HOMOLOGICAL ALGEBRA II

"I guess this is Homological Algebra II..."

0

~ DR. PAVLOV

Recall,

a presheaf on a small category $\mathcal{C}$ is a functor

$$F: \mathcal{C}^{op} \longrightarrow \text{set}$$

"speaking of functors as presheaves indicates operations that one wants to do apply to these functors, or certain properties one wants to check"

– nLab

DEFINITION Given $\mathcal{C} \in \text{Cat}$ (small category)

presheaves $PSh(\mathcal{C}) := Fun(\mathcal{C}^{op}, \text{set})$

simplicial presheaves $sPSh(\mathcal{C}) := Fun(\mathcal{C}^{op}, \text{sSet})$

CONSTRUCTION!

There is a FORGETFUL FUNCTOR "U" ⟹

simplicial presheaves
$Fun(\mathcal{C}^{op}, \text{sSet})$

$\mathcal{U}$

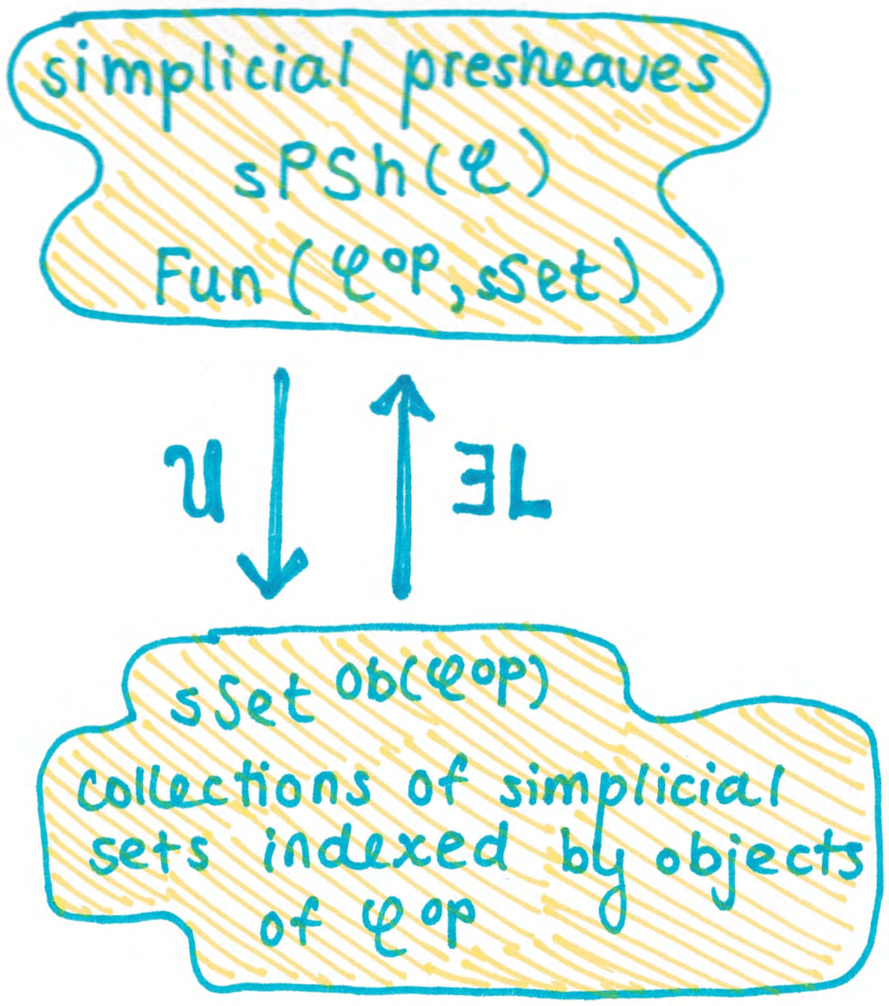
sSet $^{ob(\mathcal{C}^{op})}$
collections of simplicial sets indexed by objects of $\mathcal{C}^{op}$

$F: \mathcal{C}^{op} \longrightarrow \text{sSet}$

forgets the data about the morphisms of $\mathcal{C}^{op}$ and indexes the sSet by the elements of $\mathcal{C}^{op}$ that map into it!

Recall, $ob(\mathcal{C}) = ob(\mathcal{C}^{op})$

This functor $\mathcal{U}$ has a left adjoint functor: constructing a free presheaf on a collection of objects in sSet $^{ob(\mathcal{C}^{op})}$.



L is a cocontinuous functor (as it is left adjoint to U), and can be computed

OBJECTWISE!

That is, the value of L on some family in $sSet^{ob(\mathcal{C}^{op})}$

can be determined as follows:

Take $\{X_i\}_{i \in ob(\mathcal{C})} \in sSet^{ob(\mathcal{C}^{op})}$

Apply $L(\{X_i\}_{i \in ob(\mathcal{C})})$

$$\| $$

$$L\left(\coprod_{i \in ob(\mathcal{C})} X_i\langle i\rangle\right)$$

$$\| $$

$$\coprod_{i \in ob(\mathcal{C})} L(X_i\langle i\rangle)$$

notation: $X_i\langle i\rangle$

The family $\{X_i\}$ can be written as the coproduct of simple families, with X_i in position i and 0 elsewhere

L is cocontinuous, so ~~sum~~ commutes with coproducts!

so, we need only to determine the value of each $L(X_i\langle i\rangle)$

$$L(X_i\langle i\rangle) = \left(j \in \mathcal{C}^{op} \longmapsto hom_{\mathcal{C}}(j,i) \times X_i\right)$$

L takes ~~a bunch of~~ elements in a sSet labeled by objects of $\mathcal{C}^{op}$, and sends each of these objects to the product $hom_{\mathcal{C}}(j,i) \times X_i$, (where $hom_{\mathcal{C}}(j,i)$ is a set)

so, we have (This is a free object!)

$$\coprod_{i \in Ob(\mathcal{C})} L(Xi\langle i\rangle) = \coprod_{i \in Ob\mathcal{C}} \left(j \in \mathcal{C}^{op} \longmapsto hom_{\mathcal{C}}(j,i) \times Xi \right)$$

[NOTE: alternatively, view the map $j \longmapsto hom_{\mathcal{C}}(j,i) \times Xi$ as $L(\{Xi\}_{i \in Ob\mathcal{C}})(j) = \coprod_{i \in Ob\mathcal{C}} hom_{\mathcal{C}}(j,i) \times Xi$, the presheaf whose value on $j \in Ob\mathcal{C}$ is some ~~product~~ coproduct ↗.]

So, we now have a very explicit description of the left adjoint functor, L, as a free presheaf on a family of objects.

END OF CONSTRUCTION!

Recall,

to define a model structure on a category, we must specify 5 classes of maps:

- weak equivalences
- fibrations
- acyclic fibrations
- cofibrations
- acyclic cofibrations

DEFINITION

Given categories $\mathcal{C}$ and $\mathcal{D}$, the "functor category" $\mathcal{D}^{\mathcal{C}}$, is a category whose objects are functors $F: \mathcal{C} \longrightarrow \mathcal{D}$ morphisms are natural transformations between these functors

There are two obvious ways to put a model structure on a functor category, called the "projective" and the "injective" model structures. - nLab

⟹ We will consider the projective model structure!

DEFINITION The projective model structure on simplicial presheaves is transferred from

$$\mathrm{sset}^{\mathrm{ob}(\mathscr{C}^{op})}$$

↑ functor category

[this structure is "right-transferred", and sset could be replaced by any arbitrary cofibrantly generated model category]

☑ weak equivalences : $W_{sPSh(\mathscr{C})} := \mathcal{U}^* W_{\mathrm{sset}^{ob\mathscr{C}}}$

pullback of weak equivalences of $\mathrm{sset}^{ob\mathscr{C}}$

☑ fibrations : $F_{sPSh(\mathscr{C})} := \mathcal{U}^* F_{\mathrm{sset}^{ob\mathscr{C}}}$

☑ acyclic fibrations : $AF_{sPSh(\mathscr{C})} := \mathcal{U}^* AF_{\mathrm{sset}^{ob\mathscr{C}}}$

and the other two classes (☑ cofibrations & ☑ acyclic cofibrations) can be described as follows:

PROPOSITION Given $X, Y \in sPSh(\mathscr{C})$ and $f: X \longrightarrow Y$

The map f is a projective cofibration iff the following two conditions are satisfied:

① degenerate simplices "split off" as a disjoint summand under coproducts

② nondegenerate simplices are a retract coproduct of representable presheaves

(and now we will clarify ① and ② in more detail →)

NOTE: For reference, see Math stack Exchange article "Necessary conditions for cofibrancy in global projective model structure" based on "Nonsimplicial presheaves" by Buggar.

① degenerate simplices "split off" as a disjoint summand under coproducts

Recall,

DEFINITION

The map $A \longrightarrow B$ is "split" if it is isomorphic to a map

$$A \xrightarrow[lA]{} A \sqcup C$$

$\forall n \in \Delta^{op}$

there is a map

$$X_n \underset{X_{n,deg}}{\sqcup} Y_{n,deg} \longrightarrow Y_n$$

($X_{n,deg} \subset X_n$ and $X_{n,deg} \hookrightarrow Y_{n,deg}$, so we can glue along $X_{n,deg}$, a common intersection)

this means we can write Y_n as a coproduct with $X_n \underset{X_{n,deg}}{\sqcup} Y_{n,deg}$ as a summand

so $Y_n = (X_n \underset{X_{n,deg}}{\sqcup} Y_{n,deg}) \sqcup Y_{n,nondeg}$

$X_{n,deg}$: degenerate simplices "split off"

⋮

② nondegenerate simplices are a retract coproduct of representable presheaves

(Recall "retract" ⇒ class of cofibrations generated by transfinite compositions of cobase changes)

In the proof of this proposition, consider the pushout diagram:

$$\begin{array}{ccc} \partial\Delta^n \times \hom(-,c) & \longrightarrow & X \\ \downarrow & \ulcorner & \downarrow \\ \Delta^n \times \hom(-,c) & \longrightarrow & X' \end{array}$$

(If $X = \emptyset$, then we have the "split" $Y_{n,deg} \longrightarrow Y_n$)

$X'_n = X_n \sqcup \hom(-,c)$

In degree n, we add a single ~~element~~ representable (nondeg. n-simplices)

* Example *

Cartesian Smooth Manifolds

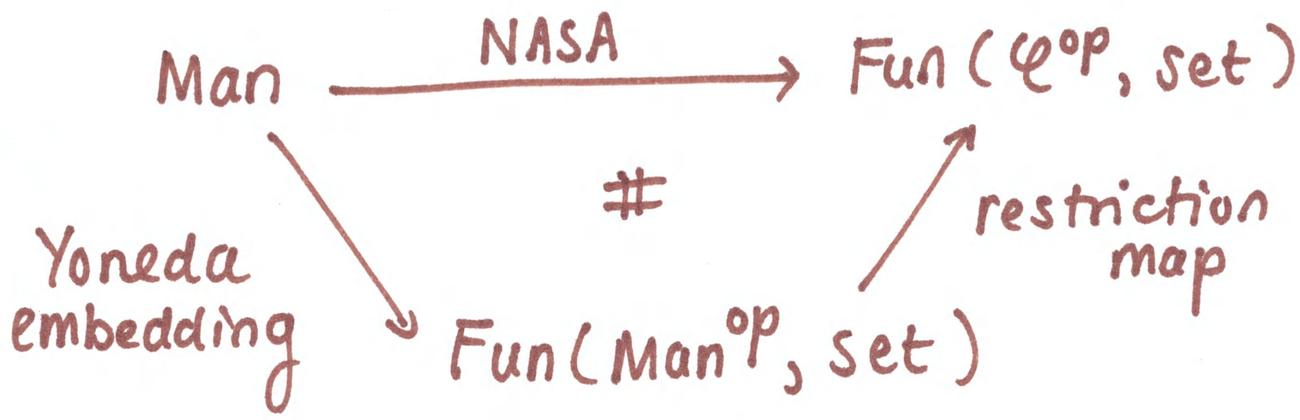
Let $\mathcal{C} = \mathrm{Cart}$

We have the functor: $\mathrm{Man} \xrightarrow{\text{NASA}} \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{set})$

the "NASA" functor, which studies spaces by launching probes into it!

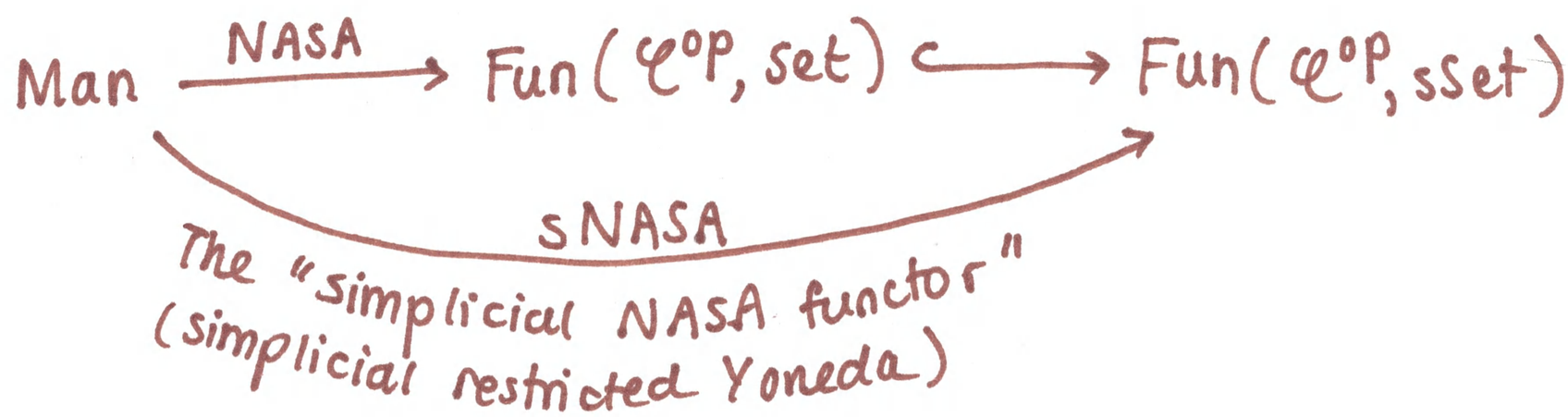
The NASA functor is also called the "restricted Yoneda embedding"

because it is the restriction of the Yoneda embedding to Cartesian manifolds.



We also have the ~~injective map~~ faithful functor, $\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{set}) \hookrightarrow \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{sSet})$ induced by the inclusion of sets into simplicial sets (set is a sset by discrete construction)

so we have the composition



For $M \in \mathrm{Man}$, is $\mathrm{sNASA}(M)$ projectively cofibrant? $\longrightarrow$

$sNASA(M)$ is projectively cofibrant

iff

$sNASA(M)$ is projective

iff

$M \in Cart$ or $M = \coprod Xi$ where $Xi \in Cart$

(M is a cartesian space)

(M is a coproduct of cartesian spaces)

[NOTE: $sNASA(M)$ does not decompose as a coproduct if M is connected. This can be seen by constructing an analog of π_0 for presheaves and considering the case: $\pi_0 sNASA(M) = \{*\}$.]

Thus, the only projective presheaves of Cart are coproducts of Cartesian spaces themselves!

and the moral of this example:

Any manifold other than a coproduct of cartesian spaces is not cofibrant!

PROPOSITION $\forall M \in \mathrm{Man}$

$\exists \{U_i\}_{i \in I}$, $U_i \subset M$ open

and $\bigcup_{i \in I} U_i = M$ ⟩ $\{U_i\}$ is a cover of M

such that

$\forall k \geq 0$, $\forall j$: $U_{j_0} \cap \cdots \cap U_{j_k}$ is either empty ($\emptyset$) or isomorphic to a cartesian space ($\cong V \in \mathrm{Cart}$)

(arbitrary intersection)

$\{U_i\}$ is called a "good cover" of M

THE PROPOSITION STATES:
Any manifold admits a good open cover



GOOD!

