# Mathematics 5399 (Introduction to Modern Algebra II)

*Spring 2021*

**Homework 2**

*First submission due February 16, 2021.*

Reminder: all claims and statements must be accompanied by full proofs. In particular, an answer to existence questions must include a full proof of existence, or a full proof of nonexistence. Likewise, an answer to a question that asks to describe something must include a full proof that the object being described is indeed the requested object.

**1.** Consider the category $\mathsf{Set}_* = \{*\}/\mathsf{Set}$ of pointed sets, defined as the slice category of the singleton set $\{*\}$. Its objects are sets equipped with a basepoint, whereas morphisms are basepoint-preserving maps of sets. Consider the forgetful functor

$$\mathsf{U}\colon \mathsf{Set}_* \to \mathsf{Set}$$

that discards the basepoint.

- Does $U$ have a left adjoint? If it does, identify it, and describe the unit and counit of the resulting adjunction.
- Does $U$ have a right adjoint? If it does, identify it, and describe the unit and counit of the resulting adjunction.

**2.** Consider the category $\mathsf{Set}^{\to}$ of maps of sets. Its objects are maps of sets. Morphsims are commutative squares of maps of sets. Consider the forgetful functor

$$\mathsf{U}\colon \mathsf{Set}_* \to \mathsf{Set}^{\to}$$

that sends a pointed set $(A, a \in A)$ to the map of sets $a\colon \{*\} \to A$ and a basepoint-preserving map of sets $f\colon (A, a) \to (B, b)$ to the commutative square with vertical maps $\{*\} \to \{*\}$ and $f$.

- Does $U$ have a left adjoint? If it does, identify it, and describe the unit and counit of the resulting adjunction.
- Does $U$ have a right adjoint? If it does, identify it, and describe the unit and counit of the resulting adjunction.

**3.** Consider the category $\mathsf{Field}$ of fields and the category $\mathsf{CRing}$ of commutative rings. Consider also the inclusion functor

$$\mathsf{U}\colon \mathsf{Field} \to \mathsf{CRing}.$$

- Does $U$ have a left adjoint? If it does, identify it, and describe the unit and counit of the resulting adjunction.
- Does $U$ have a right adjoint? If it does, identify it, and describe the unit and counit of the resulting adjunction.

Warning: the category $\mathsf{Field}$ has few limits or colimits. Although adjoint functors preserve any (co)limits that already exist, they do not guarantee the existence of any (co)limits.

**4.** Suppose $\mathsf{C}$ is a category that has all small coproducts. Fix an object $A \in \mathsf{C}$.

- Show that the functor

  $$\mathtt{hom}(A, -)\colon \mathsf{C} \to \mathsf{Set}, \qquad B \mapsto \mathtt{hom}(A, B)$$

  has a left adjoint functor, identify it, and describe the unit and counit of the resulting adjunction. What is the value of the left adjoint on the singleton set?
- Show that the functor

  $$\mathtt{hom}(-, A)\colon \mathsf{C}^{\mathsf{op}} \to \mathsf{Set}, \qquad B \mapsto \mathtt{hom}(B, A)$$

  has a left adjoint functor, identify it, and describe the unit and counit of the resulting adjunction. What is the value of the left adjoint on the singleton set?

**5.** Consider a homomorphism of rings $f\colon R \to S$. Consider also the induced functor $f^*\colon \mathsf{Mod}_S \to \mathsf{Mod}_R$ that sends an $S$-module $M$ to the $R$-module $f^*M$. The underlying abelian group of $f^*M$ is the same as that of $M$, whereas the $R$-action is given by the composition of $f\colon R \to S$ and the $S$-action $S \to \mathsf{End}(\mathsf{U}(M))$. (You may freely use the hom-tensor adjunction for abelian groups, modules, and bimodules as necessary.)

- Does $f^*$ have a left adjoint? If it does, identify it, and describe the unit and counit of the resulting adjunction.
- Does $f^*$ have a right adjoint? If it does, identify it, and describe the unit and counit of the resulting adjunction.

**6.** Consider a map of sets $f\colon X \to Y$ and two induced morphisms of posets: $f_*\colon 2^X \to 2^Y$, which sends a subset of $X$ to its image in $Y$, and $f^*\colon 2^Y \to 2^X$, which sends a subset of $Y$ to its preimage in $X$. For each of the functors $f_*$ and $f^*$ determined whether they have left or right adjoints, identify them, and describe their units and counits.

**7.** Consider the inclusion of complete metric spaces into all metric spaces, with continuous maps as morphisms. Identify the left adjoint of this functor and compute the unit and counit.

**8.**

- Construct an injective map $f\colon M \to N$ of $R$-modules (where $R$ is some commutative ring) such that the induced map $\mathsf{T}(M) \to \mathsf{T}(N)$ of tensor algebras is not injective in some degree.
- Construct a nonzero $R$-module $M$ such that the canonical quotient map $q\colon \mathsf{T}(M) \to \mathsf{Sym}(M)$ is an isomorphism.

**9.** Suppose $R$ is an integral domain and $M$ is an $R$-module. Show that $\mathsf{T}(M)$ has no zero divisors if and only if $\mathsf{T}(M)_n$ is torsion free for all $n \geq 1$. (An $R$-module $M$ is *torsion-free* if $r \cdot m = 0$ implies $r = 0$ or $m = 0$.)

**10.** Recall the canonical quotient map $q\colon \mathsf{T}(M) \to \mathsf{Sym}(M)$, which is defined for any module $M$ over a commutative ring $R$.

- Show that $q$ does not have a section in general.
- Suppose $R$ is an algebra over $\mathbf{Q}$. Consider the forgetful functor $\mathsf{U}\colon \mathsf{GrAlg}_R \to \mathsf{GrMod}_R$. Show that there is a unique section $s$ of $\mathsf{U}(q)$ such that $s_n\colon \mathsf{Sym}^n(M) \to \mathsf{T}^n(M)$ is invariant under the action of $\Sigma_n$ on $\mathsf{T}^n(M)$.
- How does the previous part fail when $R = \mathbf{Z}/p$ for a prime $p$?