Last time

$Cart_0 = \{ \mathbb{A}^n \}$ $\quad Cart_1 = $ polynomials, C^∞, $\mathcal{O}$

$Cart \hookrightarrow Spaces \qquad Fun^\times(Cart; Set)^{op} = Space$

$\qquad\qquad\qquad\qquad\qquad Fun^\times(Cart; Set) = Alg$

$\mathbb{A}^n = \mathbb{A}^n_{\mathbb{Z}} \qquad Fun^\times(Cart; Set) \ni F \qquad F(\mathbb{A}^1) \qquad Hom(\mathbb{Z}^n; A) = A(\mathbb{A}^n)$

$\mathbb{A}^0 \to \mathbb{A}^1$

$* \longrightarrow 0$

$* \longrightarrow 1$

$- \quad \mathbb{A}^1 \to \mathbb{A}^1$

$+ \quad \mathbb{A}^2 \to \mathbb{A}^1$

$\cdot$

We want: Spaces $\hookrightarrow$ Gen Spaces

Polynomial: Affine sch. $\to$ R-Schemes; Algebraic spaces etc.

C^∞ $\nearrow$ Smooth mflds $\to$ $Hom(M, N)$; $Aut(M)$ etc.

$\mathcal{O}$

$Spaces \overset{\text{Yoneda}}{\hookrightarrow} Fun(Spaces^{op}; Set)$

We want to define sheaves on spaces.

$X : Spaces \qquad Hom_{Spaces}(-, X)$

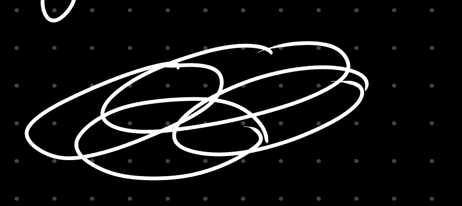
Defn: Coverage on a category $\mathcal{C}$ consists of a function assigning to each object $c : \mathcal{C}$ a collection of families of morphisms $\{\mathfrak{U}_i\}_{i:J}$ $\mathfrak{A}$ $\mathfrak{U}_i = \{ f_\alpha : U_i \to c \}_{\alpha : A}$ mathfrak{A}

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \ddot{\mathcal{C}}$

- $\mathfrak{U}$ is a covering family and $g : V \to c$

$\{ h_j : V_j \to V \}$ s.t. covering family

$$\begin{tikzcd} V_j \arrow[r] \arrow[d, "h_j"'] & U_i \arrow[d, "f_i"] \\ V \arrow[r, "g"'] & c \end{tikzcd}$$



Defn: A site is a category + coverage. Categories should be small or you should use T.T.

$F : PSh(\mathcal{C})$ is a sheaf if $F(c) \xrightarrow{\sim} \lim \left(\prod_i F(\mathfrak{U}_i) \rightrightarrows \prod_{i,j} F(\mathfrak{U}_{ij}) \right)$

$hom(-, c)$ is a sheaf (in our case).

$Gen\ Spaces = Sh(Spaces) \subset PSh(Spaces)$. Finest topology s.t. $hom_{Spaces}(-, X)$ is a sheaf

$Y : Spaces \hookrightarrow Gen\ Spaces.$ Or via localization of affine schemes.

Examples:

$Hom_{C^\infty}(M, N)$ not a manifold

$Hom_{C^\infty}(M \times W, N) = Hom_{C^\infty}(M, N)(W).$

$Sh(Man) \qquad Aut(M) : Sh(Man)$

$\eta^* \gamma \longrightarrow E\gamma \longleftarrow$ it is not unique

$\downarrow \qquad\qquad \downarrow \mathbb{R}$

$X \xrightarrow[\eta]{} Gr_1(\infty) = \mathbb{R}P^\infty \qquad \mathbb{R}^\infty = \bigcup_{n \geq 0} \mathbb{R}^n$

$VB(X) = Hom(X, Gr_1(\infty)) \qquad \widehat{\mathbb{R}}^\infty = \prod_\infty \mathbb{R}$

Computations:

$TS = Hom_{Gen\ Space}\left(\underline{Spec\ \mathbb{R}[x]/x^2}, S \right)$

$C^\infty(M) \xrightarrow{\psi} \mathbb{R} \oplus \underbrace{\mathbb{R}\langle x \rangle}$

$f \longmapsto \otimes$

$m : M \qquad\qquad \xrightarrow{\psi} \mathbb{R} \oplus \underbrace{\mathbb{R}\langle x \rangle} \xrightarrow{p_1} \mathbb{R}$

$\qquad\qquad\qquad\qquad\qquad\qquad \downarrow p_2$

$\qquad\qquad\qquad\qquad\qquad\qquad \mathbb{R}$

$C^\infty(M)_m \ni f \xmapsto[p_2 \psi]{} \partial_3(f) : \mathbb{R}$

$M, N : Man \qquad Hom_{C^\infty}(M, N) \qquad T\, Hom_{C^\infty}(M, N) =$

$= Hom\left(Spec\ \mathbb{R}[x]/x^2, Hom(M, N) \right).$

$$\begin{tikzcd} f^* T Hom(M,N) \arrow[r] \arrow[d] & T Hom(M, N) \arrow[d] \\ Spec\ \mathbb{R} \arrow[r, "f"'] & Hom(M, N) \end{tikzcd}$$

$f^* TN \qquad W : Man$

$\Gamma(f^* T Hom(M, N)) = f^* T Hom(M, N).$

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & hom(W, Hom(Spec\ \mathbb{R}[x]/x^2 ; Hom(M, N))) \arrow[d] \\ hom(W, pt) \arrow[r, "f_*"'] & hom(W, Hom(M, N)) \end{tikzcd}$$

$hom(W \times Spec\ \mathbb{R}[x]/x^2 \times M, N)$

$\qquad \|$

$\underbrace{hom(W}, Hom(M, TN))$

$$\begin{tikzcd} \Gamma(f^* TN) \arrow[r] \arrow[d] & Hom(M, TN) \ni \varphi \arrow[d] \\ pt \arrow[r] & Hom(M, N) \end{tikzcd}$$

$\varphi : M \to TN \xrightarrow{\pi} N$

$\underline{\pi \varphi =} f$

$Lie(Aut(M)) = Der(C^\infty(M)) = Vect(M)$

$\qquad \|$

$Aut(C^\infty(M)) \ni 1 \qquad = \underline{Hom(\mathbb{R}[x]/x^2; Aut(M))}$

$0 \to \underbrace{Lie(Aut\,M)}_{1 + xX} \to Aut(M)(\mathbb{R}[x]/x^2) \longrightarrow \underbrace{Aut(M)}_{\ni\, id} \to 1$

$X : \Gamma(TM)$

$id + xX$

$$\begin{tikzcd} Lie\,Aut(M) \arrow[r] \arrow[d] & T Aut(M) \arrow[d] \\ * \arrow[r, "id"'] & Aut(M) \end{tikzcd}$$

Differential forms:

M : Man $\quad (\Omega^*(M), d)$ $\quad \Omega^* : \mathrm{Sh}(\mathrm{Man})$

Ω^* : Gen Space

Proposition: $\Omega^*(\Omega^*) =$

$$\mathbb{R} \xrightarrow[0]{} \mathbb{R} \xrightarrow[1]{} \mathbb{R} \xrightarrow[0]{} \mathbb{R} \xrightarrow[1]{} \mathbb{R} \to \cdots$$

$$H^n_{dR}(\Omega^*) = \begin{cases} \mathbb{R}, & n = 0; \\ 0, & n \neq 0. \end{cases}$$

degree 1

$$\Omega^q(\Omega^1) = \mathbb{R}\langle (d\omega)^{\wedge q/2} \rangle$$
$$\mathbb{R}\langle (d\omega)^{\wedge \frac{q-1}{2}} \rangle$$

$$\Lambda^* \Omega^1_{X/\mathbb{R}}$$

Defn: $\exists\, \mathcal{F} : \mathrm{PSh}(M)$ is a sheaf if $\forall\, M = \mathcal{U}_1 \cup \mathcal{U}_2$

$$\begin{tikzcd} \mathcal{F}(M) \arrow[r] \arrow[d] & \mathcal{F}(\mathcal{U}_1) \arrow[d] \\ \mathcal{F}(\mathcal{U}_2) \arrow[r] & \mathcal{F}(\mathcal{U}_1 \cap \mathcal{U}_2) \end{tikzcd}$$

$$\mathcal{F}^m_* = \operatorname{colim}_{r \to 0} \mathcal{F}(B^m(r)).$$

Connections: $\Theta : \Omega^1(P; \mathfrak{g})$ $\quad \mathfrak{g} = \mathrm{Lie}(G)$