In the proof, recall that any smooth manifold admits a smooth triangulation (Cairns and Whitehead, 1930's)

Let $I :=$ the set of vertices in the triangulation

$U_i :=$ the open star of $i \in I$

Then we can compute the intersections:

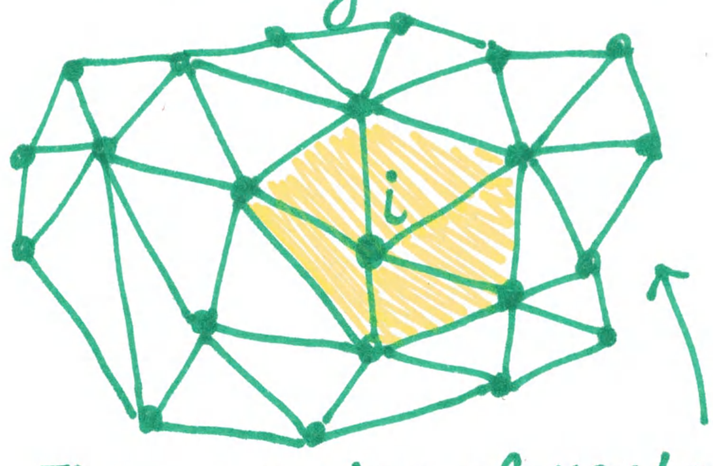
$$U_{j_0} \cap \cdots \cap U_{j_k} = \begin{cases} \emptyset & \text{if there is no simplex } \sigma \text{ with vertices } j_0, \cdots, j_k \\ \text{open star of } \sigma, & \text{otherwise} \end{cases}$$

The open star of σ is a cartesian space, namely

$$\cong \mathbb{R}^{\dim(M)}$$

hence, intersection is either empty or a cartesian space. ▨

Recall, the open star of a vertex is the union of all simplices containing i as a vertex:



The open star of vertex i, and the open star of a simplex σ is the union of the interiors of all simplices that have σ as a face.

PROPOSITION

$\check{C}(\mathcal{U}) \longrightarrow NASA(M)$

is a projective cofibrant replacement of $Y(M)$ (Yoneda)

[in sPSh(Cart)]

Recall, $(\check{C}(\mathcal{U}))_k := \coprod_j Y(\mathcal{U}_{j_0} \cap \cdots \cap \mathcal{U}_{j_k})$

↑ only nonempty intersections omit any empty!

where $j_0, \ldots, j_k$ are vertices of a k-simplex

Čech nerve construction:

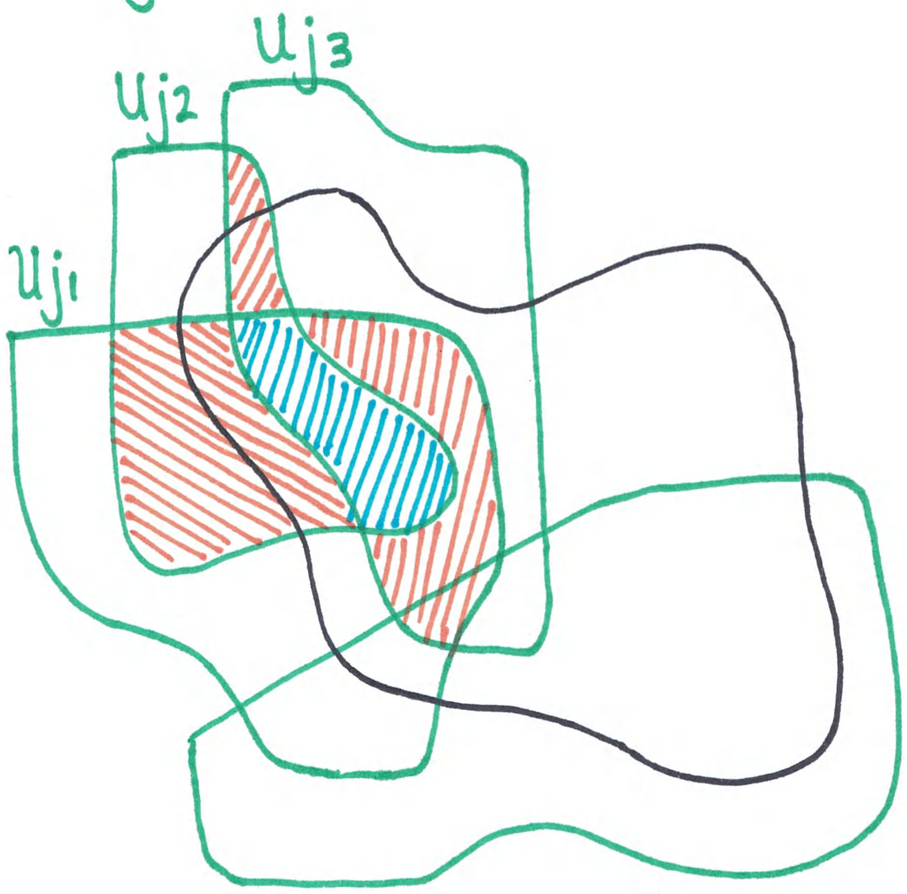
the intersection will be included into the intersection with the removed factor:

$$\coprod_j Y(\mathcal{U}_{j_0} \cap \cdots \cap \widehat{\mathcal{U}_{j_i}} \cap \cdots \mathcal{U}_{j_k})$$

removed

$$\cup\!| \quad \cup\!|$$

$$\coprod_j Y(\mathcal{U}_{j_0} \cap \cdots \cap \mathcal{U}_{j_i} \cap \cdots \mathcal{U}_{j_k})$$



$$\coprod_j Y(\mathcal{U}_{j_1} \cap \mathcal{U}_{j_2} \cap \mathcal{U}_{j_3})$$

$$\cap\!| \quad \cap\!|$$

$$\coprod_j \mathcal{U}_{j_1} \cap \mathcal{U}_{j_2} \quad \text{(or any other two!)}$$

NOTE: face maps will remove a vertex, hence, remove a factor

~~(face intersection with the removed factor will be included into the intersections)~~

→ degeneracy maps will repeat a vertex, hence, duplicate a factor (the intersection will be unchanged)

Actually writing it out:

$k=0$ $\quad (\check{C}(\mathcal{U}))_0 = \coprod_j Y(\mathcal{U}_{j_0})$

$k=1$ $\quad (\check{C}(\mathcal{U}))_1 = \coprod_j Y(\mathcal{U}_{j_0} \cap \mathcal{U}_{j_1})$

NOTE: The intersection is only taken when nonempty; the Yoneda functor is not defined the same way when applied to $\emptyset$, that is,

$$Y(\emptyset) \not\cong \emptyset$$

$Y(\emptyset)$: initial object in Man

$\emptyset$: initial object in sPSh

PROOF By the previous proposition, we need only verify two conditions:

① Degenerate simplices "split off"

$Y_{n,deg}$

Here, degenerate simplices occur when $j_p = j_{p+1}$ for some p

(repeating factor means that you are in the image of some degeneracy map)

These simplices can be "split off" from the others in the coproduct

② Nondegenerate simplices can be written as a retract of a coproduct of representable presheaves.

$Y_{n,nondeg}$

Here, nondegenerate simplices occur whenever $j_p \neq j_{p+1}$, $\forall p$

This is, already, a coproduct of representables!

so, the Čech nerve is cofibrant (in the projective model structure).

Now, to describe (local) acyclic fibrations we will show the lifting property with respect to cofibrations.

$$\check{C}(\mathcal{U}) \longrightarrow NASA(M)$$

is cofibrant

Consider the diagram:

$$\begin{array}{ccc} \partial\Delta^n \otimes Y(V) & \longrightarrow & \check{C}(\mathcal{U}) \\ \downarrow & \nearrow & \downarrow \\ \Delta^n \otimes Y(V) & \longrightarrow & NASA(X) \end{array}$$

$\forall V \in Cart$

$\forall n \geq 0$

We need to show the existence of a lifting map

$$\Delta^n \otimes Y(V) \longrightarrow \check{C}(\mathcal{U})$$

[Recall, $Y(V) = \hom(-, V)$, the Yoneda embedding]

[NOTE: for $n=0$, this holds only locally!]

(proof continued)

we can take the adjoint of the diagram:

$$\begin{array}{ccc} \partial\Delta^n \otimes Y(V) & \longrightarrow & \check{C}(\mathcal{U}) \\ \downarrow & & \downarrow \\ \Delta^n \otimes Y(V) & \longrightarrow & NASA(X) \end{array} \qquad \xrightarrow{\text{ADJOINT DIAGRAM}} \qquad \begin{array}{ccc} \partial\Delta^n & \longrightarrow & \hom(Y(V), \check{C}(\mathcal{U})) \\ \downarrow & & \downarrow \\ \Delta^n & \longrightarrow & \hom(Y(V), NASA(X)) \end{array}$$

since $\hom(Y(V), \check{C}(\mathcal{U})) = \check{C}(\mathcal{U})(V)$

and $\hom(Y(V), NASA(X)) = C^\infty(V, X)$

REWRITE

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & \check{C}(\mathcal{U})(V) \\ \downarrow & \nearrow_{\Psi} & \downarrow \\ \Delta^n & \longrightarrow & C^\infty(V, X) \end{array}$$

We can equivalently show the existence of the lifting map

$$\Delta^n \longrightarrow \check{C}(\mathcal{U})(V)$$

① Define a map $\Psi: \Delta^n \longrightarrow \check{C}(\mathcal{U})(V)$

Δ^n is a connected simplicial set

$C^\infty(V, X)$ is a discrete simplicial set

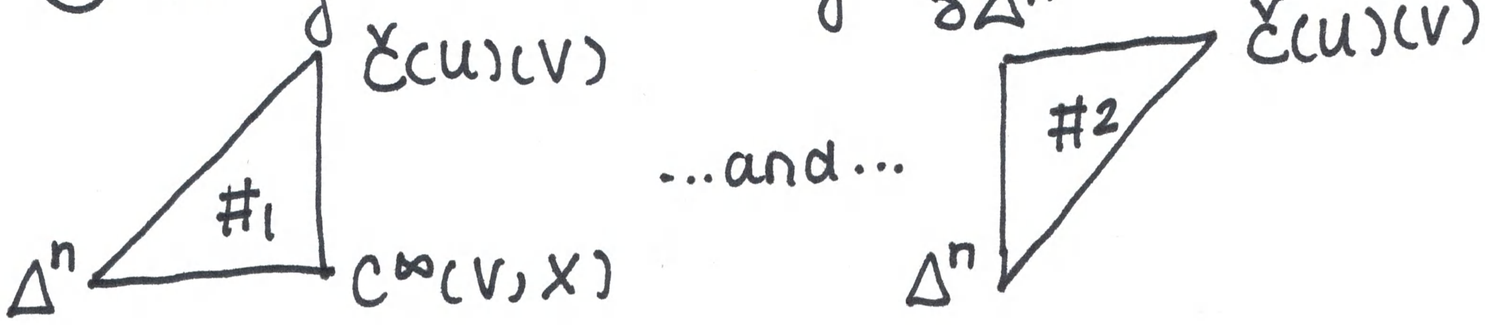
so, the map $\Delta^n \longrightarrow C^\infty(V, X)$ must factor through a point, Δ^0

$$\Delta^0 \longmapsto \varphi: V \to X \longrightarrow \text{define } \Psi = \varphi|_{U_{i_0} \cap \cdots \cap U_{i_n}}$$

An n-simplex in the Cech nerve is a pair (i, Ψ)

where $V \xrightarrow{\Psi} U_{i_0} \cap \cdots \cap U_{i_n}$

② Verify commutativity



So, in the projective model structure, we have a very nice way to define sheafifications!

We have the following Quillen adjunction! → see next page(s) for review of Quillen adjunctions →

$$sPSh(Cart)_{proj} \underset{R}{\overset{L}{\rightleftarrows}} sPSh(Cart)_{local,proj}$$

PROJECTIVE — LOCAL PROJECTIVE

Here, fibrant replacement = sheafification

derived

where the left-adjoint functor is ∞-sheafification

and the right-adjoint functor is the inclusion of sheaves into presheaves

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DEFINITION LOCAL SHEAVES (∞-SHEAVES)

Given $X \in sPSh(Cart)$

X is "local" if

$$RHom_{sset}(Y(X), \mathcal{F}) \xrightarrow{\sim} RHom_{sset}(\check{C}(u), \mathcal{F})$$

is a weak equivalence

NOTE: $Hom_{sset} : sPSh^{op} \times sPSh \longrightarrow sset$

and the above map can also (equivalently) be computed:

$$\mathcal{F}(X) \xrightarrow{\sim} Hom_{sset}(\check{C}(u), \mathcal{E}\tilde{x}\mathcal{F})$$

(cofibrant ~~fibrant~~ replacement of source and ~~cofibrant~~ fibrant replacement of target)

DEFINITION LOCAL WEAK EQUIVALENCES

The map $G \longrightarrow H$ is a "local weak equivalence" if $\forall \mathcal{F}$ (local objects)

$$RHom(H, \mathcal{F}) \xrightarrow{\sim} RHom(G, \mathcal{F})$$

is a weak equivalence

NOTE: Need to define Left Bousfield localization local weak equivalences & projective cofibrations! (see later section) →

Example* $\check{C}(u) \longrightarrow Y(X)$ is a local weak equivalence.

ADJUNCTIONS & EQUIVALENCES

Given an adjunction
(a pair of adjoint functors)

notation: $(L \dashv R)$

$$\mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}$$

$\forall X \in \mathcal{C}$, the **unit map** of X is the map

$$\eta_X : X \longrightarrow R \circ L(X)$$

$R \circ L$ = monad

that is adjoint
to the identity map: $id_{L(X)} : L(X) \longrightarrow L(X)$

$\forall Y \in \mathcal{D}$, the **counit map** of Y is the map

$$\varepsilon_Y : L \circ R(Y) \longrightarrow Y$$

$L \circ R$ = comonad

that is adjoint
to the identity map: $id_{R(Y)} : R(Y) \longrightarrow R(Y)$

Example:

$$\text{set} \underset{\mathcal{U}}{\overset{\text{free } \exists}{\rightleftarrows}} \text{Group}$$

forgetful $\mathcal{U}$

The unit map

$\forall X \in$ set

$\eta_X : X \longrightarrow \mathcal{U}(\exists(X))$

includes X as a
singleton word

The counit map

$\forall G \in$ Group

$\varepsilon_G : \exists(\mathcal{U}(G)) \longrightarrow G$

evaluates (via
homomorphism of
groups)

DEFINITION

QUILLEN ADJUNCTIONS

$$\mathcal{C} \underset{G}{\overset{\mathcal{F}}{\rightleftarrows}} \mathcal{D}$$ } Quillen adjunction

$[\mathcal{F} \dashv G]$ $\mathcal{F}$ is a left Quillen functor if it preserves cofibrations and acyclic cofibrations

G is a right Quillen functor if it preserves fibrations and acyclic fibrations

[Recall, Ken Brown Lemma, left/right Quillen functors admit left/right derived functors]

DEFINITION

QUILLEN EQUIVALENCES

$$\mathcal{C} \underset{G}{\overset{\mathcal{F}}{\rightleftarrows}} \mathcal{D}$$

is a Quillen equivalence
if it is a Quillen adjunction
and
the derived unit/counit maps
are weak equivalences

NOTE: η_X = unit map $\quad \varepsilon_Y$ = counit map

$$\mathcal{C} \underset{G}{\overset{\mathcal{F}}{\rightleftarrows}} \mathcal{D}$$

$X \in \mathcal{C}$ and $Y \in \mathcal{D}$

$\mathcal{F} \dashv G$ is an equivalence
if and only if
η_X and ε_Y are isomorphisms
$\forall X, \forall Y$

PROJECTIVE MODEL STRUCTURE & INJECTIVE MODEL STRUCTURE

There are two ways to put a model structure on a functor category $\mathcal{D}^{\mathcal{C}}$: PROJECTIVE & INJECTIVE

① THE PROJECTIVE MODEL STRUCTURE

- right-transferred
- $\mathcal{C}$ cofibrantly generated
- projective weak equivalences & projective fibrations are the natural transformations that are objectwise such morphisms of $\mathcal{C}$
  (maps of functors that are W and F when evaluated at each object of $\mathcal{C}$)
- nice criterion for cofibrations / acyclic cofibrations

② THE INJECTIVE MODEL STRUCTURE

- injective weak equivalences & injective cofibrations are the natural transformations that are objectwise such morphisms of $\mathcal{C}$
  (maps of functors that are W and C when evaluated at each object of $\mathcal{C}$)
- no nice criterion for fibrations / acyclic fibrations
  (must be determined using lifting properties)

PROPOSITION The projective and injective model structures on $Fun(\mathcal{C}^{op}, sset)$ are Quillen equivalent.

$$[Fun(\mathcal{C}^{op}, sSet)]_{proj} \underset{R}{\overset{L}{\rightleftarrows}} [Fun(\mathcal{C}^{op}, sSet)]_{inj}$$

[sketch of proof: the functors $L = id$ and $R = id$ are already derived: $\mathbb{L}L = L$ and $\mathbb{R}R = R$, so the derived unit and counit are the identity maps $\eta_X = id_X$ and $\varepsilon_Y = id_Y$]

PROPOSITION There is an adjunction

$$sPSh(Man^{op}) \underset{R}{\overset{L}{\rightleftarrows}} sPSh(Cart)$$

where the left adjoint functor, L, restricts ~~takes coverdictions~~ from Man^{op} to Cart

When considered in the ~~projective~~ local projective model structure,
the adjunction becomes a Quillen equivalence:

$$sPSh(Man^{op})_{\text{local, inj}} \underset{R'}{\overset{L'}{\rightleftarrows}} sPSh(Cart)_{\text{local, inj}}$$

LIMITS / COLIMITS and their generalization to the LAN / RAN FUNCTORS

Given a "bicomplete" category, $\mathscr{C}$
($\mathscr{C}$ has all small limits / colimits)

and given a small indexing category, I

Recall, $\mathscr{C}^I = \mathrm{Fun}(I, \mathscr{C})$

(where $\mathrm{Fun} : \mathrm{Cat}^{\mathrm{op}} \times \mathrm{CAT} \longrightarrow \mathrm{CAT}$)

A functor $I \longrightarrow *$

induces the "diagonal functor"

$$\mathscr{C}^I \longleftarrow \mathscr{C}^*$$

and since $*$ is a category with only a single object
(and identity morphism)

$$\begin{array}{ccc} \mathscr{C}^* & & \\ \| & & \\ \mathscr{C} & \xleftarrow{\Delta} & \mathscr{C}^I \end{array}$$

and we call this functor "Δ".

