MATH 6330 Notes

8-24

Resources:

- An Introduction to Manifolds - Tu
- Differential Geometry - Tu
- Manifolds and Differential Geometry - Lee
- Gauge Fields, Knots, and Gravity - Baez, Munian
- Introduction to Smooth Manifolds - Lee
- Differential Topoloy - Guillemin, Pollack
- Smooth Manifolds and Observables - Nestruev
- Mathematical Gauge Theory - Hamilton
- Principal Bundles the Classical Case - Sontz
- Mathematical Aspects of Classical Mechanics - Arnold

Recall:

Definition: A *Real Vector Space* is a module V over the ring $\mathbb{R}$. A canonical example of a real vector space is $\mathbb{R}^n$ where $n \in \mathbb{N}$.

Definition: A real vector space V is *finitely generated* if there exists $v_1, ..., v_d \in V$ such that given any $u \in V$, there exists real numbers $r_1, ..., r_d \in \mathbb{R}$ such that

$$u = \sum_{i=1}^{d} r_i v_i,$$

note that the *dimension* of the vector space V is defined to be the smallest such d that the above condition holds.

The appropriate morphisms between vector spaces are *linear maps*.

Definition: Given two vector spaces V_1 and V_2, a *linear map* $f : V_1 \longrightarrow V_2$ is a map of underlying sets that preserves all operations, i.e. such that

- $f(v + w) = f(v)$
- $f(0) = 0$
- $f(r \cdot v) = r \cdot f(v)$
- $f(-v) = -f(v)$

8-26

PROPOSITION: A d dimensional vector space is isomorphic to $\mathbb{R}^d$.
Proof: Look in any linear algebra book.

Given a d-dimensional vector space V, a specific choice of isomorphism $\mathbb{R}^d \longrightarrow V$ is referred to as a *basis*.

PROPOSITION: If $f : V \longrightarrow V'$ is a linear map, we can use the previous propsition to obtain from the diagram below, a bijection $\text{VECT}(V, V') \cong \text{VECT}(\mathbb{R}^d, \mathbb{R}^{d'}) \cong \{\text{matrices of size } d' \times d\}$ given by $f \mapsto h'^{-1} \circ f \circ h$

$$\begin{array}{ccc} \mathbb{R}^d & \xrightarrow{h'^{-1}\circ f\circ h} & \mathbb{R}^{d'} \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h'} \\ V & \xrightarrow[f]{} & V' \end{array}$$

DEFINITION: Insert definition of a topological space here.

EXAMPLE: Blah blah blah metric spaces blah blah blah

DEFINITION: Insert definition of continuity here.

EXAMPLE: If V is a finite dimensional vector space, then we can define its underlying topological space $(V, \mathcal{U})$ where $\mathcal{U}$ is defined as follows:

- Option 1: Pick a metric (norm induced by an inner product)(inner product: a bilinear, symmetric, and positive definite map) on V, and do the usual open ball business.
- Option 2: Let $A \in \mathcal{U}$ iff $A = \bigcup_{\alpha \in J} W_\alpha$ such that $W_\alpha \subseteq V$ and $W_\alpha = \bigcap_{i=1}^n Z_i$ for every $\alpha \in J$ and each Z_i is the form $f^{-1}((-\infty, a))$ for some $a \in \mathbb{R}$, f is assumed to be a linear map $f : V \longrightarrow \mathbb{R}$, and $n \in \mathbb{N}$

HW Problem: I(a)Prove the above are equivalent, (b) any linear map is continuous using Option 2

8-31

DEFINITION: Given finite dimensional $\mathbb{K}$ vector spaces V and V', open subsets $U \subseteq V$ and $U' \subseteq V'$, and a map $f : U \longrightarrow U'$, we can define two "types of derivatives" on these sets.

