n) \\ \downarrow 1 \\ \mathbb{R}\,(n-1)\end{array} \right\}$

Then,

$\mathrm{codom}(L(I)) = \mathbb{R}[x,y]/(x^2, dy = x)$ for n even

$\mathrm{codom}(L(I)) = \mathbb{R}[x,y]/(y^2, dy = x)$ for n odd $\longrightarrow$

COMPUTING "PUSHOUTS"

$$\begin{array}{ccc} \mathbb{R}[x]/x^2 & \hookrightarrow & \mathbb{R}[x,y]/(x^2, dy = x) \\ \downarrow & & \downarrow \\ A & \longrightarrow & A \otimes_{\mathbb{R}[x]/x^2} \mathbb{R}[x,y]/(x^2, dy = x) \end{array}$$

$$\| $$

$$A[y]/(dy = x)$$

new free variable (y) — we require its boundary is x (x becomes "exact")

So, now a cobase change is

$$A \longrightarrow A[y]/(dy = x)$$

Performing this operation many times adjoins many elements!

Transfinite compositions of cobase changes of $L(I)$

$$A \longrightarrow A[y_1, y_2, y_3, \dots]/(dy_i = \dots \in A[y_j]_{j<i})$$

"Relative Sullivan Algebra"

Sullivan & Quillan Founders of Relative Homotopy Theory

Example: Compute the derived intersection

Given a smooth function between two Cartesian spaces, we want to intersect with 0.

derived pullback

$$\begin{array}{ccc} P & \longrightarrow & \mathbb{R}^0 \\ \downarrow & \lrcorner h & \downarrow 0 \\ \mathbb{R}^a & \xrightarrow{f} & \mathbb{R}^b \end{array}$$

Recall,

Derived C^∞-Loci are by definition,

$$C^\infty\text{-Loci} := (DGC^\infty\text{Ring})^{op}$$

That is, derived C^∞-Loci are the opposite category of differential graded C^∞-Rings. And smooth manifolds embed into C^∞-Loci

$$C^\infty\text{-Loci} \underset{C^\infty(-)}{\hookleftarrow} \text{Man}$$

free C^∞-Ring on b-variables — free C^∞-Ring on a-variables

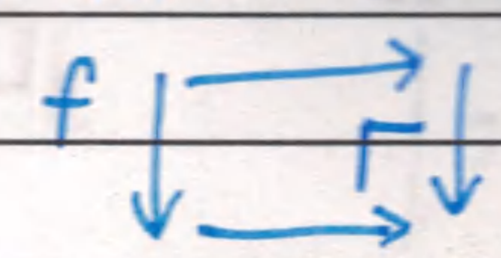
$$\begin{array}{ccc} \mathbb{R}\{y_1, \dots, y_b\} & \longrightarrow & \mathbb{R}\{x_1, \dots, x_a\} \\ y_i \downarrow 0 & & \downarrow \\ \mathbb{R} & \xrightarrow{\ \ulcorner h\ } & P \end{array}$$

$y_i \longmapsto f_i$ where $y_i : \mathbb{R}^b \to \mathbb{R}$

compute the derived pushout

$(DGC^\infty R)^{opp}$ as a model category is left proper.

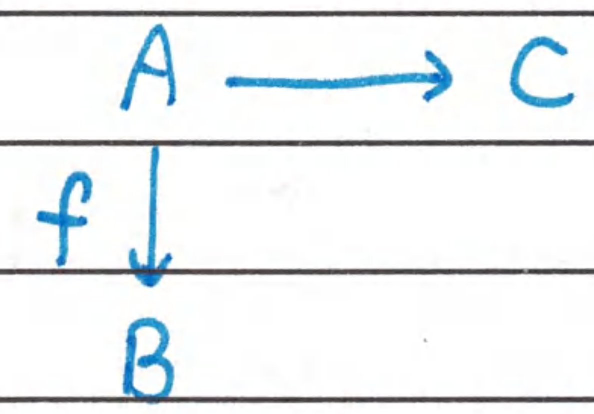
Which means that if



is an ordinary pushout square, and f is a cofibration, then it is also a homotopy pushout square (same universal properties, but instead of set of maps, we have spaces of maps)

Equivalent to the standard def of "left proper" (Dr. Pavlov's favorite)