NOTE: Δ is continuous and cocontinuous, meaning it has both left and right adjoints, namely, the colimit and limit functors!

PROPOSITION The left / right adjoint functors of the diagonal functor Δ are the colimit / limit functors

$$\mathscr{C}^I \underset{\xleftarrow{\Delta}}{\overset{\xrightarrow{\mathrm{colim}}}{\xrightarrow{\lim}}} \mathscr{C}$$

NOTE: This proposition packages all the standard info about colimits / limits into info about adjoint functors!

PROOF: The colimit functor is left adjoint to Δ

If $\text{colim} \dashv \Delta$

$$\mathcal{C}^{\mathcal{I}} \underset{\Delta}{\overset{\text{colim}}{\rightleftarrows}} \mathcal{C}$$

then,

$\forall \mathcal{D} \in \mathcal{C}^{\mathcal{I}}$
$\forall X \in \mathcal{C}$

maps $\text{colim}\,\mathcal{D} \longrightarrow X$
correspond to maps

$$\mathcal{D} \longrightarrow \Delta X$$

that is, $\forall i \in \mathcal{I}$, $\mathcal{D}(i) \longrightarrow (\Delta X)(i)$ (define map ti)

and $(\Delta X)(i) = X$, so $\mathcal{D}(i) \xrightarrow{ti} X$.

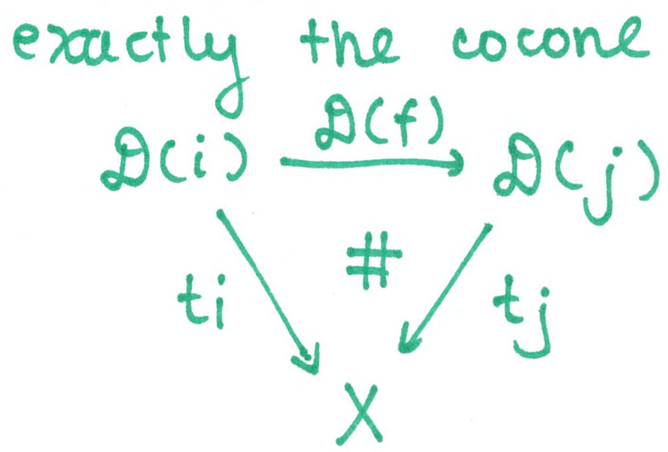
(verify naturality condition:)

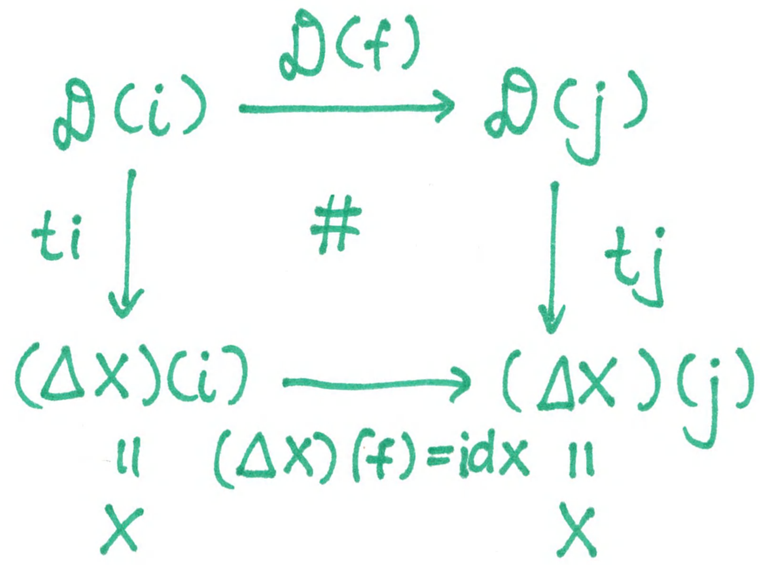
choose map $i \xrightarrow{f} j$

$$\mathcal{D}(i) \underset{\mathcal{D}(f)}{\longrightarrow} \mathcal{D}(j)$$

and we have a diagram

which is exactly the cocone





defined in the colimit! [colim $\mathcal{D}$]

[cocones are just natural transformations to constant functors]

and the proof for limits (as the right adjoint of Δ) is exactly the same. ▨

This proposition allows us to reformulate limits and colimits as adjoints of the functor Δ; from this reformulation, we can immediately derive homotopy limits and colimits
(pun fully intended)

But, for this, we need Quillen adjunctions
[that is, we need the left adjoint to preserve C, AC, and the right adjoint to preserve F, AF]

PROPOSITION $\Delta : \mathcal{C} \longrightarrow \mathcal{C}^{\mathbb{I}}$

(a) Δ is a right Quillen functor if $\mathcal{C}^{\mathbb{I}}$ is equipped with the projective model structure

(b) Δ is a left Quillen functor if $\mathcal{C}^{\mathbb{I}}$ is equipped with the injective model structure

Δ is a right Quillen functor

$\mathcal{C}^{\mathbb{I}} \underset{\Delta}{\overset{\text{colim}}{\rightleftarrows}} \mathcal{C}$ proj.

(a)

Recall, $\mathcal{C}^{\mathbb{I}}_{\text{proj.}}$ exists if $\mathcal{C}$ is cofibrantly generated

... and ...

$\mathcal{C} \underset{\lim}{\overset{\Delta}{\rightleftarrows}} \mathcal{C}^{\mathbb{I}}$ inj.

(b)

Recall, $\mathcal{C}^{\mathbb{I}}_{\text{inj.}}$ exists if $\mathcal{C}$ is combinatorial (cofibrantly generated and locally presentable, that is, cocomplete and accessible)

Δ is a left Quillen functor

DEFINITION

$\text{hocolim} := \mathbb{L}\,\text{colim}$ – left derived – practical in use!
$\text{holim} := \mathbb{R}\,\lim$ – right derived – impractical (to compute this way), requires fibrant replacement in injective model structure

The previous argument can be generalized (as follows!)

Previously,

$$I \longrightarrow *$$

induces

$$\mathcal{C}^I \longleftarrow \mathcal{C}^*$$

Consider another indexing category J, with functor

$$I \xrightarrow{g} J$$

This functor induces

$$\mathcal{C}^I \xleftarrow{\mathcal{C}g} \mathcal{C}^J$$

which is both continuous and cocontinuous

Let $g^* = \mathcal{C}g$

$$\mathcal{C}^I \xleftarrow{g^*} \mathcal{C}^J$$

Then g^* has a left/right adjoint functor, the left/right Kan extensions over g

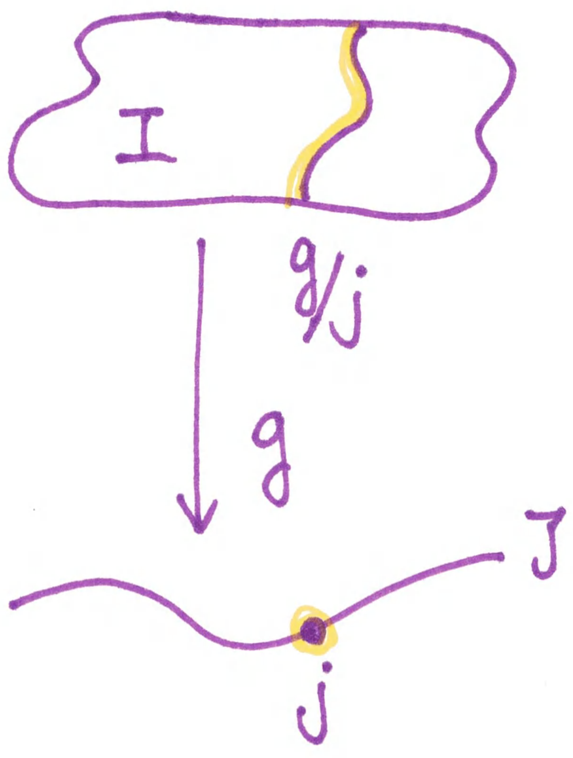
$$\mathcal{C}^I \underset{g^*}{\overset{\text{Lan}_g}{\rightleftarrows}} \mathcal{C}^J \underset{\text{Ran}_g}{\overset{g^*}{\rightleftarrows}}$$

The Kan extension is the analog of colimits in families. Rather, limits/colimits are special cases of Kan extensions!

PROPOSITIONAL EXAMPLE

Given $I = \text{Cart}$, $J = \text{Man}$ and the inclusion

$$g: I \longrightarrow J$$
$$\text{Cart} \hookrightarrow \text{Man}$$



There is a pointwise formula for the (left) Kan extension:

$$\text{Lan}_g : \mathcal{C}^I \longrightarrow \mathcal{C}^J$$
$$\Psi$$
$$F$$

$$(\text{Lan}_g(F))(j) = \operatorname{colim} F$$

fiber in I: g/j

→ that is, given an I-indexed diagram and an element $j \in J$, Lang returns the colimit of the diagram over the fiber of g over j,

$$g/_j := \begin{cases} \mathrm{Ob}(i \in I,\ g(i) \xrightarrow{f} j) \\ \mathrm{Mor}((i,f),(i',f') = i \xrightarrow{h} i') \end{cases}$$

such that

$$\begin{array}{ccc} g(i) & & \\ {\scriptstyle g(h)}\downarrow & \searrow^{\#} & \\ g(i') & \longrightarrow & j \end{array}$$

(and the (right) Kan extension is similarly defined)

PROPOSITION The following adjunctions are Quillen adjunctions with the model structures as shown:

Quillen adj. when ...

$$\mathcal{C}^I \underset{\text{proj}}{\overset{\mathrm{Lan}_g}{\rightleftarrows}} \mathcal{C}^J \qquad \mathcal{C}^J \underset{\mathrm{Ran}_g}{\overset{g^*}{\rightleftarrows}} \mathcal{C}^I_{\text{inj}}$$

$\mathcal{C}^I$ has projective model structure

$\mathcal{C}^I$ has injective model structure

DEFINITION

$$\mathrm{ho}\,\mathrm{Lan}_g = \mathbb{L}\mathrm{Lan}_g \quad \text{(left derived homotopy left Kan extension)}$$

$$\mathrm{ho}\,\mathrm{Ran}_g = \mathbb{R}\mathrm{Ran}_g \quad \text{(right derived homotopy right Kan extension)}$$

Recall, $sPSh(Cart)_{\text{local, proj.}}$ = Left Bousfield localization of $sPSh(Cart)_{\text{proj.}}$ at

$$\mathcal{C}(U) \longrightarrow X$$

for any good cover U of $X \in Cart$

(those for which mapping from $\mathcal{C}(U)$ into X produces a weak equivalence)

Recall, the previous PROPOSITION:

there is an adjunction

$$sPSh(Man^{op}) \underset{R}{\overset{L}{\rightleftarrows}} sPSh(Cart)$$

where L is the restriction functor that becomes a Quillen adjunction when we consider the correct model structure (another PROPOSITION):

$$sPSh(Man^{op})_{proj.} \underset{L}{\overset{Lan_L}{\rightleftarrows}} sPSh(cart)_{proj.}$$ } Quillen!

$$sPSh(Man)_{inj.} \underset{Ran_L}{\overset{L}{\rightleftarrows}} sPSh(Cart)_{inj}$$ } Quillen!

PROPOSITION Both Quillen adjunctions ↑ survive Left Bousfield localization

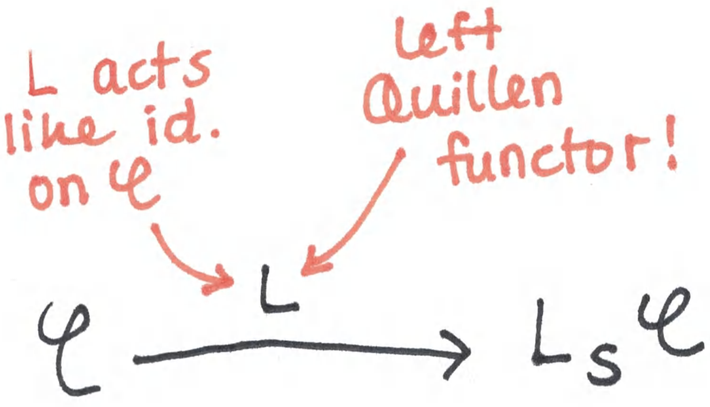
LEFT BOUSFIELD LOCALIZATION

(replaces model structure w/ another model structure so that cofibrations are the same but structure has more weak equivalences)

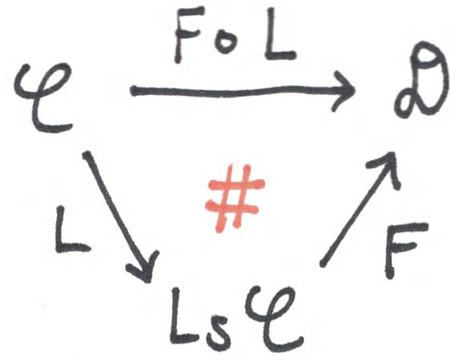
$\mathcal{C}$: model category

$S \subset Mor(\mathcal{C})$

Left Bousfield localization at S



→ so we have



and the assignment

$F \longmapsto F \circ L$

establishes an equivalence (even an isomorphism!) of categories

F: Left Quillen functors from $L_S\mathcal{C}$ to $\mathcal{D}$

F∘L : Left Quillen functors from $\mathcal{C}$ to $\mathcal{D}$ w/property that:

Left Quillen Functors from $L_S\mathcal{C}$ to $\mathcal{D}$ are mapped to left Quillen Functors from $\mathcal{C}$ to $\mathcal{D}$ such that

$\mathcal{C} \xrightarrow{G} \mathcal{D}$, $(\mathbb{L}G)(S) \subset W\mathcal{D}$

(weak equivalences when derived and applied to $S \subset Mor(\mathcal{C})$)

essentially, elements of S are inverted up to weak equivalences

Proof of Proposition ...