- The *directional derivative* of f, if it exists, is a map $V \times U \longrightarrow V'$ which maps a vector $v \in V$ and a point $x \in U$ to the vector denoted by $(\partial_v f)(x) = (D_v f)(x) = (\nabla_v f)(x) = f'_v(x)$ where each of these denotes $\lim_{t\to 0} \frac{f(x+t\cdot v)-f(x)}{t} = g'(0)$ where $g(t) = f(x + t \cdot v) - f(x)$ and $t \in \mathbb{K}$.
- The *differential* of f, if it exists, is a map $U \longrightarrow \text{HOM}(V, V')$ denoted by either Df or Tf. Given $x \in U$, we define Df by asserting that $Df(x) : V \longrightarrow V'$ is the unique linear map with the property that: given $h : u \mapsto f(u) - f(x) - (Df)(x)(u - x)$, we have $\lim_{u\to x} \frac{h(u)}{||u-x||} = 0$. We can generalize this definition (no mention of inner product/norm) by taking $h = \sum_i s_i \cdot r_i$ where $s_i : V \longrightarrow V'$ are linear, and r_i are continuous at u and $r_i(u) = 0$, and we only have finite i.

LEMMA: If V, V', U, U', f are as in the definition, Df exists, and $(\partial_v f)(x)$ exists for all x, v then

$$Df(v) = (\partial_v f)(x)$$

for all $v \in V$ and $x \in U$.

REMARK: If we only assume that $(\partial_v f)(x)$ exists for all x, v we cannot recover the above equality.

NOTE: If $V = V' = \mathbb{R}$, then we recover the usual notion of the derivative, $Df : U \longrightarrow \text{HOM}(\mathbb{R}, \mathbb{R}) \cong \mathbb{R}$.

HW: Compute the differentials of the following maps (the V_i are real vector spaces)

- $\textsc{Hom}(V_2, V_3) \times \textsc{Hom}(V_1, V_2) \longrightarrow \textsc{Hom}(V_1, V_3)$ which is defined by $(B, A) \mapsto B \circ A$
- $\textsc{Hom}(V_1, V_2) \times \textsc{Hom}(V_1, V_2) \longrightarrow \textsc{Hom}(V_1, V_2)$ which is defined by $(A_1, A_2) \mapsto A_1 + A_2$
- Consider $\mathrm{GL}(V) := \{f : V \longrightarrow V \mid f \text{ is iso.}\} \cong \{\text{invertible square matrices}\} \subseteq \textsc{Hom}(V, V)$ which is defined by $A \mapsto A^{-1}$.
- Bonus: Prove the equivalence of the second part of the above definitions of the differential.

9-2

The symmetry of higher differentials

Recall our setup from last time with V, V', U, U' and $f : U \longrightarrow U'$. We defined the differential

$$Df : U \longrightarrow \textsc{Hom}(V, V')$$

If the differential exists and we evaluate it for some $u \in U$, and evaluate the linear map $Df(u)$ at some $v \in V$, then we obtain $(Df)(u)(v) = (\partial_v f)(u)$. Suppose we take the differential of the differential, then we obtain a map

$$D(Df) = D^2 f : U \longrightarrow \textsc{Hom}(V, \textsc{Hom}(V, V'))$$

Evaluating on some $u \in U$ and subsequently on some $v_1 \in V$, and after this some other $v_2 \in V$, we obtain

$$(D^2 f)(u)(v_1(v_2)) = (\partial_{v_2} \partial_{v_1} f)(u)$$

A natural question is "what happens if you swap v_1 and v_2?" We know that nothing happens:

Proposition(Schwarz, Clairaut): $(D^2 f)(u)(v_1)(v_2) = (D^2 f)(u)(v_2)(v_1)$

Proof sketch: Apply the mean value theorem twice.

"Recall" the following proposition:

Proposition: Let V_1, V_2, V_3 be real vector spaces, then we have canonical isomorphisms

$$\textsc{Hom}(V_1, \textsc{Hom}(V_2, V_3)) \cong \textsc{Hom}(V_2, \textsc{Hom}(V_1, V_3)) \cong \textsc{Bilin}(V_1, V_2; V_3) \cong \textsc{Hom}(V_1 \otimes V_2, V_3)$$

Definition: Insert the definition of a bilinear map here.

Definition: Insert definition of tensor product here.

Insert proof that the tensor product exists here.

