Spec functor

Algebraic category $\qquad \mathcal{A}^{op} \xrightarrow[\text{Spec}]{} \text{Spaces}$

k-Alg ; C^∞ Rings ; Sheaves of rings over S

<<Affine>> schemes

$k\text{-Alg}^{op} \xrightarrow[\text{Spec}]{} \text{Ringed Top} \qquad A \mapsto (\underset{\text{Zariski top.}}{\text{Spec } A}, A)$

$\mathbb{R}\text{-Alg}^{op} \xrightarrow[\text{Spec}]{} \underline{\text{Topological spaces}} \qquad |A| = \text{Hom}_{\mathbb{R}}(A; \mathbb{R})$

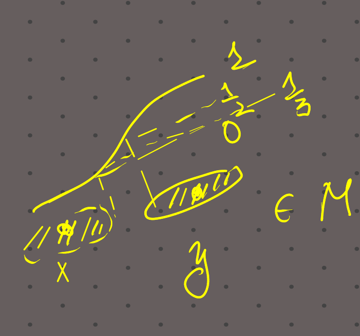
Manifolds from R-algebras and vice versa

$\mathbb{R}$-algebra $\mathcal{F}$ $\quad \text{Hom}_{\mathbb{R}}(\mathcal{F}; \mathbb{R}) =: |\mathcal{F}|$; $\quad$ Variety $\quad \mathbb{A}^n_k$

$f : \mathcal{F} \quad \mathbb{V}(f) = \{\psi : |\mathcal{F}| \mid \psi(f) = 0 \in \mathbb{R}\} \qquad \mathbb{V}(f)$

$U \subset |\mathcal{F}| \quad U = \bigcup_{W \subset \mathbb{R}} f^{-1}(W) \quad f : |\mathcal{F}| \to \mathbb{R} \qquad C^\infty(M)$ manifold

<<Zariski>> topology



Dg

$U \subset M$ is Zariski open $\iff U = \bigcup_{V \subset \mathbb{R}} f^{-1}(V)$

$\mathbb{R}$ is Hausdorff

Alg

$\mathbb{R}$ is not Hausdorff $\to$ topology on M is <<bad>>

Theorem (Milnor's exercise)

M manifold $\quad C^\infty(M) =: \mathcal{F} \quad |\mathcal{F}| \cong_{\text{homeo}} M$

Proof: $\theta : M \to |\mathcal{F}| \qquad p \mapsto (f \xrightarrow[ev_p]{} f(p))$

① θ is injective

② θ is surjective $\quad f : M \to \mathbb{R} \longleftrightarrow \rho(p_0, q)^2 =: f(q) \qquad p_0 \in M \qquad p : |\mathcal{F}|$

$\lambda := p(f) \quad \forall m : L := f^{-1}(\lambda) \quad f(m) \neq \lambda.$

$\{f_x \mid x : L\} \quad f_x(x) \neq p(f_x) \quad U_x = \{q : M \mid f_x(q) \neq p(f_x)\}$

$U_{x_1} \dots U_{x_n} \quad g = (f - \lambda)^2 + \sum_{k=1}^{m} (f_{x_k} - p(f_{x_k}))^2$

$g \neq 0$ on M

$p(g) = (p(f) - \lambda)^2 + \sum (p(f_{x_k}) - p(f_{x_k}))^2 = 0.$

$1 = p(1) = p(g \cdot \frac{1}{g}) = p(g)\, p(\frac{1}{g}) = 0, \quad W$

$U = f^{-1}(\mathbb{R}_+) \quad \square$

$\Phi : M \to N$ smooth maps between manifold

$\Phi^* : C^\infty(N) \to C^\infty(M)$

$\{x_1 \dots x_n\}$ - local coordinates

$\mathbb{R}^n, \mathbb{RP}^n, \mathbb{CP}^n$

$\mathbb{R}^n = C^\infty\{x_1 \dots x_n\} \quad \mathbb{R}[x_1 \dots x_n]$

$\mathcal{F}$ is an $\mathbb{R}$-algebra of functions somewhere

$f(p_1, \dots p_n) \qquad f : \mathbb{R}^k \to \mathbb{R} \qquad \overline{\mathcal{F}} := \underline{C^\infty\text{-completion}}$

$\mathbb{RP}^n = \text{Spec}\left(\mathbb{R}\{x_1 \dots x_{n+1}\} / (\sum x_i^2 = 1)\right)^{\mathbb{Z}/2}$

Liouville's theorem $\quad \mathcal{O}(\mathbb{CP}^n) = \mathbb{C}$

Spec $\mathbb{C} \neq \mathbb{CP}^n$

$\mathbb{CP}^n$ is a smooth manifold

?-Points of !-space = homomorphisms to ? over !

