# Mathematics 6322 (Homological Algebra II) and 6332 (Geometric Mechanics)

## Midterm 1

Complete this take-home midterm in groups of at most 5 people. Write down your solutions (one solution per group) and put them in my mailbox.

**1.** Suppose $M$ is a smooth manifold and $L$ is a locally constant sheaf of abelian groups on $M$. The latter means that $L$ is a contravariant functor from the poset of open sets of $M$ to the category of abelian groups that satisfies the descent condition and the following local constancy condition: locally on $M$ (with respect to some open cover $V$ of $M$) the sheaf $L$ is isomorphic to the sheafification of the constant presheaf that sends any open set to a fixed abelian group $A$ and any inclusion of open sets to the identity map on $A$.

- Show that there is an equivalence between the category of locally constant sheaves of abelian groups on $M$ and the category of local systems of abelian groups on $\mathsf{Sing}(M)$. (Recall that the latter can be defined, for example, as functors $\pi_{\le 1}(\mathsf{Sing}(X)) \to \mathsf{Ab}$.)
- Show that the $n$th sheaf cohomology of $M$ with coefficients in the sheaf $L$ is isomorphic to the $n$th simplicial cohomology of the simplicial set $\mathsf{Sing}(X)$ with coefficients in the local system that corresponds to $L$ under the equivalence in the previous item.

**2.** Explain how you would define the cup product of two sheaf cohomology classes. Specifically, suppose $M$ is a manifold, $A$ is a commutative ring and $x \in \mathsf{H}^m(M, A)$, $y \in \mathsf{H}^n(M, A)$ are two sheaf cohomology classes.

- Define a class $x \cup y \in \mathsf{H}^{m+n}(M, A)$. Hint: the level of technicality in this problem may vary substantially depending on whether you work with presheaves of chain complexes or presheaves of simplicial sets.
- Establish a relationship between $x \cup y$ and $y \cup x$.

**3.** Suppose $M$ and $N$ are smooth manifolds. Recall the internal hom $\mathsf{Hom}(M, N)$, which is a presheaf of sets on the cartesian site defined as follows:

$$\mathsf{Hom}(M, N)(S) = \mathsf{hom}(S \times M, N),$$

where the right side denotes the set of smooth maps $S \times M \to N$. The presheaf structure maps are given by precomposition.

- Compute the tangent space to $\mathsf{Hom}(M, N)$ at some point $f\colon M \to N$ in terms of relative vector fields along $f$.

Recall also the presheaf $\Omega^k$ of differential $k$-forms, which is defined as follows: $\Omega^k(S)$ is the set of differential $k$-forms on $S$. The presheaf structure maps are given by pullbacks of differential forms. A differential $k$-form on $\mathsf{Hom}(M, N)$ is by definition a morphism of presheaves of sets

$$\mathsf{Hom}(M, N) \to \Omega^k.$$

- Explain (with a proof) what such a differential $k$-form is in concrete terms, without making any references to sheaves.

**4.** Suppose $X$ is a topological space (not necessarily a manifold) and $A$ is a sheaf of abelian groups on $X$.

- Explain how to define the sheaf cohomology of $X$ with coefficients in $A$.
- Give a concrete description (with proof) of the 0th sheaf cohomology group of $X$ with coefficients in $\mathbf{Z}$. (Be sure not to impose any additional conditions on the topological space $X$.) Hint: in the category of simplicial presheaves (or presheaves of nonnegatively graded chain complexes) any sheaf of abelian groups (placed in simplicial degree 0 or chain degree 0) automatically satisfies the homotopy descent condition.