9-7

Proposition: If $\{e_i\}_{i=1}^n$ is a basis for V and $\{e'_j\}_{j=1}^k$ is a basis for V', then $\mathcal{B} := \{e_i \otimes e'_j\}_{i \in \mathbb{N}_{\leq n},\ j \in \mathbb{N}_{\leq k}}$ is a basis for $V \otimes V'$

Proof: The proof proceeds in two steps

1. We would like to show that $\{e_i \otimes e'_j\}_{i \in \mathbb{N}_{\leq n},\ j \in \mathbb{N}_{\leq k}}$ spans $V \otimes V'$. It suffices to show that for $v \in V$ and $v' \in V$, we can express $v \otimes v'$ as a linear combination of elements of $\mathcal{B}$. Since both V and V' have bases, we can express either vector in terms of their respective basis elements:

$$v = \sum_{i=1}^n v_i e_i \quad \wedge \quad v' = \sum_{j=1}^k v'_j e'_j,$$

therefore we recover that:

$$v \otimes v' = (\sum_{i=1}^{n} v_i e_i) \otimes (\sum_{j=1}^{k} v'_j e'_j) = \sum_{i,j} v_i v'_j (e_i \otimes e_j),$$

as desired.

2. We would like to show that the elements of $\mathcal{B}$ are linearly independent. To show this recall that $\{f_i\}_{i \in \mathbb{N}_{\leq k}} \subseteq V$ are linearly independent if and only if $\{g_i\}_{i \in \mathbb{N}_{\leq k}} \subseteq \textsc{Hom}(V, \mathbb{R})$ such that

$$g_i(f_i) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Take the dual bases $\{f_i\}_{i \in \mathbb{N}_{\leq n}} \subseteq V^*$ and $\{f'_i\}_{i \in \mathbb{N}_{\leq k}} \subseteq V'^*$. Consider the map $h_{i,j}$ which we define as the composition:

$$V \otimes V' \xrightarrow{f_i \otimes f'_j} \mathbb{R} \otimes \mathbb{R} \xrightarrow{\cong} \mathbb{R}$$

Observe that $h_{i,j} \in (V \otimes V')^*$. Now one can see that

$$h_{i,j}(e_{i'} \otimes e'_{j'}) = (f_i \otimes f'_j)(e_{i'} \otimes e'_{j'}) = f_i(e_{i'}) \cdot f'_j(e'_{j'}) \quad \text{(here we use the isomorphism)}$$

where

$$f_i(e_{i'}) \cdot f'_j(e'_{j'}) = \begin{cases} 1, & (i, i') = (j, j') \\ 0, & \text{otherwise} \end{cases}$$

This completes the proof.

Now that that's over, let's see how tensors behave under "change of coordinates". Suppose $e_1, ..., e_n$ is a basis for V and $e'_1, ..., e'_{n'}$ is a basis for V'. Then

$$e_i = \sum_k a_{i,k} e'_k$$

where the $a_{i,k}$ assemble into the "change of basis" matrix. Then

$$t = \sum_{i,j} t_{i,j} \cdot (e_i \otimes e_j) = \sum_{i,j} t_{i,j} \cdot (\sum_k a_{i,k} e'_k) \otimes (\sum_l a_{l,k} e'_l) =$$

$$= \sum_{i,j,k,l} t_{i,j} \cdot a_{i,k} \cdot a_{j,l} \cdot (e'_k \otimes e'_l)$$

So if our old coordinates are $t_{i,j}$ (this is our coefficient before $e_i \otimes e_j$) then our new coordinates (coefficients) are $\sum_{i,j} t_{i,j} \cdot a_{i,k} \cdot a_{j,l}$ This should satisfy the question "how do physical tensors correspond to mathematical tensors?" Physical tensors are simply mathematical tensors expressed in coordinates.

HW: V and W are arbitrary real vector spaces (a) Construct a canonical map $V^* \otimes W \longrightarrow \textsc{Hom}(V, W)$. (b) prove that the image of this map coincides with finite rank maps from V to W ($\text{rank}(f : V \longrightarrow W) := \dim(\text{Im}(f))$). (c) Prove that rank = tensor rank where tensor $\text{rank}(t) := \min\{n \in \mathbb{N} \mid t = \sum_{i=1}^{n} a_i \otimes b_i\}$

Definition: Let $V \in \textsc{Vect}_{\mathbb{R}}$ and let $k \in \mathbb{N}$

- $V^{\otimes k} := V \otimes ... \otimes V$ (k-times)
- $Sym^k := V^{\otimes k} / (... \otimes v \otimes v' \otimes ... - ... \otimes v' \otimes v \otimes ...)$
- $\bigwedge^k V := V^{\otimes k} / (... \otimes v \otimes v' \otimes ... + ... \otimes v' \otimes v \otimes ...)$

9-14

Definition: A *chart* on a set X is given by the following data:

- $U \subseteq X$
- W a finite dim $\mathbb{R}$ vector space
- $V \subseteq W$ open subset
- $f : U \longrightarrow V$ a bijection

Definition: The transition map t from a chart (U, V, W, f) to a chart (U', V', W', f') is defined as the composition

$$f(U \cap U') \to U \cap U' \to f'(U \cap U'),$$

which we can write succinctly as $t = f' \circ f^{-1}$, where it is understood we are (co)restricting to $U \cap U'$ where necessary.