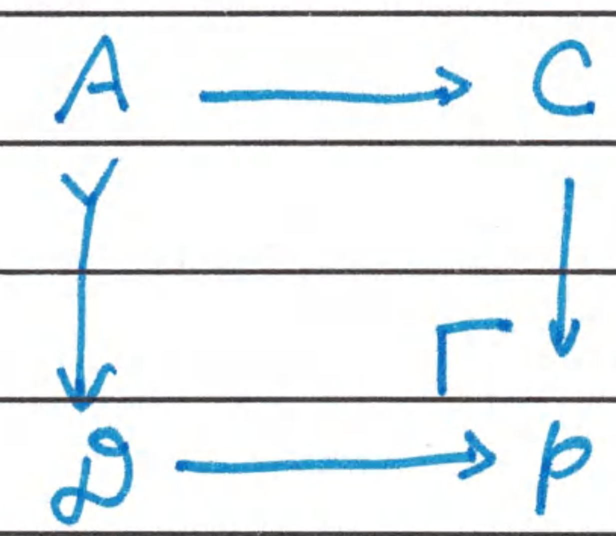
A more practical "recipe" for computing homotopy pushouts in left proper model categories; to compute the homotopy pushout of



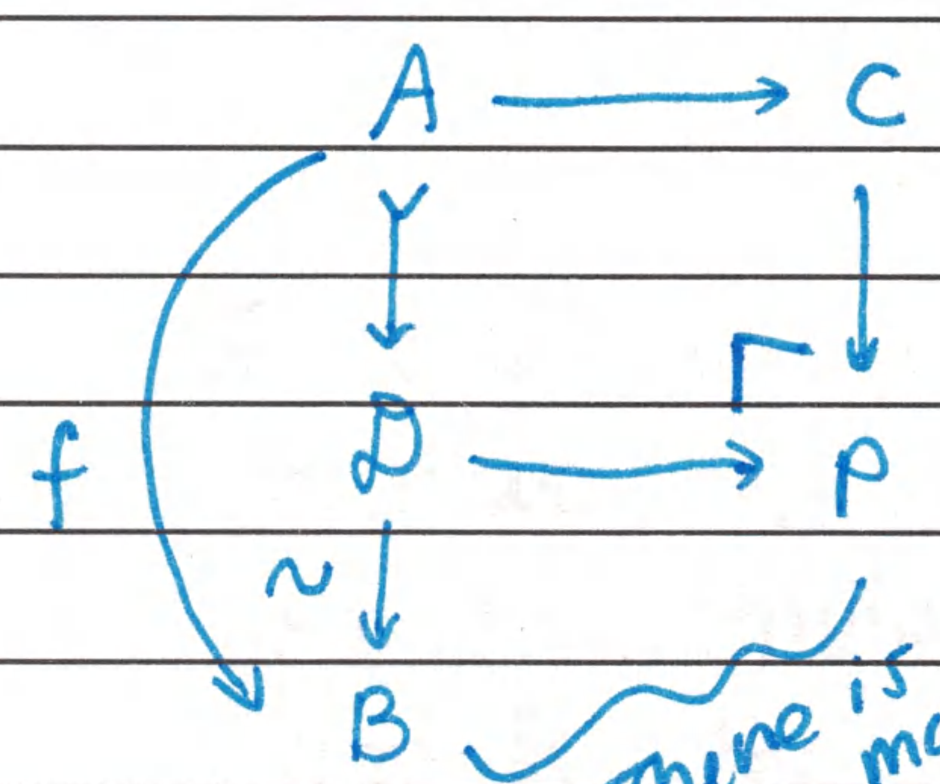
factor f as a composition of a cofibration followed by a weak equivalence

$A \rightarrowtail D \xrightarrow{\sim} B$

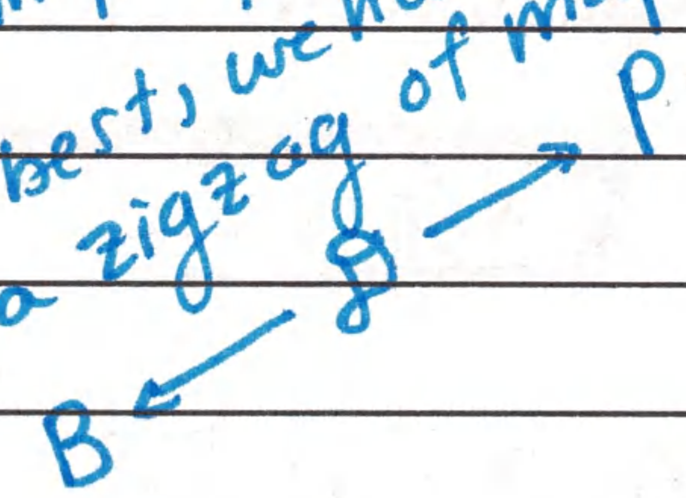
and compute the ordinary pushout of ...



... then P is the ordinary pushout!



At best, we have a zigzag of maps



Let's apply it! (The recipe) to our derived intersection example!