We want to show that the adjunction survives Bousfield localization (Quillen!)

so, we need to show:

$$sPSh(Man^{op})_{\text{local, proj}} \underset{L}{\overset{Lan_L}{\rightleftarrows}} sPSh(Cart)_{\text{local, proj.}}$$

is still a Quillen adjunction!

It suffices to show that $sPSh(Man^{op})_{\text{local, proj}} \xleftarrow{Lan_L} sPSh(Cart)_{\text{local, proj.}}$ is a left Quillen functor

(for then the adjunction will be Quillen).

① show $sPSh(Man^{op})_{\text{local, proj.}} \longleftarrow sPSh(Cart)_{\text{proj.}}$ is left Quillen. →

No local condition here!

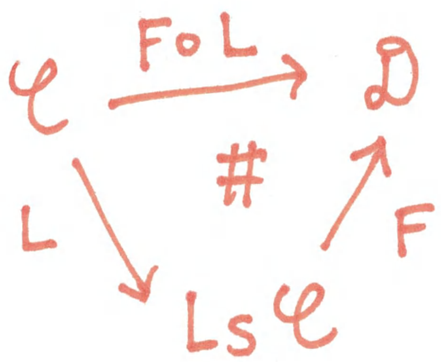
we know $sPSh(Cart)_{proj} \xrightarrow[Lan_L]{} sPSh(Man)_{proj}$

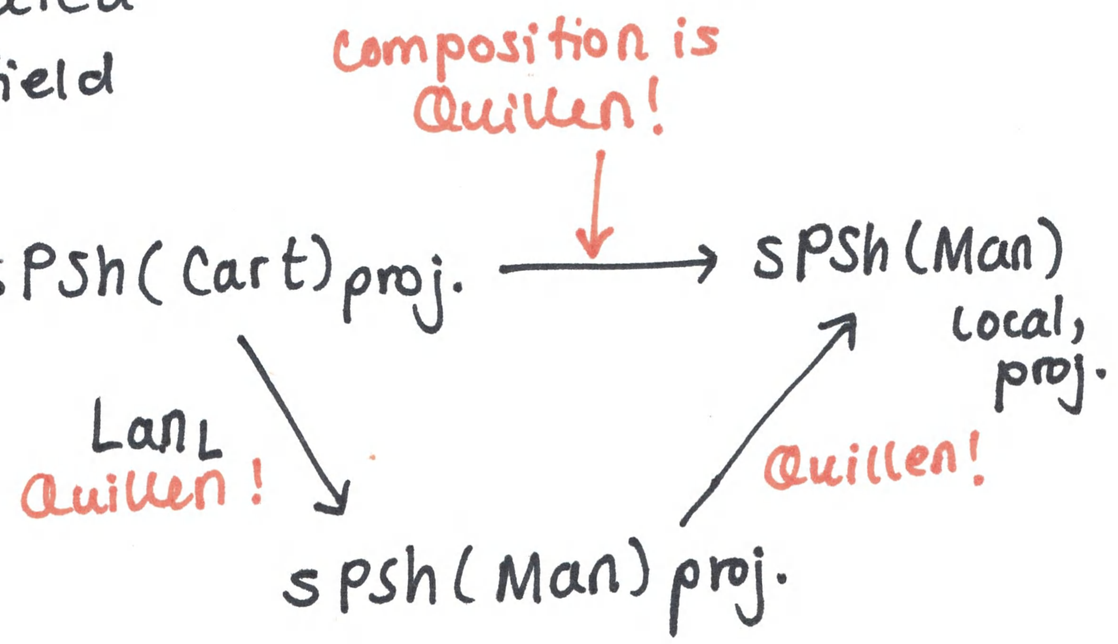
is a left Quillen functor,

as is

$sPSh(Man)_{proj} \longrightarrow sPSh(Man)_{\substack{local,\\ proj.}}$

so, by the previously stated properties of left Bousfield localization:





② we need to show that, extending from Cart to Man, the left Kan extension sends simplicial presheaves in Cart to simplicial presheaves in Man.

$$\mathbb{L}(Lan_L)\left(\underset{cart}{\check{C}}(\mathcal{U}) \longrightarrow Y(X)\right)$$

projectively cofibrant (doesn't need to be derived!)

$$\| $$

$$Lan_L\left(\underset{cart}{\check{C}}(\mathcal{U}) \longrightarrow Y(X)\right)$$

$$\| \quad \leftarrow$$ THIS STEP REQUIRES SOME EXPLANATION → (see lemma next page)

$$\underset{man}{\check{C}}(L(\mathcal{U})) \longrightarrow Y(L(X))$$

↑ a weak equivalence!

"THE POINT OF CATEGORY THEORY IS TO SHOW THAT TRIVIAL THINGS ARE TRIVIAL, SOMETIMES IN A NONTRIVIAL WAY."

—DR. PAVLOV

used in previous proposition proof ←

LITTLE LEMMA

$$\mathcal{C}^{I^{op}} \underset{g^*}{\overset{Lan_g}{\rightleftarrows}} \mathcal{C}^{J^{op}}$$

$\forall X \in I,$

$$Lan_g(Y_I(X)) \cong Y_J(g(X))$$

---

proof (sketch)...

By the Yoneda lemma, showing $Lan_g(Y_I(X)) \cong Y_J(g(X))$ is the same as showing that there is a natural isomorphism:

$$hom(Lan_g(Y_I(X)), \mathcal{F}) \longrightarrow hom(Y_J(g(X)), \mathcal{F})$$

$\forall \mathcal{F} \in \mathcal{C}^{J^{op}}$ (presheaf)

Recall,

$$\mathcal{C}^{op} \xrightarrow{\hat{Y}} Fun(\mathcal{C}, Set)$$

$$X \longmapsto (\mathcal{F} \longmapsto hom(X, \mathcal{F}))$$

by the Yoneda lemma (again...)

And this functor $\hat{Y}$ is a fully faithful functor, which means that there is a bijection

$$hom(X, X') \xrightarrow{\cong} hom(\hat{Y}(X), \hat{Y}(X'))$$

so, to show $X \cong X'$ it suffices to show $\hat{Y}(X) \cong \hat{Y}(X')$

To show $X \cong X'$

$(Lang(Y_I(x)) = Y_J(g(x)))$

We need to show $\hat{Y}(X) \cong \hat{Y}(X')$

$(hom(Lang(Y_I(X)), \mathcal{F}) = hom(Y_J(g(X)), \mathcal{F})$

and we can show this easily by: (expanding definitions)

$hom(Lang(Y_I(X)), \mathcal{F})$

IIS adjoint

$hom(Y_I(X), g^*\mathcal{F})$

IIS Yoneda

$(g^*\mathcal{F})(X)$

II

$\mathcal{F}(g(X))$

IIS Yoneda

$hom(Y_J(g(X)), \mathcal{F})$ ∎

some additional material...

LOCAL WEAK EQUIVALENCES

Recall, a simplicial map

$$f: X \longrightarrow Y$$

is a **simplicial weak equivalence**

if

$$\mathcal{E}^{\infty}(f) : \mathcal{E}^{\infty} X \longrightarrow \mathcal{E}^{\infty} Y$$

is a simplicial homotopy equivalence

A **left Bousfield localization** of a relative category $(\mathcal{C}, W)$ w/respect to a class of morphisms

$$S \subset \mathrm{Mor}(\mathcal{C})$$

is a relative category $(\mathcal{C}, W')$

where W' is the class of

"local weak equivalences"

(S-local weak equivalences)

An **S-local object**

$$X \in \mathcal{C}$$

has the property that

$\forall s \in S$, the induced morphism map

map (s, X)

[simplicial set of maps]

is a simplicial weak equivalence.

An **S-local weak equivalence** is a morphism

$$f \in \mathcal{C}$$

with the property that

$$\forall W \in \mathcal{C}$$

W, S-local object

map (f, W)

[simplicial set of maps]

is a simplicial weak equivalence.

PROPOSITION [Dugger and Isaksen]

(a) A morphism in $sPSh(Cart, Kan)$ (actually, for any site, S)

$$\mathcal{F} \xrightarrow{f} G$$

is a local weak equivalence

iff

$\forall X \in Cart$
$\forall n \geq 0$
$\forall$ diagrams of the form

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & \mathcal{F}(X) \\ \downarrow & \# & \downarrow \\ \Delta^n & \longrightarrow & G(X) \end{array}$$

$\exists \{U_i\}_{i \in I}$
open cover of X

such that

$\forall i$, that is,
restricting the
given diagram to
individual elements
of the open cover

$$\begin{array}{ccccc} \partial\Delta^n & \longrightarrow & \mathcal{F}(X) & \longrightarrow & \mathcal{F}(U_i) \\ \downarrow & \# & \downarrow \exists & \nearrow & \downarrow \\ \Delta^n & \longrightarrow & G(X) & \xrightarrow{\#^h} & G(U_i) \end{array}$$

the diagonal map exists,
making the upper triangle commute
and the lower triangle commute up to a homotopy

[NOTE: Here there need only be a local lift, on each U_i, not necessarily a global lift, on X itself)

(b) For a morphism in $sPSh(Cart)$
(rather than $sPSh(Cart, Kan)$... that is, NOT valued in Kan complexes)

(i) compose w/ Ex^∞
$\mathcal{F}, G \in sPSh(Cart)$ $\quad Cart^{op} \xrightarrow{\mathcal{F}} sSet \xrightarrow{Ex^\infty} Kan$

(ii) Apply (a) to $Ex^\infty(\mathcal{F}) = Ex^\infty \circ \mathcal{F}$ and $Ex^\infty(G) = Ex^\infty \circ G$.

* Example *

Consider the morphism $\check{C}(\mathcal{U}) \xrightarrow{f} Y(W)$

To show f is a local weak equivalence in $sPSh(Cart, Kan)$
(by the previous proposition)

$\forall X \in Cart$
$\forall n \geq 0$
$\forall$ diagrams of the form

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & \check{C}(\mathcal{U})(X) \\ \downarrow & & \downarrow \\ \Delta^n & \longrightarrow & Y(W)(X) \\ & & \| \\ & & C^\infty(X, W) \end{array}$$

by Yoneda lemma

consider the adjoint of the diagram

$$\begin{array}{ccc} \partial\Delta^n \otimes Y(X) & \longrightarrow & \check{C}(\mathcal{U}) \\ \downarrow & \exists \nearrow & \downarrow \\ \Delta^n \otimes Y(X) & \longrightarrow & W \end{array}$$

and this lift map was previously constructed!
(for each i indexing a good cover of X)

For $n \geq 1$, $\partial\Delta^n \longrightarrow \check{C}(\mathcal{U})(X)$ is defined on $0 \leq k \leq n$ vertices of Δ^n by maps $X \longrightarrow \mathcal{U}_{ik}$, $\forall k$

Then, $\partial\Delta^n \longrightarrow Y(W)(X)$ is defined by composition: $X \to \mathcal{U}_{ik} \to W$

For $n=0$, locality is crucial.

There are no maps $\partial\Delta^0 \otimes Y(X) \longrightarrow \check{C}(\mathcal{U})$, so the upper triangle commutes trivially
$(\partial\Delta^0 \longrightarrow \check{C}(\mathcal{U})(X))$

Restricting to fixed $\mathcal{U}_i$:

$$\begin{array}{ccc} & & \check{C}(\mathcal{U})(X)_0 = \coprod_j C^\infty(X, U_j) \\ & \exists \nearrow & \downarrow \\ \Delta^0 & \xrightarrow[\forall g]{} & C^\infty(X, W) \end{array}$$

Take $\{g^* \mathcal{U}_i\}_{i \in I}$ as open cover of X
Then, for fixed i,

$$\begin{array}{ccc} & C^\infty(g^*\mathcal{U}_i, \mathcal{U}_i) \hookrightarrow & \coprod C^\infty(g^*\mathcal{U}_i, \mathcal{U}_j) \\ \Delta^0 \nearrow & & \searrow \\ & \longrightarrow & C^\infty(g^*\mathcal{U}_i, W) \end{array}$$

∎

DEFINITION

$M \in Man$ [fix a manifold]

$\mathcal{A} \in Sh(M, Ab)$ [sheafs on Man w/ values in an abelian group]

$\mathcal{A}$ is called a "coefficient system"

We can define chain presheaves as:

$$cPSh(M) := Fun(open(M)^{op}, Ch)$$

replacement for Cart, since we are working on a fixed manifold, M

Within this category $cPSh(M)$, we can construct a new chain presheaf ...

$$\mathcal{A}[0] \in cPSh(M)$$

$\mathcal{A}$: coefficient system

[0]: the degree

$\mathcal{A}[0]$ sends

$$U \longmapsto \mathcal{A}(U)[0]$$

U: element of $Open(M)$

$\mathcal{A}(U)[0]$: turns into a chain by concentrating in deg 0; all other components (degrees) of the chain = 0

PROPOSITION

$LCS \subset Sh(M, Ab)$

"locally constant sheaves"

$\mathcal{F} \in LCS$ if $\forall U \subset V$

homotopy equivalence

the induced restriction map is an isomorphism:

$$\mathcal{F}(V) \xrightarrow{\cong} \mathcal{F}(U)$$

∴ There is a functor

$$LCS \xrightarrow{\text{takes sheaves}} \text{Local systems on } Sing M;\ Fun(\pi_{\leq 1} Sing M, Ab)$$

(coverings on $Sing M$, together with abelian group structures on fibers)

takes sections

Sheaves of sets form TOPOS and sheaves of simplicial sets form HIGHER TOPOS!

NOTE: $\mathcal{A}[0]$ is typically NOT local (not an ∞-sheaf!)
This is because the sheafification process creates elements in negative degrees (and abelian groups cannot have neg. degrees)

Chain complexes have higher homotopy groups!

→ We equip $cPSh(M)$ with a local projective model structure (defined as before) and perform Left Bousfield localization.

DEFINITION The "sheaf cohomology" of $M \in Man$ w/ coefficients in $A \in sh(M, Ab)$

is the evaluation on M of the fibrant replacement of $A[0]$

$$R(A[0])$$

in the local projective model structure.

(i) sheafify
(ii) replace w/ Kan (if necessary)
(iii) Evaluate on M

There is a functor...

$$\begin{array}{ccc} cPSh & \xrightarrow{eval_M} & coCh \\ \mathcal{F} & \longmapsto & \mathcal{F}(M) \end{array}$$

} "evaluation on M"

and $\mathbb{H}^n(\mathbb{R}(eval_M))(A[0]) = \mathbb{H}^n_{sh}(M, A)$

↑ right-derived functor evaluation on M, applied to $A[0]$

↑ SHEAF COHOMOLOGY

$H^n[(\mathbb{R}(eval_M))(A[0])]$

WARNING! This term "sheaf cohomology" is used both when defining homology and cohomology

SHEAF COHOMOLOGY

$$H^n(M, \mathcal{A})$$

FOR CHAIN COMPLEXES		FOR COCHAIN COMPLEXES
$H^n(M, \mathcal{A}) =$	$H_0(\mathbb{R}\mathrm{Reval}_M(\mathcal{A}[n]))$	$= H^0(\mathbb{R}\mathrm{Reval}_M(\mathcal{A}[-n]))$
$H^n(M, \mathcal{A}) =$	$H_{-n}(\mathbb{R}\mathrm{Reval}_M(\mathcal{A}[0]))$	$= H^n(\mathbb{R}\mathrm{Reval}_M(\mathcal{A}[0]))$

BUNDLE GERBES



$\mathcal{A} \in Sh(Man, Ab)$
"coefficient systems"

Fix $M \in Man$

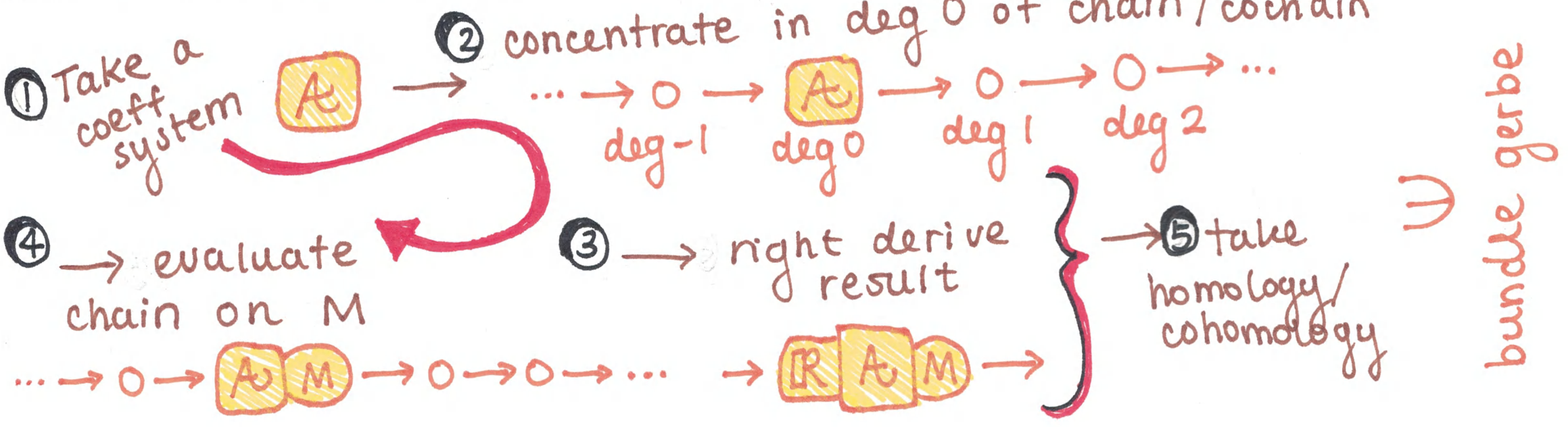
$\mathcal{A} \in Sh(M, Ab)$
"coefficient systems over M"

A "bundle gerbe" is an element in the sheaf cohomology

DEFINITION

A bundle $(n-1)$-gerbe
with coefficients in $\mathcal{A}$ over M
is an element in the sheaf cohomology

$$H^n(M, \mathcal{A})$$



MORE ON SHEAF COHOMOLOGY...