Definition: Two charts C_1 and C_2 on a set X are said to be *compatible* if the transition maps $t_{1,2}$ and $t_{2,1}$ are smooth (C^∞) maps between open subsets of W and W', note that as maps of sets $t_{1,2}^{-1} = t_{2,1}$.

Definition: An *Atlas* on a set X is a collection of cahrts $\{C_\alpha\}_{\alpha \in J}$ on X such that for any $\alpha, \beta \in J$, C_α and C_β are compatible. Moreover, we require that if given a chart D on X such that D is compatible with C_α for all $\alpha \in J$, then $D \in \{C_\alpha\}_{\alpha \in J}$, this is equivalently stated: we require that $\mathcal{A}$ be maximal.

Definition: A *Smooth Manifold* is a set X together with an atlas $\mathcal{A} = \{C_\alpha\}_{\alpha \in J}$.

Definition: The underlying topological space of a smooth manifold $(X, \mathcal{A})$ is the topological space (X, τ) where $U \in \tau$ if

$$U = \bigcup_{\alpha \in J} f_i^{-1}(V_i)$$

where every $V_i \subseteq W_i$ is open and each W_i is some finite dim vector space over $\mathbb{R}$.

In practice, we can construct an atlas on a set X as follows:

1. Take a collection of charts $\{C_\alpha = (U_\alpha, W_\alpha, V_\alpha, f_i)\}_{\alpha \in J}$ on X such that the collection $\{U_\alpha\}$ covers X and C_α, C_β are compatible for any $\alpha, \beta \in J$.
2. Define $\mathcal{A} \coloneqq \{D \mid D \text{ is a chart } \wedge\ D \text{ is compatible with } C_\alpha \text{ for every } \alpha \in J\}$
3. $\mathcal{A}$ is the unique atlas containing $\{C_\alpha\}_{\alpha \in J}$

Examples:

- Every finite dimensional vector space V is a smooth manifold with a single trivial chart.
- If M is a smooth manifold and $G \subseteq M$ is open, then G is itself a smooth manifold. To see this, select those charts C_α on M for which $U_\alpha \subseteq G$.
- Any open subset of $\mathbb{R}^n$ is a smooth manifold using the above two examples.

The following example gets its own subheading:

Example: The Sphere

Take a finite dimensional real vector space V with inner product $\langle \cdot, \cdot \rangle : \mathrm{Sym}^2(V) \longrightarrow \mathbb{R}$. We define

$$S^V \coloneqq \{v \in V \mid \langle v, v \rangle = 1\},$$

and claim that S^V is a smooth manifold. Charts on S^V can be constructed using the stereographic projection. Details on this next time!

9-16

Example: The Sphere

Take $p \in S^V$ where S^V is defined as last time, we define the stereographic projection as

$$S_p^V : S^V \setminus \{p\} \longrightarrow \langle p \rangle^\perp \quad \text{where} \quad S_p^V(q) = \frac{1}{1 - \langle q, p \rangle} \cdot (q - \langle q, p \rangle \cdot p),$$

recall that $\langle p \rangle^\perp$ is the orthogonal complement of p. We can then interpret the above formula as the projection of q onto the orthogonal complement of p. We have an inverse

$$S_p^{V\,-1} : \langle p \rangle^\perp \longrightarrow S^V \setminus \{p\} \quad \text{where} \quad S_p^{V\,-1}(w) = \frac{2}{1 + \langle w, w \rangle} \cdot w + \frac{-1 + \langle w, w \rangle}{1 + \langle w, w \rangle} \cdot p.$$

Recall the reason we're interested in this map: it defines a chart. For any $p \in S^V$ we have a chart C_p, we need only verify (to obtain a smooth structure) that $p, p' \in S^V$, the charts C_p and $C_{p'}$ are compatible. To do this we write down the transition map

$$t : \langle p \rangle^\perp \setminus \{S_p^V(p')\} \longrightarrow \langle p' \rangle^\perp \setminus \{S_{p'}^V(p)\},$$

which is simple enough granted that we can make the definition $t = S_{p'}^V \circ S_p^{V\,-1}$, and observe that t is smooth because it is defined as the composition of smooth functions. Thus given any two $p, p' \in S^V$, we obtain compatible charts $C_p, C_{p'}$ which can be combined to provide a smooth structure on the sphere S^V.