P is a total space of prin. G-bundle.

(i) $\Theta|_{\pi^{-1}(p)} = MC$ on G

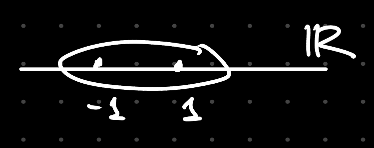
Maurer Cartan

(ii) $R_g : P \to P$ $\quad R_g^* \Theta = \mathrm{Ad}_{g^{-1}}(\Theta)$

Problem: $E_\nabla G \to B_\nabla G$

$H^*_{dR}(B_\nabla G) \leftarrow$ classify G-bundles

$\mathbb{Z}/2$ - bundles with connection over M



$\mathbb{R}$ - bundle (1-vector bundle) over M

$$\uparrow$$

$\mathrm{Gr}_1(\mathbb{R}^\infty) = \mathbb{R}P^\infty$ $\qquad H^1(M; \mathbb{Z}/2)$

$\qquad \| $

$\quad K(\mathbb{Z}/2\mathbb{Z}; 1)$

$\mathbb{Z}/2$ - coverings are classified by

$$\mathcal{G}_0 = \mathcal{G}_0(M) = \left\{ \begin{matrix} P \\ \downarrow \mathbb{Z}/2 \\ M \end{matrix} \right\} \qquad (\pi_0(\mathcal{G}))$$

$$\mathcal{G}_1(P, P') = \left\{ \begin{tikzcd} P \arrow[rr, "\varphi", "\simeq"'] \arrow[dr, "\mathbb{Z}/2"'] & & P' \arrow[dl, "\mathbb{Z}/2"] \\ & M & \end{tikzcd} \right\}$$

$$\begin{tikzcd} & \mathrm{Set}_\Delta \arrow[dr] & \\ \mathrm{Gpd} \arrow[ur] \arrow[rr, "\pi_0"'] & & \mathrm{Space} \end{tikzcd}$$

$F : \mathrm{Gpd} \to \mathrm{Set}_\Delta$ $\qquad F(\mathcal{G})_0 = \mathcal{G}_0$; $F(\mathcal{G})_1 = \mathcal{G}_1$;

$F(\mathcal{G}) = N(\mathcal{G}).$ $\qquad F(\mathcal{G}_n) = \{ \bullet \to \bullet \to \cdots \to \bullet \}$

G-bundle with connection $\rightsquigarrow$ Gpd

$$\mathcal{G}_0 = \begin{matrix} P \\ \downarrow /G \\ M \end{matrix} \qquad \Theta : \Omega^1(P; \Theta)$$

$$\mathcal{G}_1 = \begin{tikzcd} P \arrow[rr, "\varphi"] \arrow[dr] & & P' \arrow[dl] \\ & M & \end{tikzcd} \qquad \varphi^* \Theta' = \\ = \Theta$$

classifying stack.

Theorem: $H^*_{dR}(B_\nabla G) = H^*(\underbrace{\mathrm{Sym}^*(\mathfrak{g}^*)^G ; d = 0}_{\text{deRham complex.}})$

$$F(X) \xrightarrow{\sim} \operatorname{holim} \left(\prod_{\mathcal{U} : \mathcal{U}} F(\mathcal{U}) \rightrightarrows \prod F(\mathcal{U} \cap \mathcal{V}) \rightrightarrows \prod F(\cap_3) \right)$$

$F : \mathrm{Space}^{op} \to \mathrm{Gpd}$ $\qquad$ cover of X $\qquad$ Gpd