NOTE: This is hypercohomology because $\mathcal{F}$ need not be concentrated in a single degree

$$\mathbb{H}^*_{Sh}(M,\mathcal{F}) = H^*(\mathbb{R}\mathrm{eval}_M)(\mathcal{F}) = (\mathbb{R}\mathrm{Hom}(M,\mathcal{F}))(\mathbb{R}^0)$$

where $\mathcal{F}$ can be a presheaf on a fixed manifold, M

derived internal hom

should be derived

point $*$

$$\mathcal{F}: \mathrm{Open}(M)^{op} \longrightarrow \mathrm{coCh}$$

or a presheaf on a cartesian space

$$\mathcal{F}: \mathrm{Cart}^{op} \longrightarrow \mathrm{coCh}$$

(cochains)

NOTE: ordinary co-simplicial homology

$$H^n(S,A) = \pi_n[\mathbb{R}\mathrm{Hom}(S, U\Gamma(A[n]))]$$

(compare to above)

WHAT IS "SHEAFIFICATION"?

"Given a site $\mathcal{C}$, the "sheafification functor" universally turns presheaves on $\mathcal{C}$ into sheaves. It is characterized as the left adjoint functor $L: \mathrm{PSh}(\mathcal{C}) \longrightarrow \mathrm{sh}(\mathcal{C})$ to the inclusion $\mathrm{Sh}(\mathcal{C}) \hookrightarrow \mathrm{PSh}(\mathcal{C})$ of sheaves into presheaves... therefore, sheafification is a special case of a very general phenomenon of localizations of categories."

-nLab

PROPOSITION Fibrant Replacement in left Bousfield Localization

$$L_S\mathcal{C}$$

($\mathcal{C}$ localized at $S \subset \mathrm{Mor}(\mathcal{C})$) can be computed as

$$R_{L_S\mathcal{C}} = R_{\mathcal{C}} \circ \mathcal{L}$$

where $\mathcal{L}$ is the localization functor (sheafification, etc...) and $R_{\mathcal{C}}$ is fibrant replacement in $\mathcal{C}$

So, given a presheaf:
(i) sheafify
(ii) fibrantly replace in (original) model structure

* Example * (∞-sheafification)

$$L_S \mathcal{C}$$

$S = \{\check{C}(U) \longrightarrow Y(X)\}$

$\mathcal{C} = \mathrm{Fun}(\mathrm{Cart}^{op}, \mathrm{set})$

$L_{\check{C}(U)\to Y(X)}\ \mathrm{Fun}(\mathrm{Cart}^{op}, \mathrm{set})$

The "localization functor"

$$\mathcal{L} : \mathcal{C} \longrightarrow \mathcal{C}$$

takes presheaves on $\mathcal{C}$ and produces sheaves which are locally weakly equivalent

$\mathrm{Im}\,\mathcal{L} \subset$ S-local objects

$\mathcal{F} \longmapsto \mathcal{L}\mathcal{F}$

local weak equivalence

so $\mathcal{L}(\check{C}(U)) \simeq Y(X)$

In general, the entire principle is called
"sheaf cohomology" regardless of whether it involves homology or cohomology.

This is because it concerns the right-derived functor:

$$(\mathbb{R}\mathrm{eval}_M)(\mathcal{F}) = (\mathbb{R}\mathrm{Hom}(M, \mathcal{F}))(\mathbb{R}^0)$$

The right-derived evaluation functor, evaluated on a presheaf, $\mathcal{F}$

The derived internal hom, evaluated on $\mathbb{R}^0$

NOTE: $\mathbb{R}^0$ is the terminal object in Cart; this holds in general, replacing $\mathbb{R}^0$ w/ the terminal object in the given category

RECALL: The internal hom functor

$$\mathrm{Hom}: sPSh^{op} \times sPSh \longrightarrow sPSh$$

is part of an adjunction:

$$\mathcal{F} \longrightarrow \mathrm{Hom}(G, H)$$

$$\mathcal{F} \times G \longrightarrow H$$

and recall that $sPSh = \mathrm{Fun}(\mathrm{Cart}^{op}, sSet)$

substitute $\mathcal{F} = Y(X) = \hom(-, X)$

The Yoneda lemma!

By the above adjunction, maps... ...which means...

$$\hom(\mathcal{F}, \mathrm{Hom}(G,H)) \longrightarrow \hom(Y(X)\times\Delta^n, \mathrm{Hom}(G,H))$$

...are the same as $\amalg S$ maps... $\longrightarrow$ $\amalg S$

$$\hom(\mathcal{F}\times G, H) \longrightarrow \hom(Y(X)\times\Delta^n \times G, H)$$

and $\hom(Y(X)\times\Delta^n, \mathrm{Hom}(G,H)) = (\mathrm{Hom}(G,H))[X]_n$ ← evaluation at X, then take the set of n-simplices!

by the Yoneda lemma!

RECALL ...

① Why is $\hom(Y(X)\times\Delta^n, \mathrm{Hom}(G,H)) = (\mathrm{Hom}(G,H))[X]_n$?

$sPSh = \mathrm{Fun}(Cart^{op}, sSet)$

$sSet = \mathrm{Fun}(\Delta^{op}, set)$

so...

$sPSh = \mathrm{Fun}(Cart^{op}, \mathrm{Fun}(\Delta^{op}, set))$

$\cong$ (by currying!)

$= \mathrm{Fun}((Cart\times\Delta)^{op}, set)$

and for $X \in Cart$
and $[n] \in \Delta$

$\hom(Y(X,[n]), \mathcal{F}) = \mathcal{F}(X)_n$

② What is "currying"?

$\mathrm{Hom}(A, \mathrm{Hom}(B,C))$

$\downarrow\cong$

$\mathrm{Hom}(A\times B, C)$

is called "currying"!

③ so, $\hom(Y(X,[n]), \mathrm{Hom}(G,H))$

$\| S$

$(\mathrm{Hom}(G,H))[$

&

④ $G, H \in sPSh$
$A \in sSet$

we have adjunction:

$A \longrightarrow \mathrm{Hom}_{sSet}(G,H)$

$A \otimes G \longrightarrow H$

Now, substitute $A = Y([n]) = \Delta^n$ ... so we see that...

$(\mathrm{Hom}_{sSet}(G,H))_n = \hom(A, \mathrm{Hom}_{sSet}(G,H))_n = \hom(\Delta^n \otimes G, H)$

These must be equal!

A PAGE OF LEMMAS ...

when life gives you lemmas...

LEMMA 1

$$\mathrm{Hom}_{\mathrm{sset}}(G,H) = (\mathrm{Hom}(G,H))(\mathbb{R}^0)$$

($\mathbb{R}^0$: terminal object in category)

sketch of proof: $\hom(\mathcal{F} \times G, H) = \hom(\mathcal{F}, \mathrm{Hom}(G,H))$

let $\mathcal{F} = Y(X) \times \Delta^n \longrightarrow$ $= (\mathrm{Hom}(G,H))(X)_n$

and $\hom(\Delta^n \otimes G, H) = \hom(\Delta^n, \mathrm{Hom}_{\mathrm{sset}}(G,H))_n$

$= [\mathrm{Hom}_{\mathrm{sset}}(G,H)]_n$

and we have:

$Y(\mathbb{R}^0, [n]) \times G \cong \Delta^n \otimes G$

since maps $S \longmapsto (\infty(S, \mathbb{R}^0) \times \Delta^n \times G(S)$

are the same as maps $S \longmapsto \Delta^n \times G(S)$

LEMMA 2

$$\mathrm{eval}_X \cong \mathrm{Hom}_{\mathrm{sset}}(Y(X), -)$$

sketch of proof: Both are functors $\mathrm{sPsh} \longrightarrow \mathrm{sset}$.

Choose Δ^n, $\mathcal{F} \in \mathrm{sPsh}$

Consider the set of maps $\Delta^n \longrightarrow \mathrm{eval}_X(\mathcal{F})$

$\hom(\Delta^n, \mathrm{eval}_X(\mathcal{F})) = \hom(\Delta^n, \mathcal{F}(X)) = (\mathcal{F}(X))_n$

and

$(\mathrm{Hom}_{\mathrm{sset}}(Y(X), \mathcal{F}))_n = \hom(\Delta^n \otimes Y(X), \mathcal{F}) = \hom(Y(X, [n]), \mathcal{F})$

$= (\mathcal{F}(X))_n$

so the two are equivalent functors.

LEMMA 3 (combining the two previous ↑ lemmas)

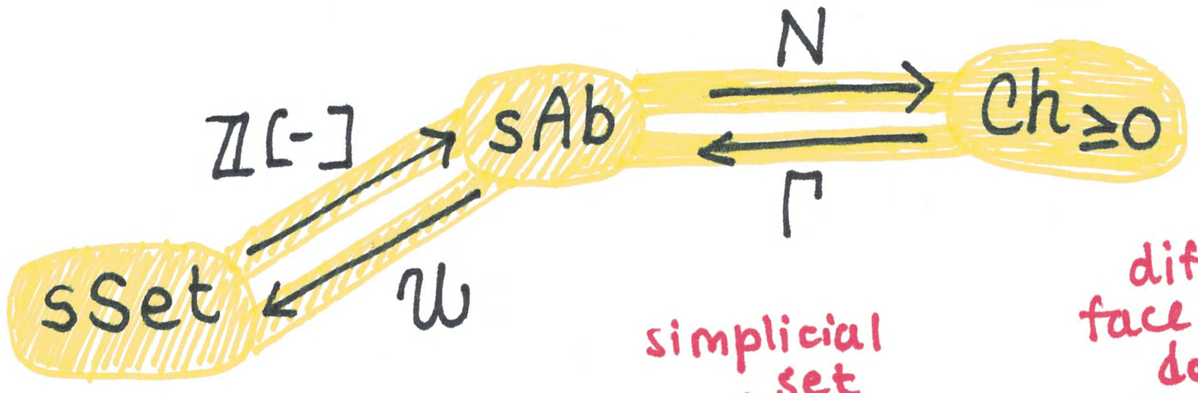
$$\mathrm{eval}_X = (\mathrm{Hom}(Y(X, [0]), -))(\mathbb{R}^0)$$

BACK TO BUNDLE GERBES ...



→ Recall, the

DOLD KAN CORRESPONDENCE ←



simplicial set



differentials: face maps di...

$A_2 \underset{d_2}{\overset{d_0}{\overset{\Rightarrow}{\underset{\Rightarrow}{d_1}}}} A_1 \underset{d_1}{\overset{d_0}{\Rightarrow}} A_0$

- ℤ[-] : "free functor" $\quad sSet \longrightarrow sAb$, $\quad A \longmapsto A_\bullet$
- N : "normalized Moore functor" $\quad sAb \longrightarrow Ch_{\geq 0}$, $\quad A_\bullet \longmapsto \left[\bigcap_{i=1}^{n} \ker(d_i)\right]$ degree n
- Γ : "gamma functor"
- 𝒰 : "underlying set functor"

$$\cdots \rightrightarrows A_n \rightrightarrows \cdots \rightrightarrows A_1 \rightrightarrows A_0$$

n-simplices — 1-simplices — 0-simplices

("throws away" differential maps)

$\left[\bigcap_{i=1}^{n+1} \ker(d_i)\right]$ degree n+1

$\left[\bigcap_{i=1}^{n-1} \ker(d_i)\right]$ degree n-1

$\cdots$ $\quad \ddots$

ANTUM HOMOTOPY SEMINAR

Enriched category"

: $\mathcal{C}$ be a category
∃ An enrichment of
in a monoidal cat.
$(\mathcal{V}, \otimes, I)$ consists of:

For each pair $X, Y \in Ob(\mathcal{C})$
$\hom(X,Y)$

For each triple $X, Y, Z \in Ob(\mathcal{C})$
$\circ : \hom(X,Y) \otimes \hom(Y,Z) \to \hom(X,Z)$

associativity, etc.
diagrams commute

For each $X \in Ob(\mathcal{C})$
there is a map $j_X : I \to \hom(X,X)$
(identity)

OREM There is a simplicial
illen adjunction:

$$Ch_+ \underset{NF}{\overset{\Gamma}{\rightleftarrows}} sSet$$

lain functors U, Γ, N, F:

all, **DOLD-KAN CORRESPONDENCE**

$Ch_+ \to sAb$ "gamma functor"
$sAb \to Ch_+$ "Normalized Moore functor"

F] The functor N

$$N(A_\bullet)_n := \bigcap_{i=1} \ker(d_i), \quad i \neq 0$$

↑
: sAb

and differential maps
$d = d_0 : (NA_\bullet)_n \to (NA_\bullet)_{n-1}$
are the face map d_0
(all differential maps!)

