MATH 6330 (Manifold Theory)

Homework Problems

Fall 2021

First Set Tentative Due Date: 9/9

1. Let V be a finite dimensional real vector space, let $\langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{R}$ be an inner product (symmetric, bilinear, and positive definite map) on V, and let $f : V \longrightarrow \mathbb{R}$ be an arbitrary linear map.
   (a) Construct a topology on V using its inner product and prove it is a topology.
   (b) Define a topology $\tau \subset \mathcal{P}(V)$ by declaring $U \in \tau$ if and only if $U = \bigcup_{\alpha \in J} W_\alpha$ where $W_\alpha \subset V$ and $W_\alpha = \bigcap_{i=1}^{n} Z_i$ for every $\alpha \in J$, and where each Z_i is the form $f^{-1}((-\infty, a))$ for some $a \in \mathbb{R}$. Prove this *is* a topology.
   (c) Show that the above topologies coincide.
   (d) Use the topology constructed in (b) to prove that any linear map is continuous.
2. Let V_1, V_2, and V_3 be finite dimensional real vector spaces. Compute the differential of the following maps:
   (a) $f : \mathrm{Hom}(V_2, V_3) \times \mathrm{Hom}(V_1, V_2) \longrightarrow \mathrm{Hom}(V_1, V_3)$, where $f(B, A) = B \circ A$.
   (b) $g : \mathrm{Hom}(V_1, V_2) \times \mathrm{Hom}(V_1, V_2) \longrightarrow \mathrm{Hom}(V_1, V_2)$, where $g(A_1, A_2) = A_1 + A_2$.
   (c) $h : GL(V) \longrightarrow \mathrm{Hom}(V, V)$, where $h(A) = A^k$ and $k \in \mathbb{Z}$. Recall that $GL(V) \coloneqq \{A : V \longrightarrow V \mid A \text{ is invertible}\}$.
3. (Bonus) Recall the definition of the differential: Given finite dimensional real vector spaces V and V', open subsets $U \subseteq V$ and $U' \subseteq V'$, and a map $f : U \longrightarrow U'$, we define the differential of f, $Df : U \longrightarrow \mathrm{Hom}(V, V')$ by asserting that for each $x \in U$, $(Df)(x) : V \longrightarrow V'$ is the unique linear map such that the function $h : U \longrightarrow V'$ defined by $h(u) = f(u) - f(x) - (Df)(x)(u - x)$, satisfies one of the following equivalent conditions:
   (a) $\lim_{u \to x} \frac{||h(u)||}{||u-x||} = 0$
   (b) h can be presented as a finite sum $\sum_i l_i(u - x) g_i(u)$, where each $l_i : V \longrightarrow \mathbb{R}$ is a linear form on V, and each $g_i : U \longrightarrow V'$ is a function that is continuous at x and $g_i(x) = 0$.

   Prove that conditions (a) and (b) are equivalent.

Second Set Tentative Due Date: 9/16

1. Let V and W be arbitrary real vector spaces.
   (a) Construct a canonical map $f : V^* \otimes W \longrightarrow \mathrm{Hom}(V, W)$
   (b) Prove that $\mathrm{Im}(f) = \{A : V \longrightarrow W \mid \mathrm{Dim}(\mathrm{Im}(A)) < \infty\}$

(c) Given an element $t \in X \otimes Y$, we define its tensor rank as:

$$\text{TRank}(t) = \text{Min}\{n \in \mathbb{N} \mid t = \sum_{i=1}^{n} x_i \otimes y_i\}.$$

Prove using the map constructed in (a) that the tensor rank coincides with the usual notion of rank (the *rank* of a linear map is the dimension of its image).

2. Suppose that V is a vector space with basis $\{e_i\}_{i \in I}$ and suppose that $k \geq 0$. Prove the following:

(a) $\{e_{i_1} \otimes ... \otimes e_{i_k}\}_{i \in I^k}$ is a basis for $V^{\otimes k}$

(b) $\{e_{i_1} \cdot e_{i_2} \cdot ... \cdot e_{i_k}\}_{i \in I^k}$ with $i_1 \leq i_2 \leq ... \leq i_k$ is a basis for $\text{Sym}^k V$

(c) $\{e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_k}\}_{i \in I^k}$ with $i_1 < i_2 < ... < i_k$ is a basis for $\bigwedge^k V$

(d) Compute the dimension of the above spaces.