Factor $\mathbb{R}\{y_1, \dots, y_6\} \longrightarrow \mathbb{R}$

$= \bigotimes_i \mathbb{R}\{y_i\} \longrightarrow \mathbb{R}$

$\mathbb{R}\{y_i\} \longrightarrow \mathbb{R}\{y_i\}[y_i'] \,/\, (y_i'^2,\ dy_i' = y_i)$

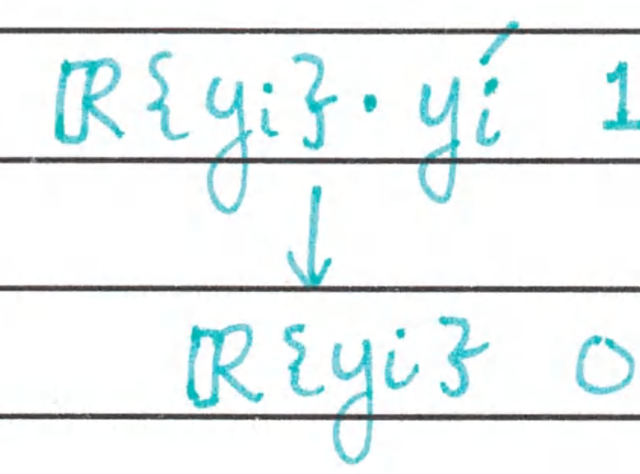
$y_i \mapsto 0$, $y_i' \mapsto 0$: $\mathbb{R}\{y_i\}[y_i'] \to \mathbb{R}$ quasiisomorphism (map to $\mathbb{R}$)

(compute) Homology:

$H_0 = \mathbb{R}\{y_i\} / y_i \mathbb{R}\{y_i\} \xrightarrow{\cong} \mathbb{R}$ evaluation at 0

$H_1 = 0$

~~functions~~



Seminar (4/1) EXAMPLES OF DERIVED INTERSECTIONS

$$\begin{array}{ccc} P & \longrightarrow & \mathbb{R}^0 \\ \downarrow & \lrcorner h & \downarrow 0 \\ \mathbb{R}^a & \xrightarrow{f} & \mathbb{R}^b \end{array}$$

We want to compute the zero locus, that is, those elements which map to zero (under f; the zero locus of f)

$\{x \in \mathbb{R}^a \mid f(x) = 0\}$

THE ZERO LOCUS

$$\begin{array}{ccc} \mathbb{R}\{y_1, \ldots, y_b\} & \xrightarrow{y_i \mapsto f_i} & \mathbb{R}\{x_1, \ldots, x_a\} \\ \downarrow & & \downarrow \\ \mathbb{R} & \longrightarrow & C^\infty D \end{array}$$

(with h marked at the corner of the square)

Resolution (based on computations last time)

$C^\infty P = \mathbb{R}\{x_1, \ldots, x_a\}\langle y_1', \ldots, y_b'\rangle$, where $dy_i' = f_i$ (the x_i in degree 0, the y_i' in degree 1; d: differentials)

and where

$H_0 \cong$ the ordinary pushout

WE WANT EXAMPLES such that $H_1 \neq 0$

Today, we will consider the case when $a = 1$, $\mathbb{R} \to \mathbb{R}$

that is, $\mathbb{R}^a = \mathbb{R}$. Additionally, consider when $b = 1$

$$C^\infty \mathbb{R} \xrightarrow[d]{(-)\cdot f_1} C^\infty \mathbb{R}$$

In addition to H_0 (the ordinary pushout),

we have $H_1 = \ker d = \{g \in C^\infty\mathbb{R} \mid g|_{\mathrm{supp} f_1} = 0\}$

(H_1 is nontrivial, and says that given curve mapping into $\mathbb{R}$, the first homology group is meaningful)

When $b = 2$, $\mathbb{R} \longrightarrow \mathbb{R}^2$

$$\mathbb{R}\{x\} \xrightarrow{-f_2 \oplus f_1} \mathbb{R}\{x_1\} \oplus \mathbb{R}\{x_1\} \xrightarrow{f_1 \oplus f_2} \mathbb{R}\{x_1\}$$

$$y_1' y_2' \qquad\qquad y_1' \qquad\qquad y_2' \qquad\qquad 1$$

since we have differential

$d(y_1' y_2') = f_1 \cdot y_2' - y_1' \cdot f_2$

So, $H_2 = \{g \in C^\infty \mathbb{R} \mid g|_{\mathrm{supp} f_1 \cup \mathrm{supp} f_2} = 0\}$

and $H_1 = Z_1 / B_1$ (cycles modulo boundaries) ... $\longrightarrow$

and what are the cycles and boundaries in this case?