$\cdots \exists A_1 \rightrightarrows A_0$
ircial abelian groups

mple

Consider, for $A \in Ab$
= nerve of the category w/ single object
and A morphisms

$BA))_0 = 0$
$BA))_1 = A$ kernel of all maps except d_0
$BA))_2 = \{(a,b) : a=0,\ ab=0\} = 0$

$\cdots A \times A \rightrightarrows A \rightrightarrows 0$
three maps
1. projects onto 1st factor (d_0)
2. multiplies w/ ab. group structure (d_1)
3. projects onto 2nd factor (d_2)

$$(\Gamma(A_\bullet))_n = \bigoplus A_k = A^n$$

$[n] \twoheadrightarrow [k]$ surjection ($k+1$, result of those!)
$[n] \twoheadrightarrow [0]$ will be $0 \cdots 0$ (there are n of those!)

w/ face maps:
$d^0 : [n-1] \hookrightarrow [n]$

$$(\Gamma(A[1]))_n = \bigoplus A_k = A^n$$

$$(\Gamma(A[1]))_1 = \bigoplus A_1 = A$$

$$(\Gamma(A[1]))_0 = A_0 = 0$$

Example $A[1] = (\cdots \to 0 \to 0 \to A \to 0 \cdots)$ $\xrightarrow{deg\,1}$

$d^0, id, d : A_k \to A_k$

($d : [k] \hookrightarrow [k+1]$)

d will induce maps s:

$[n-1] \twoheadrightarrow [k+1] \hookrightarrow [k]$ (d / A_k)

Every map can be factored as epimorphism followed by monomorphism

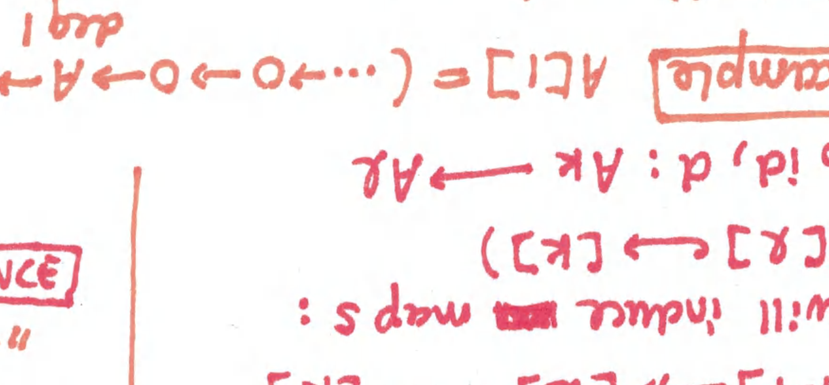
$[k]$ ↑ A_k

This is a map $[n-1] \hookrightarrow [n]$ — miss i^{th}

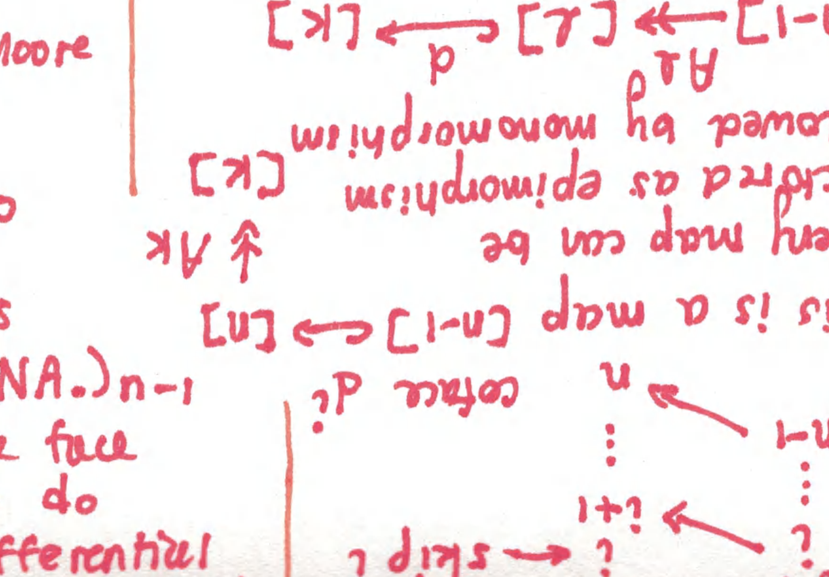
$$(\Gamma(A_*))_n = \bigoplus A_k$$

$[n] \twoheadrightarrow [k]$ surjections (order-preserving)

face maps induced by coface maps (maps that miss the vertex i, as shown):



Γ maps $[n]$ to $[n]$ — coface d^i



skip i

so, $N(BA) = A[1]$

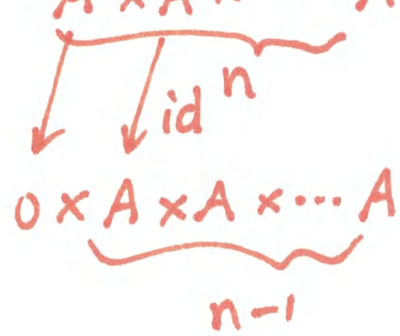
$= (\to 0 \cdots \to 0 \to A \to 0 \cdots)$ $\xrightarrow{deg\,1}$ def of Γ



$[n-1] \twoheadrightarrow [1] \xrightarrow{id} [1]$

and for the face map d^0, we have

$A \times A \times \cdots A$

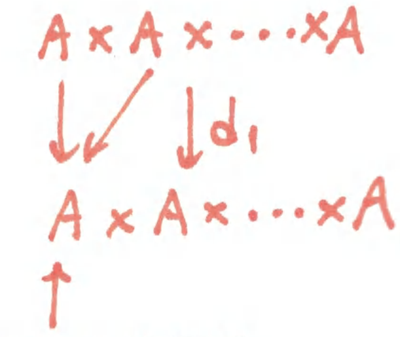


$0 \times A \times A \times \cdots A$

$n-1$

The face map d^1

$A \times A \times \cdots \times A$



$A \times A \times \cdots \times A$

↑ uses group operation

$A \hookrightarrow A \times A \hookleftarrow A$

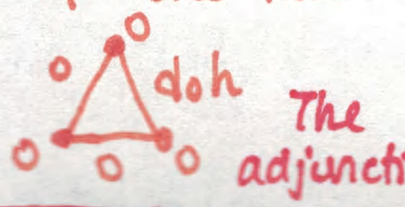


universal property } group operation!

$[n-1] \twoheadrightarrow [n] \to [1]$
d^1

REMARK

Think of elements in $(N(A_\bullet))_n$ as being labeled simplices where everything is zero except one face



The adjunction:

THEOREM $\Gamma \dashv N$

(or rather $N \dashv \Gamma$) is an equivalence of categories:

sAb are the SAME as chain complexes!

→

"Model Categories"

eak equivalences (w.e.)
ibrations (F)
trations (C)
yclic fibrations (AF)
cyclic cofibrations (AC)

] 1. W.e: quasi-isos
2. F: levelwise epimorphiims
3. C: levelwise monos w/ projective cokernel

] } NOT equivalence, but still an adjunction

W.e. as usual
F: Kan fibrations
C: monomorphisms, levelwise $X_n \hookrightarrow Y_n$

romote adjunction to a Quillen
nction, we need to show
adjoint preserves
t adjoints preserves

mA $\pi_k(U\Gamma(A_*)) \cong H_k(A_*)$

2] $f: A_* \to B_*$ quasi-iso
(chain map)
uces $U\Gamma(f): U\Gamma(A_*) \to U\Gamma(B_*)$
is a weak equivalence
$H_k(A_*)$ w.e. to $H_k(B_*)$

]
DK = UΓ
old Kan

nA $A_* \twoheadrightarrow B_*$ levelwise epimorphism
induces
$DK(A_*) \twoheadrightarrow DK(B_*)$
Kan fibration $\cdots \leftarrow \Omega^1(-, \mathcal{C}x) \xleftarrow{\deg n \mapsto \deg n-1} \Omega^n(-, \mathcal{C}x) \leftarrow (\cdots \to 0 \to \mathcal{C}x(-, \mathcal{C}x) \xrightarrow{\deg n} \cdots \to \Omega^n)$

R Prop
$DK: Ch_+ \rightleftarrows sSet : NF$
is a Quillen adjunctions, so it preserves the homotopy theory

p(cof., fib)
Left ↗ right
reserves preserves
ibration fibrations
ce since fibrations
ibrations are good in
good in codomain
omain

Even more, it is a simplicial Quillen
junction (between simplicial model
s w/ simplicial enrichment)

By Dold Kan

$\mathrm{Map}(\check{C}(\{U_\alpha\}), DK(\Omega^{\leq n}(-, \mathcal{C}x)))$
$\cong \mathrm{Map}(NF(\check{C}(\{U_\alpha\})), \Omega^{\leq n}(-, \mathcal{C}x))$
$\cong \mathrm{Map}(CF(\check{C}(\{U_\alpha\}), \Omega^{\leq n}\ldots))$

↑ similar to functor N, but takes alternating face maps

where
$C(A_\bullet) \cong N(A_\bullet) \oplus D(A_\bullet)$
(a natural split exact sequence between of $C(A_\bullet)$, where $D(A_\bullet)$ is deg. of $A_\bullet$
and $N(A_\bullet) \xrightarrow{\simeq} C(A_\bullet)$
is an acyclic cofibration

so
$\mathrm{Map}(CF(\check{C}(\cdots), \cdots)$
is a map out of an acyclic cofibration
is an acyclic fibration; hence, a weak eq.

so
$\cong DK(\hom(CF(\check{C}(\{U_\alpha\}), \Omega^{\leq n})$
$= DK(\prod_{i \in \mathbb{Z}} \hom(CF(\check{C}(\{U_\alpha\}), \Omega^{\leq n})$
$= \mathrm{Tot}(\check{C}(\{U_\alpha\}), \Omega^{\leq n})$

$\mathbb{Z}[-]$

* Applications to Gerbes *

Construct the complex
$\Omega^{\leq n}(-, \mathcal{C}x)$ (in $Fh(Man, Ch+)$)
$= (\cdots \to 0 \to \mathcal{C}x(-, \mathcal{C}x) \xrightarrow{d \log} \Omega^1(-, \mathcal{C}x) \to \cdots \to \Omega^n)$
shift of chain complexes

Recall, an $(n-1)$-gerbe w/ connection is an element of the totalization of the bicomplex:

$\mathrm{Tot}^0(\check{C}(\{U_\alpha\}), \Omega^{\leq n}(-, \mathcal{C}x))$

(and elements in Tot^1 will be morphisms between gerbes)

THEOREM We have an equivalence of spaces
$\mathrm{Map}(M, \mathbb{B}^n \mathcal{C}_X^\Delta) \simeq \mathrm{Map}(M, \mathbb{B}^n \mathcal{C}_X^\Delta)$

Proof $\mathrm{Map}(M, \mathbb{B}^n \mathcal{C}_X^\Delta) := \mathrm{Map}(\check{C}(\{U_\alpha\}), \mathbb{B}^n \mathcal{C}_X^\Delta)$
where $\check{C}(\{U_\alpha\}) = \coprod_{\alpha_0 \cdots \alpha_n} U_{\alpha_0 \cdots \alpha_n}$ for some good open cover
Čech nerve!
$\mathbb{B}^n \mathcal{C}_X^\Delta = DK(\Omega^{\leq n}(-, \mathcal{C}x))$
$\simeq DK(\mathrm{Tot}^\bullet(\check{C}(\{U_\alpha\}), \Omega^{\leq n}(-, \mathcal{C}x)))$ ← w.e.
$= (\mathrm{Map}(M, \mathbb{B}^n \mathcal{C}_X^\Delta))$
to embed into presheaves using Yoneda
$M \longmapsto C(-, M)$
α

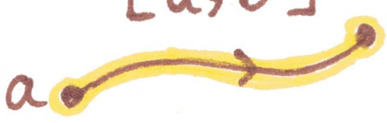
deRham COHOMOLOGY

Recall, $\Omega^0(M)$:= space of smooth functions on M

$\Omega^1(M)$:= space of 1-forms on M

$f(x)\,dx$ — segment

integrate over interval $[a,b]$ — $\int_a^b f(x)\,dx$



d

$\Omega^2(M)$:= space of 2-forms on M

$f(x,y)\,dx\wedge dy$ — area element

integrate over surface S — $\int_S f(x,y)\,dx\wedge dy$



d

$\Omega^3(M)$:= space of 3-forms on M

$f(x,y,z)\,dx\wedge dy\wedge dz$ — volume element

integrate over region R — $\int_R f(x,y,z)\,dx\wedge dy\wedge dz$



d

$\Omega^n(M)$:= space of n-forms on M

d = exterior derivative takes k-form to (k+1)-form

define Ω as a presheaf on manifolds:

$\Omega : Man^{op} \longrightarrow Ch$

$M \longmapsto \cdots \leftarrow \underbrace{\Omega^n(M)}_{\deg -n} \leftarrow \Omega^{n-1}(M) \leftarrow \cdots \leftarrow \overbrace{\Omega^0(M)}^{\deg 0} \leftarrow 0 \leftarrow \cdots$

Reverses degrees so that differential maps go in the correct direction ($\partial^2 = 0$)

FACT

Ω is an ∞-sheaf

Recall, all chain complexes are fibrant in the projective model structure

will be proved later!

... so...

FACT

Ω is a fibrant object in $Fun(Man^{op}, Ch_{proj.})_{proj.\,local}$

where $Ch_{proj.}$ has
W = quasi-isomorphisms
F = degreewise surjections

PROPOSITION $M \in Man$

$$\mathbb{R}Hom_{ch}(Y(M), \Omega) = \overbrace{\Omega(M)}^{\text{evaluate codomain on domain!}}$$

representable presheaf of arbitrary manifold is projectively cofibrant!

fibrant! (see above)

technically, should be $N\mathbb{Z}[Y(M)]$ so argument is a presheaf on Ch

$\mathbb{Z}[-]$: presheaf sset $\longrightarrow$ presheaf sAb

$N[-]$: presheaf sAb $\longrightarrow$ presheaf Ch

so $N\mathbb{Z}[Y(M)]$ is a presheaf of chain complexes.

RECALL Locally constant sheaves

"A locally constant sheaf A over a topological space is a sheaf of sections of a covering space of X: there is a cover of X by open subsets $\{U_i\}$ such that the restrictions $A|_{U_i}$ are constant sheaves"
— nLab

DEFINITION

$$\mathbb{R}_{\delta} : \mathrm{Man}^{op} \longrightarrow \mathrm{Ch}$$

$$M \longmapsto C(M,\mathbb{R})[0] = \mathbb{R}_{\delta}[0]$$

(locally constant)

NOTE
$\mathbb{R}_{\delta}[0]$ is not a local object; does not satisfy ∞-sheaf property

[locally constant ≡ ordinary cohomology w/ local system coefficients]

Example: sheaf cohomology w/ coeffs in constant sheaves is ordinary cohomology:



$$H^1(S^1, \mathbb{R}_{\delta}) = \mathbb{R} \neq 0$$

Actually, $H^1(N\mathbb{Z}[S^1], \mathbb{R}_{\delta}] = H^0(S^1, \mathcal{U}\Gamma(\mathbb{R}_{\delta})[1])$

$$= H^0 \underbrace{\mathbb{R}\mathrm{Hom}_{sset}}(Y(S^1), \mathcal{U}\Gamma(\mathbb{R}_{\delta})[1])$$

cofibrant replacement of $Y(S^1)$ (already cofibrant) and fibrant replacement of $\mathcal{U}\Gamma(\mathbb{R}_{\delta})[1]$ means ∞-sheafification!

PROPOSITION

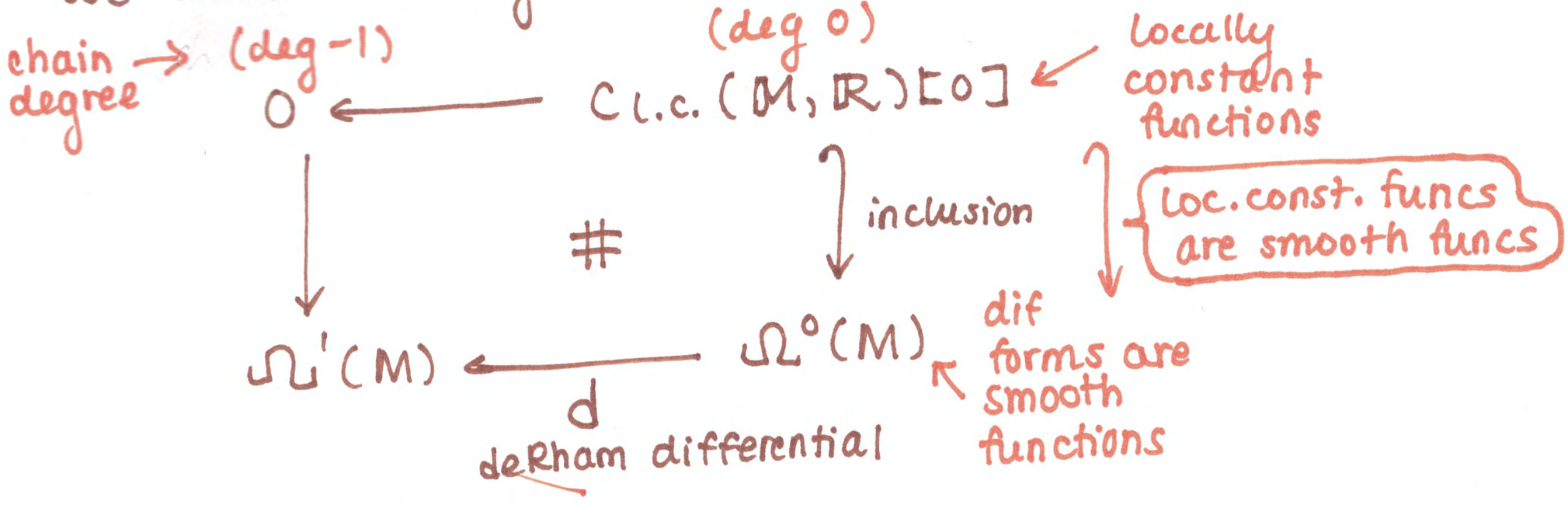
There is a map $\mathbb{R}_{\delta} \longrightarrow \Omega$
(presheaf) (chain complex)

[specifically, $M \longmapsto$ ($C(M,\mathbb{R})[0]$ (locally constant) $\downarrow$ $\Omega(M)$)] $\longrightarrow$

the only nontrivial map here is in deg 0!

