# Quantum Homotopy 9-2

The definition of a functorial field theory

Let's briefly recall what we would like to formalize. We would like to formalize functors

$$\textsc{Bord}_{d-1,d} \longrightarrow \textsc{Hilb}$$

where these categories are defined as in the previous lecture. We run into trouble defining composition in the category $\textsc{Bord}_{d-1,d}$, the usual thing to do is "glue" bordisms, however one quickly runs into problems with this definition as the usual notion of gluing destroys geometric structures on our manifolds. We can remedy this solution by equipping our objects with $d-$dimensional collars. That is given a manifold $M$, we replace it with a manifold $M' \cong M \times (-1,1)$, and we only define our geometric structure on the $d$-dimensional open manifold. The takeaway is that now everything in sight becomes $d$-dimensional. Morphisms between these "collared" manifolds are still bordisms but now together with embeddings from the source and target manifolds into the ambient bordism between them. Composition follows from identifying along the images of these embeddings which are open. We come to the question: "What is a geometric structure?"

To every $M \in \textsc{Man}_d$ we would like to assign geometric structures on $M$ such that we can restrict to geometric structures along open embeddings, and such that these geometric structures can be glued with respect to open covers. To formalize this we can say we would like geometric structures to be sheaves on the site $\textsc{Man}_d$, where the site structure comes from asserting that the objects of $\textsc{Man}_d$ are $d$-dim smooth manifolds, morphisms are open embeddings, and covering families are given by open covers. Well, "a sheaf of what??"

- Sets? This can encode: Riemannian metrics, smooth maps to a fixed smooth manifold (spacetime for example), symplectic/conformal/complex/Kähler structures, differential forms of a fixed degree. The first things that we fail to encode however are: principal G-bundles with connection (known in physics as gauge fields). Although to each manifold $M$ we can assign to it the set of all principal $G$-bundles on it in a compatible way, this assignment does not encode morphisms. Well, principal $G$-bundles form a groupoid so maybe we should take a sheaf of:
- Groupoids? These will work, however we need to formalize the idea of a sheaf of groupoids properly, this idea will be referred to as a stack. Even these do not encode: geometric string structures and bundle gerbes with connection. To encode these we need the semi-ultimate sheaf, a sheaf of:
- Simplicial Sets! However one must take care to interpret this as a *homotopy coherent* sheaf.

Here we switch gears for a moment, we need to require that field theories be smooth. However, in order for us to assert our functor is "smooth", we need the appropriate notion of "smoothness" to be present in our arbitrary target category. The solution to this problem is to replace our usual symmetric monoidal category with *sheaves* of symmetric monoidal categories on the site $\textsc{Man}$ which has objects: smooth manifolds, morphisms: smooth maps, together with covering families given by open covers. A field theory then becomes not just a symmetric monoidal functor but, a morphism of homotopy coherent sheaves of symmetric monoidal categories on the site $\textsc{Man}$.