Definition: Given a smooth manifold M, we define it's *dimension* as a map of sets $\dim_M : \pi_0(M) \longrightarrow \mathbb{N}$ where $\pi_0(M)$ is the set of connected components of M. For any $x \in \pi_0(M)$ (any open/closed connected subset of M), we define $\dim_M(x) = \dim(W)$ where W is a vector subspace in some chart $C = (U, W, V, f)$, such that $x \in U$.

Definition: Insert the definition of connectedness, local connectedness, and connected components.

Example:

- If V is a real vector space, then $\dim_M(V) = \dim(V)$
- If $U \subseteq V$ is open then $\dim_M(U) = \dim_M(V)$
- If V is a real vector space, then $\dim_M(S^V) = \dim(\langle p \rangle^\perp) = \dim(V) - \dim(\langle p \rangle^\perp) = \dim(V) - 1$

9-21

Example: Orientable surface of genus g

We discussed in detail the smooth structure on an orientable surface of genus g, charts were drawn and I didn't know how to typeset them!

HW: Prove that the non-orientable surface with $c > 0$ cross-caps is a smooth manifold.

Definition: Let $X, X' \in \text{Man}$, then a *smooth map* $g : X \longrightarrow X'$ is a map of underlying sets such that for any charts $\mathcal{C}_X$ and $\mathcal{C}_{X'}$ and elements of these charts $U \in \mathcal{C}_X$ and $U' \in \mathcal{C}_{X'}$, we require that the composition

$$f(U \cap g^{-1}(U')) \xrightarrow{\quad f^{-1} \quad} U \cap g^{-1}(U') \xrightarrow{\quad g \quad} U' \xrightarrow{\ f' \ } f'(U')$$

is C^∞, where f, f' are the usual bijections in a chart. That is, we require that the map $f' \circ g \circ f^{-1} : f(U \cap g^{-1}(U')) \longrightarrow f'(U')$ is C^∞, and that its domain is an open subset of W (W here is the understood vector space on which we model $\mathcal{C}_X$).

PROPOSITION(S):

- Smooth maps are continuous

- The identity map is smooth

- The composition of smooth maps between smooth manifolds is a smooth map

DEFINITION: Given $X, X' \in$ MAN, we can define their product in MAN denoted $X \times X'$ as follows: We take the product of underlying sets X and X', and construct charts on $X \times X'$ by taking the product of charts $\mathcal{C}_X \times \mathcal{C}_{X'}$, which are defined by taking products of all of their data (including the pairing of the canonical bijections).

HW: Prove that the elements of $\mathcal{C}_X \times \mathcal{C}_{X'}$ are compatible so that $X \times X'$ is actually a manifold. Secondly, prove that the projection maps $\pi_X : X \times X' \longrightarrow X$ and $\pi_{X'} : X \times X' \longrightarrow X'$ are C^∞. Finally, prove that if given smooth maps $h : Y \longrightarrow X$ and $h' : Y \longrightarrow X'$, then $(h, h') : Y \longrightarrow X \times X'$ defined by $y \mapsto (h(y), h'(y))$ is a smooth map.

Preview for Thursday's class:
DEFINITION: A Lie group is a group object in the category of smooth manifolds.

9-23

HADAMARD'S LEMMA: Given a smooth function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $f(0) = 0$, there exists $g_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $f = \sum_i x_i g_i$

PROOF: Observe that

$$f(x) - f(0) = \int_0^1 f_t'(t \cdot x)dt,$$

we can then write $h(t) = f(t \cdot x)$, and further note that

$$h(1) - h(0) = \int_0^1 h'(t)dt,$$

from which it follows that

$$f(x) = \int_0^1 \sum_i x_i \frac{\partial f}{\partial x_i}(t \cdot x_i)dt,$$

we can "pull the x_i's out" and let

$$g_i(x) = \int_0^1 \frac{\partial f}{\partial x_i}(t \cdot x_i)dt,$$

from which the result follows.

