Mathematics 5399 (Introduction to Modern Algebra II)

Spring 2021

Homework 1

First submission due February 2, 2020.

1. Suppose C is a category that admits equalizers (i.e., limits for the diagram $I = \{0 \rightrightarrows 1\}$) and small products.

(a) Prove that C admits all small limits, and the limit of a small diagram $D\colon I \to \mathsf{C}$ can be computed using an equalizer of small products,

$$\mathtt{lim}_{i\in I} D(i) \to \mathtt{eq}\left(\prod_{k\in I} D(k) \rightrightarrows \prod_{f: i\to j} D(j)\right),$$

where the map to the equalizer is induced by the map

$$(p_k)_{k\in I}\colon \mathtt{lim}_{i\in I} D(i) \to \prod_{k\in I} D(k)$$

and the two maps in the equalizer are

$$(D(f)\circ p_i)_{f:i\to j}, (p_j)_{f:i\to j}.$$

(Hint: it may be helpful to expand the above formulas using the known formulas for limits and equalizers in the category Set, to see what is going on concretely.)

(b) Formulate an analogous statement for colimits and prove it by applying part (a) to the category C^{op}.

2. Suppose I and J are categories and $D\colon I \times J \to \mathsf{V}$ is a diagram.

(a) Assuming all limits and colimits below exist, construct a canonical map

$$\mathtt{colim}_{i\in I}\mathtt{lim}_{j\in J} D(i,j) \to \mathtt{lim}_{j\in J}\mathtt{colim}_{i\in I} D(i,j).$$

(b) Give an example of I, J, V, and D such that the map in part (a) is not an isomorphism.

3. Consider a category I with a single object 0 and a single nonidentity morphism $e\colon 0 \to 0$, where $e\circ e = e$. Suppose $D\colon I \to \mathsf{V}$ is a diagram. Show that the limit of D exists if and only if the colimit of D exists. Show that the limit of D is isomorphic to the colimit of D (more specifically, the corresponding apices are isomorphic). Describe, in concrete terms, the (co)limit for the case $\mathsf{V} = \mathsf{Set}$.

4. A morphism $e\colon A \to A$ in a category C is *idempotent* if $e \circ e = e$. A *splitting* of an idempotent is a pair $(s\colon A \to B, r\colon B \to A)$ such that $rs = \mathrm{id}_A$ and $sr = e$.

(a) Express the data of an idempotent morphism as a diagram $D\colon I \to \mathsf{C}$ (i.e., define I and explain how to construct D from e).

(b) Show that A is the colimit of D, with $r\colon B \to A$ being the injection map.

(c) Show that A is the limit of D, with $s\colon A \to B$ being the projection map.

5. A category I is *filtered* if the following conditions are satisfied:

- I is nonempty;
- for any object $i, j \in I$ there is $k \in I$ such that there exist morphisms of the form $i \to k$ and $j \to k$;
- for any pair of parallel arrows $f, g\colon i \to j$ there is an arrow $h\colon j \to k$ such that $hf = hg$.

(a) Show that any directed poset gives rise to a filtered category.

(b) Give an example of a filtered category that does not arise from a construction described in part (a). (Hint: look at other problems.)

(c) Show that the canonical map in Problem (2a) is an isomorphism if $\mathsf{V} = \mathsf{Set}$, I is filtered, and J is finite (i.e., has finitely many morphisms). Hint: the concrete descriptions of limits and colimits in Set may be useful.

6. A *graph* is a quadruple (V, E, s, t), where V is a set of *vertices*, E is a set of *edges*, $s, t \colon E \to V$ are the *source* and *target* maps. Any small category C has an underlying graph $\mathsf{U}(C) = (O, M, s, t)$, with O the set of objects, M the set of morphisms, and s and t being the source and target maps.