$Z_1 = \{(g_1, g_2) \mid f_1 g_1 + f_2 g_2 = 0\}$ cycles

$B_1 = \{(-f_2 \cdot h, f_1 \cdot h) \mid h \in C^\infty \mathbb{R}\}$ boundaries

If $f_1 \neq 0$ (nonvanishing), then $f_1 g_1 + f_2 g_2 = 0$

$\Rightarrow g_1 = -f_1^{-1} f_2 g_2$

$\Rightarrow g_1 = -f_2 (\underbrace{f_1^{-1} g_2}_{h})$

and $g_2 = f_1 \cdot h$

thus, $H_1 = 0$

(the first homology group $= 0$)

Interesting Case: f_1 & f_2 vanish somewhere

It doesn't pass through 0

$f_1 = x^{n_1} \hat{f}_1$, $\hat{f}_1(0) \neq 0$, $n_1 \leq n_2$, $n_1 > 0$

$f_2 = x^{n_2} \hat{f}_2$ (In fact, $\hat{f}_1(t) \neq 0, \forall t$)

Now, cycles and boundaries

Z_1: $f_1 g_1 + f_2 g_2 = 0$, $x^{n_1} \hat{f}_1 g_1 = -x^{n_2} \hat{f}_2 g_2$

CYCLES and divide by x^{n_1}

$\hat{f}_1 g_1 = -x^{n_2 - n_1} \hat{f}_2 g_2$

since $\hat{f}_1$ is nonvanishing, g_1 is a multiple of $x^{n_2 - n_1}$

so...

$g_1 = x^{n_2 - n_1} \hat{g}_1$

and the equation becomes

$g_1 = -\hat{f}_1^{-1} x^{n_2 - n_1} \hat{f}_2 g_2$

$Z_1 = \{(-x^{n_2 - n_1} \hat{f}_2 h, \underbrace{\hat{f}_1 \cdot h}_{g_2})\}$

and

BOUNDARIES

$B_1 = \{(-x^{n_2} \hat{f}_2 \cdot h, x^{n_1} \hat{f}_1 \cdot h)\}$

comparing cycles and boundaries (not the same, so we have a nontrivial homology group)

$B_1 = x^{n_1} \cdot Z_1$

and

$H_1 = Z_1 / B_1 = Z_1 / x^{n_1} \cdot Z_1$

$H_1 = C^\infty \mathbb{R} / x^{n_1} \cdot C^\infty \mathbb{R}$

$H_1 = \mathbb{R}[x] / x^{n_1}$ (truncated polynomials!)

EXAMPLE: This fits:

$\{y = x^{3/2}\} \cap \{y = 0\}$

$\{y^2 = x^3\}$ $(x, y) = (t^3, t^2)$

$H_1 = \mathbb{R}[x] / x^2$

(This result is helping to answer some questions about Dr. Weinburg's research)

EXAMPLE:

Compute a surface $z^2 = (x^2+y^2)^n$ (for low cases of n) intersected with $z = 0$

[illegible]

$$\begin{array}{ccccc} P & \longrightarrow & S & \longrightarrow & \mathbb{R}^0 \\ \downarrow & \lrcorner h & \downarrow & \lrcorner h & \downarrow 0 \\ \mathbb{R}^2 & \longrightarrow & \mathbb{R}^3 & \longrightarrow & \mathbb{R} \end{array}$$

$(x_1, x_2, x_3) \longmapsto x_3^2 - (x_1^2 + x_2^2)^n$

$(x_1, x_2) \longmapsto (x_1, x_2, 0)$

The pushout square

$$\begin{array}{ccc} P & \longrightarrow & \mathbb{R}^0 \\ \downarrow & \lrcorner & \downarrow 0 \\ \mathbb{R}^2 & \longrightarrow & \mathbb{R} \end{array}
\qquad
\begin{array}{ccc} \mathbb{R}\{y_1\} & \longrightarrow & \mathbb{R}\{x_1, x_2\} \\ \downarrow & & \downarrow \\ \mathbb{R} & \xrightarrow{\ h\ } & \mathbb{R}\{x_1, x_2\}\langle y_1' \rangle \end{array}$$

$(x_1, x_2) \longmapsto (x_1^2 + x_2^2)^n$

where $dy_1' = (x_1^2 + x_2^2)^n$

Compute the homology

[illegible]

$H_1 = \{ g \in \mathbb{R}^2 \mid g|_{\text{supp} f} = 0 \}$

and even in general ($\mathbb{R}^a$)

$H_1 = \{ g \in \mathbb{R}^a \mid g|_{\text{supp} f} = 0 \}$

(detects open sets on which function vanishes)

$H_0 = C^\infty \mathbb{R}^a / (f)$