DEFINITION: Recall here the definition of a group.

EXAMPLES: (OF LIE GROUPS)

- Fix a finite dimensional real vector space V, then $\mathrm{GL}(V)$ is a Lie group. The reason this is a group should be clear, why is it a Lie group? First observe that $\mathrm{GL}(V) \subseteq \mathrm{End}(V)$ and $\mathrm{End}(V)$ has the structure of a real vector space. We claim that $\mathrm{GL}(V)$ is an open subset of $\mathrm{End}(V)$, from which it follows that $\mathrm{GL}(V)$ is a smooth manifold. To see that $\mathrm{GL}(V)$ is open, consider the map

$$\det : \mathrm{End}(V) \longrightarrow \mathbb{R}$$

observe that $\mathrm{GL}(V) = \det^{-1}(\mathbb{R} \setminus \{0\})$, and since we can express det as a horrible polynomial function, in particular it is continuous. Thus, $\mathrm{GL}(V)$ is the continuous pre-image of an open subset of $\mathbb{R}$, so it is an open subset of $\mathrm{End}(V)$, and in particular it is a manifold. This is not sufficient to show that $\mathrm{GL}(V)$ is a Lie group, as we must still be certain that the operations are smooth. Well, one can easily realize the coordinates of the multiplication/composition map as polynomials:

$$(B, A) \mapsto (BA)_{ij} = \sum_k B_{i,k} A_{k,j}$$

which are smooth, moreover it was proven in the homework that this multiplication/composition map has a differential, one could argue by induction that this map is C^∞. An entirely analogous argument follows for the inverse and identity maps (it is an important point that a point realized as a map from a point is a smooth map).

- The special linear group: $\mathrm{SL}(V) \leq \mathrm{GL}(V)$ is the subset of all invertible matrices A with $\det(A) = 1$.
- We can fix an inner product $\langle \cdot, \cdot \rangle$ on V and define the orthogonal linear group: $\mathrm{O}(V) = \{A \in \mathrm{GL}(V) \mid \langle Av, Aw \rangle = \langle v, w \rangle\}$
- The special orthogonal linear group: $\mathrm{SO}(V) \leq \mathrm{O}(V)$ is the subset of all orthogonal matrices of determinant one.
- Define the Hermitian inner product on a complex vector space V as $\langle \alpha, \beta \rangle = \overline{\alpha} \cdot \beta$. This inner product is a *real* bilinear map $V, V \longrightarrow \mathbb{C}$ that is complex linear in the second variable and complex anti-linear in the first variable, anti-symmetric, and positive definite. We define the unitary group: $\mathrm{U}(V) = \{A \in \mathrm{GL}_{\mathbb{C}}(V) \mid \langle Av, Aw \rangle = \langle v, w \rangle\}$ where $\langle \cdot, \cdot \rangle$ is the Hermitian inner product on V
- There's of course a special version $\mathrm{SU}(V)$

Definition: A tangent vector to a point $x \in U \subseteq M$, where $M \in$ Man, is an equivalence class of trajectories which we require to be smooth maps $p : \mathbb{R} \longrightarrow M$ such that $p(0) = x$, under the following identification:

$$p \sim q \quad \Longleftrightarrow \quad \text{in some, and hence all charts, containing } x \text{ we have :}$$

$$(f \circ p)'(0) = (f \circ q)'(0)$$

where $f : U \longrightarrow f(U)$ is the chart mentioned.

HW: Show that if the above equality holds in a single chart, it must hold in all charts.

9-28

Recall our definition of the tangent vector from last time:

Definition: A tangent vector to a point $x \in U \subseteq M$, where $M \in$ Man, is an equivalence class of trajectories which we require to be smooth maps $p : \mathbb{R} \longrightarrow M$ such that $p(0) = x$, under the following identification:

$$p \sim q \quad \Longleftrightarrow \quad \text{in some, and hence all charts, containing } x \text{ we have :}$$

$$(f \circ p)'(0) = (f \circ q)'(0)$$

where $f : U \longrightarrow f(U)$ is the chart mentioned.