M is a smooth manifold

$M \hookrightarrow \mathbb{R}^N$ (Whitney's thm)

$* \to M \qquad \mathbb{F}_p \qquad \mathbb{F}_{p^n}$

$\uparrow$

Spec $A \qquad \text{Spec } \mathbb{F}_{p^n} \to C$

$\text{Spec } \mathbb{R} \to M \overset{1-1}{\longleftrightarrow} \text{Hom}_{\mathbb{R}}(C^\infty(M); \mathbb{R})$

$\mathfrak{m} \triangleleft C^\infty(M) \quad \{0\}$

Generalization: homomorphisms of algebras = maps

$\text{Hom}_{\mathbb{R}}(C^\infty(M); A) \overset{1-1}{\longleftrightarrow} |A| \longrightarrow M$

$A = C^\infty(N)$

Smooth, geometric $\mathbb{R}$-algebras

Serre-Swan

Line bundles over manifolds

$E \xrightarrow[\pi]{} M \longrightarrow \Gamma(\pi)$ it is a $C^\infty(M)$-module

$\bigcup_{h : |A|} P_h \equiv P/hP$

$\bigcup_{z : M} P/\{f : C^\infty(M) \mid f(z) = 0\} \cdot P = E_{set}$

$|S^*(P^\vee)| \qquad P$ f.g. projective

Serre-Swan theorem

$\text{VB}_M \underset{|-|}{\overset{\Gamma(-)}{\rightleftarrows}} C^\infty(M)\text{-Mod}_{pf}$

~~Proof~~: Sketch of proof in one direction

$e_1, \dots e_N$ - base of sections of E_η

$$\begin{array}{ccc} E_\eta & \to & E \\ \downarrow & & \downarrow \\ M & \xrightarrow[\eta]{} & G_2 \end{array} \qquad 0 \to \ker \Phi \to \mathbb{R}\langle e_1, \dots e_N \rangle \xrightarrow[\Phi]{} E_\eta \to 0$$

$\square$

Pullback of bundles = induction on modules

$$\begin{array}{ccc} E_f & \to & E \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & N \end{array} \qquad m : M$$

$S^*(\Gamma(E_f)) \longleftarrow S^*(\Gamma(E)) \quad f(m)$

$\uparrow \qquad\qquad \uparrow$

$C^\infty(M) \longleftarrow C^\infty(N)$

$\Gamma(E_f) \qquad \Gamma(E)$

$C^\infty(M) \qquad f^*$

$C^\infty(M) \otimes_{f^*, C^\infty(N)} \Gamma(E) = \Gamma(E_f)$

Tangent bundle as bundle of derivations

$C^\infty(M)$ $\quad$ Def: derivation of $C^\infty(M)$ at a point $z \in M$ is an $\mathbb{R}$-linear map $\xi : C^\infty(M) \to \mathbb{R}$ s.t. the local Leibniz rule is satisfied:

$\xi(fg) = \xi(f)\, g(z) + f(z)\, \xi(g)$

$\lambda^1 \dots \lambda^n$

Lemma: ξ - der. of $C^\infty(M)$ at $z \qquad \sum_{i=1}^n \lambda^i \frac{\partial}{\partial x_i}$

$f \in C^\infty(M) \qquad f(t) = f(z) + \sum (t_i - z_i) \left.\frac{\partial f}{\partial x_i}\right|_z + \sum (t_i - z_i)(t_j - z_j) g_{ij}(t)$

$\dim \text{Der}_z(C^\infty(M)) = \dim M. \qquad \underline{\text{Der}(C^\infty(M))} \qquad C^\infty(M) \to C^\infty(M)$

$\partial(fg) = \partial(f) g + f \partial(g).$

Kähler differentials (= universal derivations)

$\updownarrow$

$T^*M = \bigcup_{z : M} (T_z M)^\vee$

$\underline{TM}$ tangent bundle of M

$TM = \bigcup_{z : M} T_z M$, $\quad T_z M = \text{Der}_z(C^\infty(M))$

$\Gamma(T^*M) = \Lambda^1(M)$

Proposition: Let P be a module over $C^\infty(M)$ then

$\text{Hom}_{C^\infty(M)}(\Lambda^1(M); P) \xrightarrow{1-1} \text{Der}_{C^\infty(M)}(P) \quad \gamma \mapsto \gamma \circ d$

$d : C^\infty(M) \to \Lambda^1(M) \qquad f \mapsto df$

$\langle dx_i, \frac{\partial}{\partial x_j} \rangle = \delta_{ij} = \begin{cases} 0 & i \neq j; \\ 1 & i = j. \end{cases}$

General approach to differential forms on <<spaces>> via universal derivations

$\Lambda^* \Lambda^1(M) = \Omega^*(M) \leftarrow$ general construction of dif. form.

Poincaré lemma

Let $M = V \qquad \underline{H^*_{dR}(M) = 0} \quad * > 0 \qquad \underline{H^0_{dR}(M) = \mathbb{R}}$

Proof: $V^\vee \xrightarrow{id} V^\vee \qquad \Lambda^*_{C^\infty}(V^\vee \xrightarrow{id} V^\vee)$

$C^\infty\{V^\vee\} \to \Lambda^1(V)$

$\text{CoCh}(\mathbb{R}) \underset{\square}{\overset{\Lambda^*_{C^\infty}}{\rightleftarrows}} \text{DGC}^\infty\text{A} \qquad \Lambda^*_{C^\infty}(0) \simeq \underset{0}{\mathbb{R}} \to 0 \to 0 \dots$

$V \to [V^\vee \xrightarrow{id} V^\vee]$

$\text{Vect}_{\mathbb{R}} \underset{\to}{\overset{\square}{\leftarrow}} \text{CoCh}(\mathbb{R})$

$\square$