$\longrightarrow$ For, $M \longrightarrow \begin{pmatrix} C^{\infty}_{l.c.}(M,\mathbb{R})[0] \\ \downarrow \\ \Omega(M) \end{pmatrix}$

we have a diagram:



POINCARÉ LEMMA

The map $\mathbb{R}_{\delta} \longrightarrow \Omega$ is a local weak equivalence

COROLLARY

$$\mathbb{R}\mathrm{Hom}_{ch}(Y(M), \mathbb{R}_{\delta}) = \mathrm{Hom}_{ch}(Y(M), \Omega) = \Omega(M)$$

Now, we can do a similar computation for Cartesian spaces:

$$\mathrm{Cart} \overset{L}{\hookrightarrow} \mathrm{Man} \qquad \underline{\text{But}} \quad \mathrm{PSh}(\mathrm{Cart}) \longleftarrow \mathrm{PSh}(\mathrm{Man})$$

~~res~~ restriction — no restriction!

There is restriction of presheaves!

and thus we have

$$\mathrm{Fun}(\mathrm{Man}^{op}, Ch) \xrightarrow{\text{res}} \mathrm{Fun}(\mathrm{Cart}^{op}, Ch)$$

local proj. — local proj.

since this functor "res" is part of a Quillen equivalence, it preserves weak equivalences, so...

$$\mathbb{R}\mathrm{Hom}_{ch}(Y(M), \mathbb{R}_{\delta}) = \mathbb{R}\mathrm{Hom}_{ch}(\underbrace{\mathrm{res}\, Y(M)}_{NASA(M)}, \mathrm{res}\, \mathbb{R}_{\delta}) \longrightarrow$$

$\mathbb{R}\mathrm{Hom}_{ch}(Y(M), \mathbb{R}_\delta) = \mathbb{R}\mathrm{Hom}_{ch}(\mathrm{res}\, Y(M), \mathrm{res}\, \mathbb{R}_\delta)$

$Y(M) = N_{ASA}(M)$

technically, $Y_{ch}(M) = \mathbb{Z}[Y(M)][0]$ so it is a chain presheaf

where $Y_{ch}: \mathrm{Man} \to \mathrm{Fun}(\mathrm{Man}^{op}, Ch)$

$M \mapsto (S \mapsto \mathbb{Z}[C^\infty(S, M)][0])$

need to cofibrantly replace

already fibrant! (prove later)

already derived

$= \mathbb{R}\mathrm{Hom}_{ch}(\check{C}_{ch}(\mathcal{U}), \mathrm{res}\, \mathbb{R}_\delta)$

where $\check{C}_{ch}(\mathcal{U}) \in \mathrm{Fun}(\mathrm{Cart}^{op}, Ch)$

whereas $\check{C}(\mathcal{U}) \in \mathrm{Fun}(\mathrm{Cart}^{op}, sSet)$

and in degree n, we define

$[\mathrm{Hom}_{ch}(\check{C}(\mathcal{U}), \mathrm{res}\, \mathbb{R}_\delta)]_n$

$= \mathrm{Hom}_{Ab}\left(\bigsqcup_i \mathbb{Z}[Y(U_{i_0} \cap \cdots \cap U_{i_n})], \mathrm{res}\, \mathbb{R}_\delta\right)$

$= \prod_i \mathrm{Hom}_{Ab}(\mathbb{Z}[Y(U_{i_0} \cap \cdots \cap U_{i_n})], \mathrm{res}\, \mathbb{R}_\delta)$

$= \prod_i \mathrm{Hom}_{Ab}(Y(U_{i_0} \cap \cdots \cap U_{i_n}), \mathcal{U}(\mathrm{res}\, \mathbb{R}_\delta))$

$= \prod_i \mathcal{U}(\mathrm{res}\, \mathbb{R}_\delta)(U_{i_0} \cap \cdots \cap U_{i_n})$

$= \prod_i \mathbb{R}$

because locally constant real-valued functions on a cartesian manifold are constant function, hence, each evaluates to a real #

(taken over all nonempty intersections $U_{i_0} \cap \cdots \cap U_{i_n} \neq 0$)

define

$(\check{C}_{ch}(\mathcal{U}))_n = \bigsqcup_i \mathbb{Z}[Y(U_{i_0} \cap \cdots \cap U_{i_n})]$

$\mathcal{U}$: good open cover $= \{U_{i_k}\}_{k \in K}$

and the differential is

$d: (\check{C}(\mathcal{U}))_{n+1} \to (\check{C}(\mathcal{U}))_n$

can be defined on individual parts (since the map is additive)

For $S \xrightarrow{\varphi} U_{i_0} \cap \cdots \cap U_{i_{n+1}}$

$S \in \mathrm{Cart}$

$d\varphi = \sum_{j=1}^{n+1} (-1)^j (\varepsilon_j \circ \varphi)$

where $\varepsilon_j: U_{i_0} \cap \cdots \cap U_{i_{n+1}} \to U_{i_0} \cap \cdots \cap \hat{U}_{i_j} \cap \cdots \cap U_{i_{n+1}}$

remove jth coord.

→ so, the differential map

$$d: \mathrm{Hom}_{Ab}((\check{C}(\mathcal{U}))_n, \mathrm{res}\,\mathbb{R}_S) \longrightarrow \mathrm{Hom}_{Ab}((\check{C}(\mathcal{U}))_{n+1}, \mathrm{res}\,\mathbb{R}_S)$$

$$\| \qquad\qquad\qquad\qquad\qquad\qquad \|$$

$$\prod_{i} \mathbb{R} \longrightarrow \prod_{j} \mathbb{R}$$

i such that intersection for i is nonempty

j such that intersection for j is nonempty

Choose some $\varphi \in \prod_i \mathbb{R}$

$$\varphi \longmapsto \left(j \longmapsto \sum_{k=0}^{n+1} (-1)^k \varphi_{j\hat{k}} \right)$$

and this is known as the Čech cochain complex (same cohomology as de Rham cohomology). ▨

and... BACK TO BUNDLE GERBES... again!



① SHEAF COHOMOLOGY:

$$H^*(M,\mathcal{F}) = (\mathbb{R}\mathrm{eval}_M)(\mathcal{F})$$
$$= (\mathbb{R}\mathrm{Hom}(M,\mathcal{F}))[\mathbb{R}^0]$$

② BUNDLE GERBE

A bundle $(n-1)$-gerbe is an element in $H^n_{Sh}(M,A)$
(with coefficients in A over M)

and for chain complexes, ← where A is a sheaf of abelian groups on Man

$$H^n_{Sh}(M,A) = H_0((\mathbb{R}\mathrm{eval}_M)(A[n]))$$
$$\| $$
$$H_0\,\mathrm{eval}_M(\mathcal{U}\Gamma(A[n]))$$

local fibrant replacement of $A[n]$

* Example *

$\mathbb{A} \in$ AbLieGroup (abelian Lie group)

$M \in$ Man

$U \subset M$ open

There is a canonical way to construct a sheaf (A):

$$A(U) := C^\infty(U, \mathbb{A})$$

we can then define bundle gerbes w/coeffs in $\mathbb{A}$ over M as a derived map (cofibrantly replace source, fibrantly replace target):

$$NASA(M) \longrightarrow \mathbb{B}^n\mathbb{A} \qquad (\text{in } sPSh(Cart))$$

where $\mathbb{B}^n$ sends $S \longmapsto \Gamma(C^\infty(S, \mathbb{A})[n])$

$S \in Cart$

or, alternatively, as chain presheaves:

$$N\mathbb{Z}(NASA(M)) \longrightarrow \mathbb{A}[n] \qquad (\text{in } cPSh(cart))$$

where $\mathbb{A}[n]$ sends $S \longmapsto C^\infty(S, \mathbb{A})[n]$

$S \in Cart$

NOTE: $\mathbb{B}^n\mathbb{A}$ is an ∞-sheaf (fibrant object in projective model structure)

and

$\mathbb{A}[n]$ is an ∞-sheaf of chain complexes on $\rightarrow$ Cartesian spaces (and NOT on manifolds)

NOTE 2: The derived map $NASA(M) \longrightarrow \mathbb{B}^n\mathbb{A}$ is an ordinary map $\check{C}(U) \longrightarrow \mathbb{B}^n\mathbb{A}$, where U is some open cover (good!) of M.

Such maps, can be computed $\rightarrow$ $\hom(\check{C}(U), \mathbb{B}^n\mathbb{A}) = \hom(\check{C}(U), \Gamma(\mathbb{A}[n])) = \hom(N\mathbb{Z}(\check{C}(U)), \mathbb{A}[n])$.

Bundle gerbes continued...

→ we can call such a derived map "$\exists$":

For $\exists \in sPSh$

$\exists : Cart^{op} \longrightarrow sSet$

and for $\mathbb{A} \in AbLieGroup$

$$\exists(s) := \mathcal{U}\Gamma(C^\infty(s, \mathbb{A})[n])$$

$\Uparrow$

Cart

PRACTICAL COMPUTATION of 0-COHOMOLOGY:

The 0th-sheaf cohomology can be computed

$$H^0_{sh}(X, A) \cong A(X)$$

by evaluating A on X, where $A \in Ab$ and $X \in Top$.

(Actually, $A \in sh(X, A)$, a sheaf of abelian groups)

Recall, the constant presheaf ($\underline{A}$)

$$\underline{A} : \text{Open}(X)^{op} \longrightarrow 1 \xrightarrow{B} Ab$$

The map B picks out an abelian group, and the terminal object is mapped to that group.

By a slight abuse of notation, we view the map $1 \xrightarrow{B} Ab$ as an abelian group $B \in Ab$.

Then A, the constant sheaf, is the sheafification of $\underline{A}$

Then, $A(X) := \operatorname{colim}_{\{U_i\}_{i \in I} \text{ open cover of } X} \underline{A}(U)$

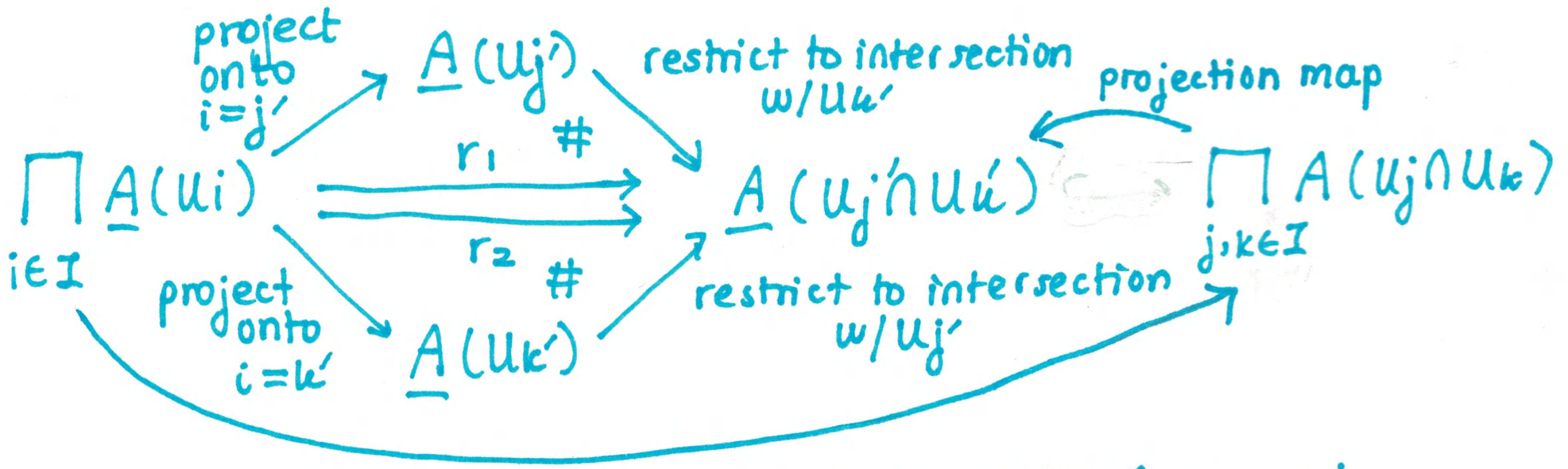
and we have the diagram

projection-restriction maps (next page)

$$\begin{array}{ccccc}
\underline{A}(U) & \longrightarrow & \prod_{i \in I} \underline{A}(U_i) & \overset{r_1}{\underset{r_2}{\rightrightarrows}} & \prod_{j,k \in I} \underline{A}(U_j \cap U_k) \\
 & & & & U_j \cap U_k \neq \emptyset \\
\| & & \| & & \| \\
\text{evaluate constant presheaf} & & \prod_{i \in I} B & \rightrightarrows & \prod_{j,k \in I} B \\
 & & & & U_j \cap U_k \neq \emptyset
\end{array}$$

i copies of the abelian group B

projection - restriction maps



Then, when we evaluate the constant presheaf, we get

$$\prod_{i\in I} B \rightrightarrows \prod_{j,k\in I} B$$

and we can compute the equalizer:

$$\prod_{\check{\pi}_0(\mathcal{U})} B \longrightarrow \prod_{i\in I} B \rightrightarrows \prod_{j,k\in I} B$$

where $\check{\pi}_0(\mathcal{U}) = I/\sim$

(the idex set modulo equivalence relation $\sim$,
$i \sim j$ whenever $U_i \cap U_j \neq \emptyset$)

Thus, this set $\check{\pi}_0(\mathcal{U})$
(a Čech-like index set)
indexes a partition of X into disjoint open subsets