This definition is functional, and motivated by physical intuition, but mathematically speaking a *curve* and a *vector* should be different concepts. We introduce the following equivalent definition to ameliorate this:

DEFINITION: Given a smooth manifold M and point $x \in M$, a tangent vector at x is a given by a family $(w_C)_{C \in \mathcal{C}_x}$ where $\mathcal{C}_x$ is the sub-collection of the atlas on M consisting of all charts that contain x, and where given a particular chart $C = (U, V, W, f)$, $w_C \in W$ and

$$w_{C'} = (Dt_{C,C'})(f(x))(w_C)$$

where $t_{C,C'}$ is the transition map from C to C'.

PROOF OF EQUIVALENCE:

(1) $\Rightarrow$ (2)

Suppose a manifold M, point $x \in U \subseteq M$, and subset U are given together with an equivalence class of curves γ with $\gamma(0) = x$. To produce a family of vectors as in definition (2) we set

$$w_C = (f \circ \gamma)'(0)$$

where $C = (U, V, W, f)$. We must show that this is well defined, to this end let $\nu \sim \gamma$. By the definition of $\sim$ we have

$$(f \circ \gamma)' = (f \circ \nu)' = w_C$$

so our choice of w_C is indeed well defined. The family $(w_C)_{C \in \mathcal{C}_x}$ is completely determined by a single choice of vector w_C, as all other members of the family can be computed using the above formula.

(2) $\Rightarrow$ (1)

Suppose a manifold M, point $x \in U \subseteq M$, and subset U are given together with a family $(w_C)_{C \in \mathcal{C}_x}$. To construct an equivalence class of curves, pick any chart $C = (U, V, W, f)$ and any smooth curve $\gamma : \mathbb{R} \longrightarrow V$ with $\gamma(0) = f(x)$ and $\gamma'(0) = w_C$, then pull this curve back onto the manifold using f. For example pick $\gamma(t) = f(x) + t \cdot w_C$ and take $[f^{-1} \circ \gamma]$. We must verify the result we have recovered is independent of our choices of γ and C', to this end suppose we have another curve $\nu : \mathbb{R} \longrightarrow V$ with $\nu(0) = f(x)$ and $\nu'(0) = w_C$. We have that $[f^{-1} \circ \gamma] = [f^{-1} \circ \nu]$ because $(f \circ f^{-1} \circ \gamma)'(0) = (f \circ f^{-1} \circ \nu)'(0)$. Now suppose we pick a different chart $C' = (U', V', W', g)$, then we consider the class $[g^{-1} \circ \gamma]$. By means of the transition map $t_{C,C'}$ we obtain that

$$[g^{-1} \circ \gamma] = [f^{-1} \circ (t_{C,C'}^{-1} \circ \gamma)],$$

so it suffices to verify that the curve $\eta \coloneqq t_{C,C'}^{-1} \circ \gamma$ satisfies $\eta(0) = f(x)$ and $\eta'(0) = w_C$. Observe that

$$\eta(0) = (t_{C,C'}^{-1} \circ \gamma)(0) = t_{C,C'}^{-1}(g(x)) = f(x),$$

and

$$\eta' = D(t_{C,C'}^{-1})(\gamma(0))\gamma'(0) = (D(t_{C,C'})(t_{C,C'}^{-1}(\gamma(0))))^{-1}(\gamma'(0)) =$$
$$= (D(t)(f(x)))^{-1}(w_{C'}) = w_C$$

HW: Complete the proof of equivalence of the above definitions by showing that, by starting with a tangent vector as in definition one, then producing a tangent vector as in definition two using the above, then producing a new tangent vector as in definition one using the above, we get the same tangent vector back. Then do it starting with a tangent vector as in definition two.

DEFINITION: Given a smooth manifold M, a point $x \in M$, and a tangent vector $v \in T_x M$, we define the directional derivative of a smooth function $f : M \longrightarrow \mathbb{R}$ in the direction of v as

$$(D_v f)(x) = (f \circ \gamma)'(0)$$

where $v = [\gamma]$ and $\gamma : \mathbb{R} \longrightarrow M$. Note that, because the vector v is tangent to x, it is somewhat meaningless to write any evaluation at x (where else would we evaluate?), so one could equivalently write $(D_v f) = (f \circ \gamma)'(0)$.

HW: Show that a different choice of representative for v produces the same directional derivative
HW: Give a definition of $D_v f$ using definition (2) of a tangent vector and prove its equivalence to the definition given above.