Quantum Homotopy 8-26

Our goal is to understand the following paper: The AKSZ Construction in derived algebraic geometry as an extended topological field theory: Calaque, Haugseng, Scheimbauer.

Definition: A functorial field theory is a symmetric monoidal functor $\text{Bord}_{d-1,d} \longrightarrow \text{Hilb}$ where Hilb is the category of Hilbert spaces: objects are Hilbert spaces and morphisms are continuous maps, the tensor product is given by the completed tensor proudct of Hilbert spaces; $\text{Bord}_{d-1,d}$ is the category of bordisms: objects are $d-1$ dimensional manifolds M, possibly with additional structure, e.g.

- a smooth map $M \longrightarrow T$
- a Riemannian metric on M
- an orientation on M
- a principle G-Bundle with connection
- a conformal structure (an equivalence class of Riemannian metrics where two R.M.'s are equivalent if they are related through multiplication by a positive real function)

Morphisms in the the category of bordisms are d dimensional bordisms, composition of bordisms is given by gluing, and the monoidal structure is given by the disjoint union.

Original Motivation: Functional integral

If we take $d = 1$ we obtain traditional QM/QFT. Objects in our bordism category will be zero dimensional manifolds (points), and a bordism between two objects becomes a "trajectory". In the above first example of a geometric structure, letting T be four dimensional spacetime, we obtain a bona fide trajectory. Applying the above functor to an object in this category produces a Hilbert space, the space of its states, and applying this functor to a morphism produces a linear map of Hilbert spaces, functoriality encodes the semigroup property. $d = 2$ encodes string theory. Objects/morphisms in the bordism category M and B, are typically equipped with a *field bundle*, a typical example of which is a principle G-bundle. A *field* is a section of such a field bundle, so we can speak of fields on M and B, as objects are the boundaries of morphisms in Bord, fields on morphisms restrict to fields on objects. We can depict this behavior in a diagram:

$$Fields(M_1) \xleftarrow{\;r_1\;} Fields(B) \xrightarrow{\;r_2\;} Fields(M_2)$$

In physics, hands begin waving some. We would like to ask ourselves "what is the Hilbert space $\mathcal{F}(M)$"? We take "L^2"(Fields(M)). In $d = 1$ we have Fields$(M) = T$ provided that M is a single point (here T is some target space e.g. 4D space-time), this follows from the fact that Fields$(M) = \text{Maps}(M, T)$, which brings us to $\mathcal{F}(M) = L^2(M)$. If M is made up of k-points, then we obtain $\mathcal{F}(M) = L^2(M)^{\otimes k}$. In $d = 1$, Fields(M) generally becomes infinite dimensional and things become very hand wavy. In keeping with the interpretation of abstract trajectories, Fields(M_-) serves as a space of possible (abstract) beginning/ending points, and Fields(B) serves as a space of possible (abstract) trajectories. Feynman suggested the following in $d = 1$: Make the definition

$$\mathcal{F}(B)(f)(t) = \int_{t:B \longrightarrow T} f(r_1(t))dt$$

Where $f : C^\infty(M_1, T) \longrightarrow \mathbb{R}$ or "f : Fields$(M_1) \longrightarrow \mathbb{R}$", and $t_2 : M_2 \longrightarrow T$ or "$t_2 \in$ Fields(M_2)", and $r_2(t) = t_2$. This encodes Feynman's path integral over all possible trajectories. While in $d = 1$, this integral can be shown to converge, in higher dimensions things go wrong (no higher dimensional Wiener measure). This leads us to take interest instead in the formal properties this integral must satisfy, thus bringing us to the axiomatic perspective of QFT.