(a) Show that any graph $G = (V, E, s, t)$ admits a *free category* $\mathsf{F}(G)$ with the following universal property: for any small category C, the set of functors $\mathsf{F}(G) \to C$ is canonically isomorphic to the set of morphisms of graphs $G \to \mathsf{U}(C)$.

(b) Given an explicit description (i.e., identify all morphisms) of the free category Ψ on the following graph (draw a picture): $V = \{0, 1\}$, $E = \{S\colon 1 \to 0, T\colon 1 \to 0, I\colon 0 \to 1\}$, where the notation $A\colon b \to c$ means that $s(A) = b$, $t(A) = c$.

(c) Formulate and prove a generalization of part (a): define the notion of a category freely generated by a graph with relations (e.g., using a universal property), and prove its existence. (For an example of relations, see part (d).)

(d) Given an explicit description (i.e., identify all morphisms) of the free category Ψ on the graph from part (a), subject to the relations $SI = \mathrm{id}_0$, $TI = \mathrm{id}_0$.

7. Consider the category Ψ from Problem 5. The colimit of a diagram $D\colon \Psi \to C$ is known as a *reflexive coequalizer*. (Hint: the universal property from Problem 5 may be useful in identifying the concrete data associated with the diagram D.)

(a) Suppose S is a set and $R \subset X \times X$ is a relation on S. Construct a reflexive coequalizer diagram $\Psi \to \mathsf{Set}$ that sends $0 \mapsto S$, $1 \mapsto R$, $S \mapsto p_0$, $T \mapsto p_1$, $I \mapsto (s \mapsto (s, s))$. Prove that X/R is the colimit of this diagram.

(b) Suppose $f\colon G \to H$ is a surjective homomorphism of groups. Take $K = G \times_H G = \{(g, g') \mid f(g) = f(g')\}$. Here $G \times_H G$ is the limit of the diagram of groups $G \to H \leftarrow G$ (both morphisms are f) and as such, automatically has a group structure. Show that the assignment $0 \mapsto G$, $1 \mapsto K$, $S \mapsto p_0$, $T \mapsto p_1$ (p_i are projection maps) and $I \mapsto (g \mapsto (g, g))$ define a reflexive coequalizer diagram. Prove that H is the colimit of this diagram.

(c) Formulate and prove an analogue of (b) for ideals of commutative rings.

8. A category I is *sifted* if the following conditions are satisfied:

- I is nonempty;
- for any objects $i, j \in I$ the category of *cospans* from i to j in I is connected. Here a cospan from i to j is a diagram $i \to k \leftarrow j$ in I; a morphism of cospans from $i \to k \leftarrow j$ to $i \to k' \leftarrow j$ is a morphism $k \to k'$ that makes the two triangles with vertices i, k, k' respectively j, k, k' commute; finally, a category is *connected* if any two objects can be connected by a chain of morphisms going in either direction (e.g., a zig-zag).

(a) Show that any filtered category is sifted.

(b) Prove that the walking coequalizer, i.e., $0 \rightrightarrows 1$ is not a sifted category.

(c) Prove that the walking reflexive coeqalizer Ψ from Problem 5 is a sifted category. (A good "formula" to keep in mind: sifted colimits = filtered colimits + reflexive coequalizers.)

(d) Show that the canonical map in Problem (2a) is an isomorphism if $\mathsf{V} = \mathsf{Set}$, I is filtered, and J is finite discrete (i.e., has finitely objects and only identity morphisms, so J-limits are finite products). Hint: the concrete descriptions of limits and colimits in Set may be useful.

9. Consider the category Δ, whose objects are finite nonempty totally ordered sets and morphisms are nondecreasing maps of sets ($x \le y$ implies $f(x) \le f(y)$). Prove that Δ^{op} is sifted (Problem 8).

10. Suppose C is a variety of algebras (take the category of groups if you want). Show that the forgetful functor $\mathsf{C} \to \mathsf{Set}$ (assume there is a single underlying set for simplicity) preserves sifted colimits (you can take just reflexive coequalizers for simplicity).