NOTE: $\check{\pi}_0(\mathcal{U})$ is different from $\pi_0(X)$: when $X =$

$|\pi_0(X)| = 2$ and $|\check{\pi}_0(\mathcal{U})| = 1$



take $\mathcal{U} = \{X\}$

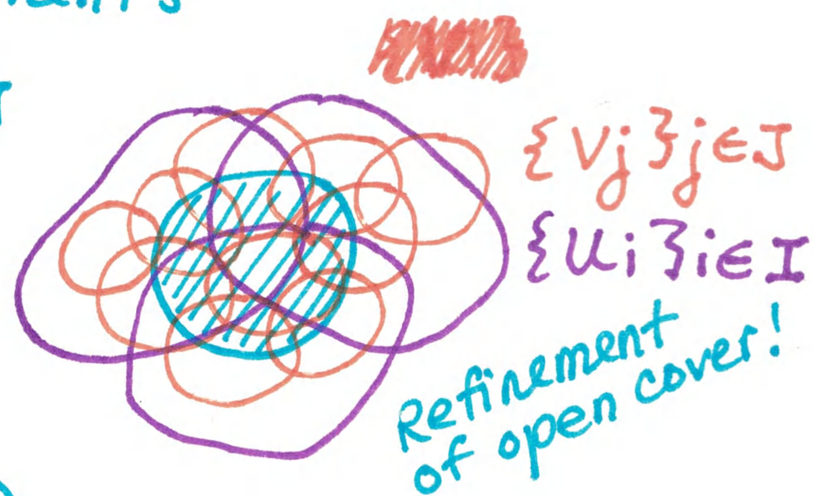
so, $\operatorname{colim}_{\mathcal{U}=\{U_i\}_{i\in I}} \underline{A}(\mathcal{U}) = \operatorname{colim}_{\mathcal{U}} \prod_{\check{\pi}_0(\mathcal{U})} B$

one copy for each connected component ⟶

The colimit is taken over the diagram where objects are open covers and morphisms are refinements

$$\{U_i\}_{i\in I} \longrightarrow \{V_j\}_{j\in J}$$

$$I \xleftarrow{\varphi} J$$



Then $U_{\varphi(j)} \supset V_j$

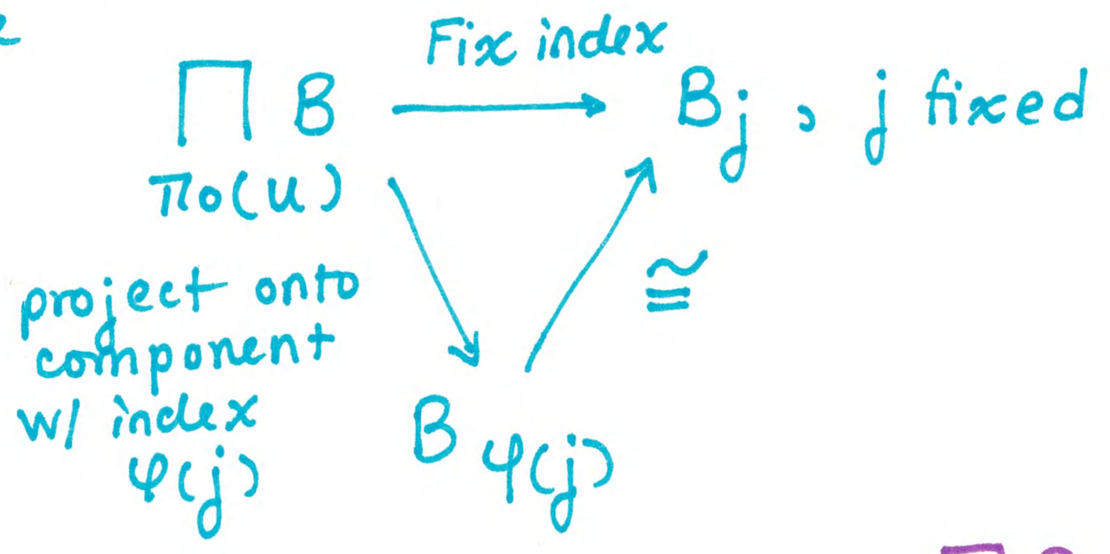
and we have a map

$$\check{\pi}_0(U) \longleftarrow \check{\pi}_0(V)$$

which induces a map of abelian groups:

$$\prod_{\pi_0(U)} B \longrightarrow \prod_{\pi_0(V)} B$$

To define a map into a coproduct ↑, we can define into each component by fixing some index j:



Now, if X is locally connected, $\operatorname{colim} = \prod_{\pi_0(X)} B$ (the usual π_0 computation), so, the sheaf cohomology reflects as many copies of the group of coeffs as there are connected components.

But now, we can compute the 0th-sheaf cohomology of a topological space which is not locally connected!

$$\operatorname*{colim}_{U} \prod_{\pi_0(U)} B = \text{locally constant, } B\text{-valued, functions on } X$$

[where B is equipped w/ the discrete topology,]
... where every element of B forms an open set...

$f: X \to B$ locally constant B-valued function $\Rightarrow$ $f^*\{b\} \subset X$ open $\forall b \in B$, $\{f^*\{b\}\}_{b\in B}$ forms a partition on X into open subsets. ▨

DESCENT PROPERTY FOR PRESHEAVES

PROPOSITION

Given functors $\Delta \xrightarrow{F} Ch$, cosimplicial objects in the category of chain complexes, we can compute the homotopy limit:

$$\text{holim}\left(\Delta \xrightarrow{F} Ch\right) = \text{Tot}^{\sqcap}(NF)$$

(Dold-Kan)

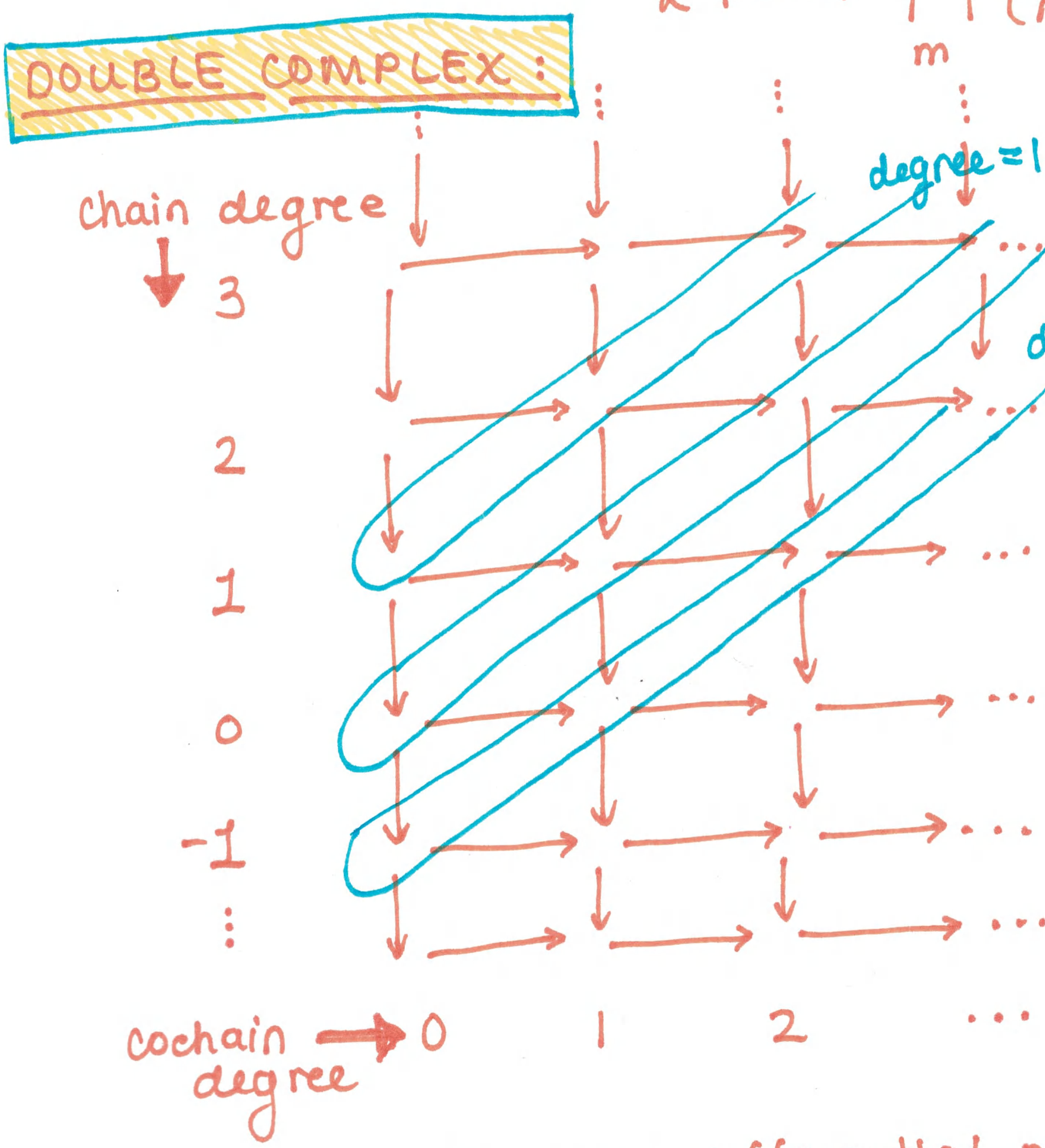
where $\text{Tot}^{\sqcap}$:= totalization

and NF: double complex, a cochain-chain complex which maps some degree

$$k \longmapsto \prod_m (NF)_{m,\,m+k}$$

(cochain index, chain index)

DOUBLE COMPLEX:



$\prod_m (NF)_{m,\,m+k}$

DEFINE!

d

$(NF)_{m,\,m+k-1}$

projection onto index $m, m+k-1$ (fixed m)

$\prod_m (NF)_{m,\,m+k-1}$

DEFINE THE MAP ABOVE AS THE COMPOSITION SHOWN

And we can define the differential map (from degree n to degree $n-1$, cochain differential) using the chain and cochain differentials,

d_V := vertical differential (chain)

d_H := horizontal differential (cochain)

and the differential is defined $\partial = d_H + d_V$

(Note that $\partial^2 = (d_H + d_V)^2 = d_H^2 + d_V^2 + d_H d_V + d_V d_H$

permute, get sign $-$

$= d_H^2 + d_V^2 + d_H d_V - d_H d_V$

$= 0 + 0 + 0 = 0$

$\partial_H^2 = 0 \quad \partial_V^2 = 0$ ↖ cancel!

and $\prod_m (NF)_{m,m+k} \underset{\varphi_2}{\overset{\varphi_1}{\rightrightarrows}} (NF)_{m,m+k-1}$

$\varphi_1 := d_V \circ$ (project onto $m, m+k$)

$\varphi_2 := d_H \circ$ (project onto $m-1, m+k-1$)

[since we have

degree $(m, m+k)$

$d_V \downarrow$

degree $(m-1, m+k-1) \xrightarrow[d_H]{}$ degree $(m, m+k-1)$]

NOTE: For $Ch_{\geq 0}$, the totalization $Tot^{\prod}: coCh_{Ch\geq 0} \longrightarrow Ch_{\geq 0}$ is an "intelligent truncation" of chains, truncating degree < 0 and taking the kernel of the differential map in degree $= 0$.

* Example! *

Take $\mathcal{A} \in Sh(Man, Ab)$

Shift $\mathcal{A}$ in degree n; $\mathcal{A}[n] \in PreSh(Man, Ch)$

Recall, this is not an ∞-sheaf ... $\mathcal{A}[0]$ is an ∞-sheaf in $Sh(Man, Ch_{\geq 0})$ but not $\mathcal{A}[n]$.

However, we would like to determine under what conditions $\mathcal{A}[n]$ is an ∞-sheaf!

That is, under what conditions we can compute homotopy colimits ... →

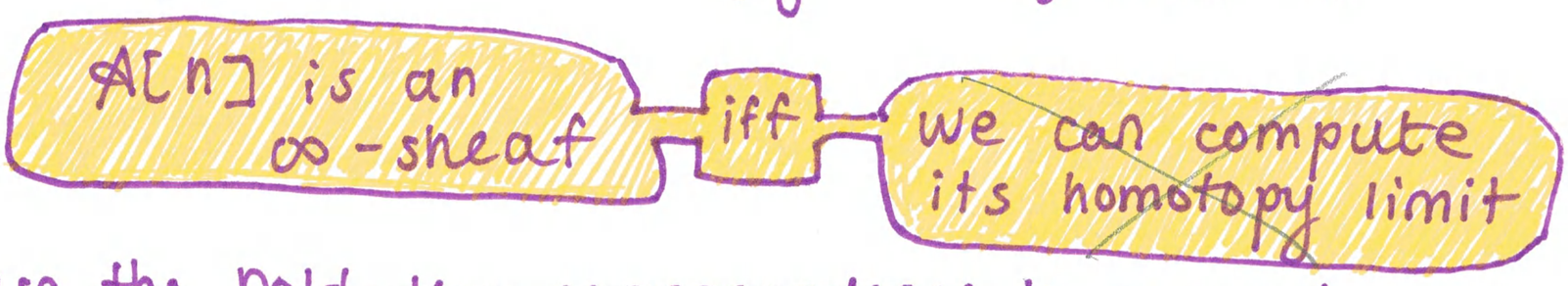
We want to compute the homotopy ~~co~~limit of the diagram:

$$\simeq \nearrow \text{holim}\left(\coprod_i \mathbb{A}[n](U_i) \rightrightarrows \coprod_{jk} \mathbb{A}[n](U_j \cap U_k) \;\substack{\rightarrow\\\rightarrow\\\rightarrow}\; \cdots \right)$$

$\mathbb{A}[n](X)$

where $\mathcal{U} = \{U_i\}_{i \in I}$ is an open cover of $X \in \mathrm{Man}$

Since there is a weak equivalence between $\mathbb{A}[n]$ and its homotopy limit, whenever $\mathbb{A}[n]$ is an ∞-sheaf, we can compute its homotopy limit, that is



☆ weak equiv
i.e., ∞-sheaf condition for

① use the Dold-Kan correspondence to convert the cosimplicial object into a bicomplex:

$$\coprod_i \mathbb{A}[n](U_i) \;\substack{\xrightarrow{f_1}\\\xrightarrow{f_2}}\; \coprod_{jk} \mathbb{A}[n](U_j \cap U_k) \;\substack{\rightarrow\\\rightarrow\\\rightarrow}\; \cdots \quad (q_1, q_2, q_3)$$

in each chain degree, place the same object as in the cosimplicial degree

$$\coprod_i \mathbb{A}[n](U_i) \longrightarrow \coprod_{jk} \mathbb{A}[n](U_j \cap U_k) \longrightarrow \cdots$$

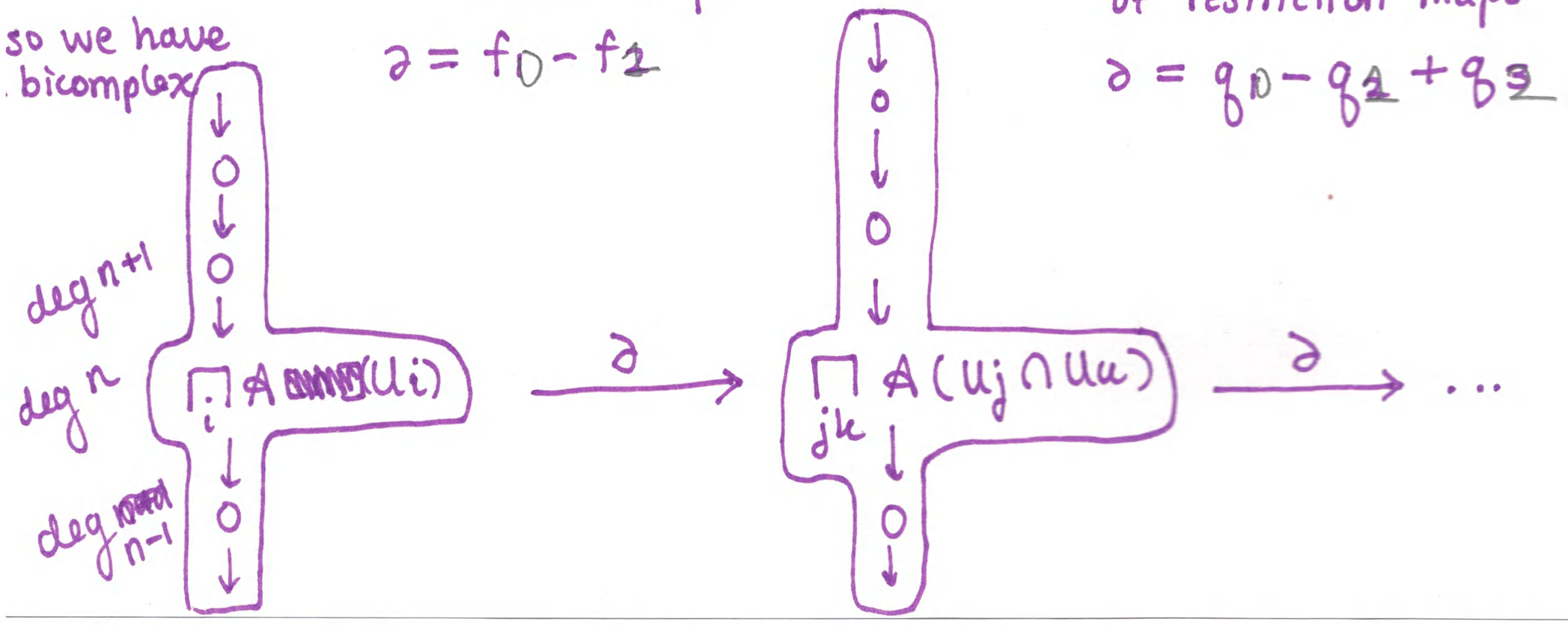
differential: alternating sum of coface maps

$$\partial = f_0 - f_1$$

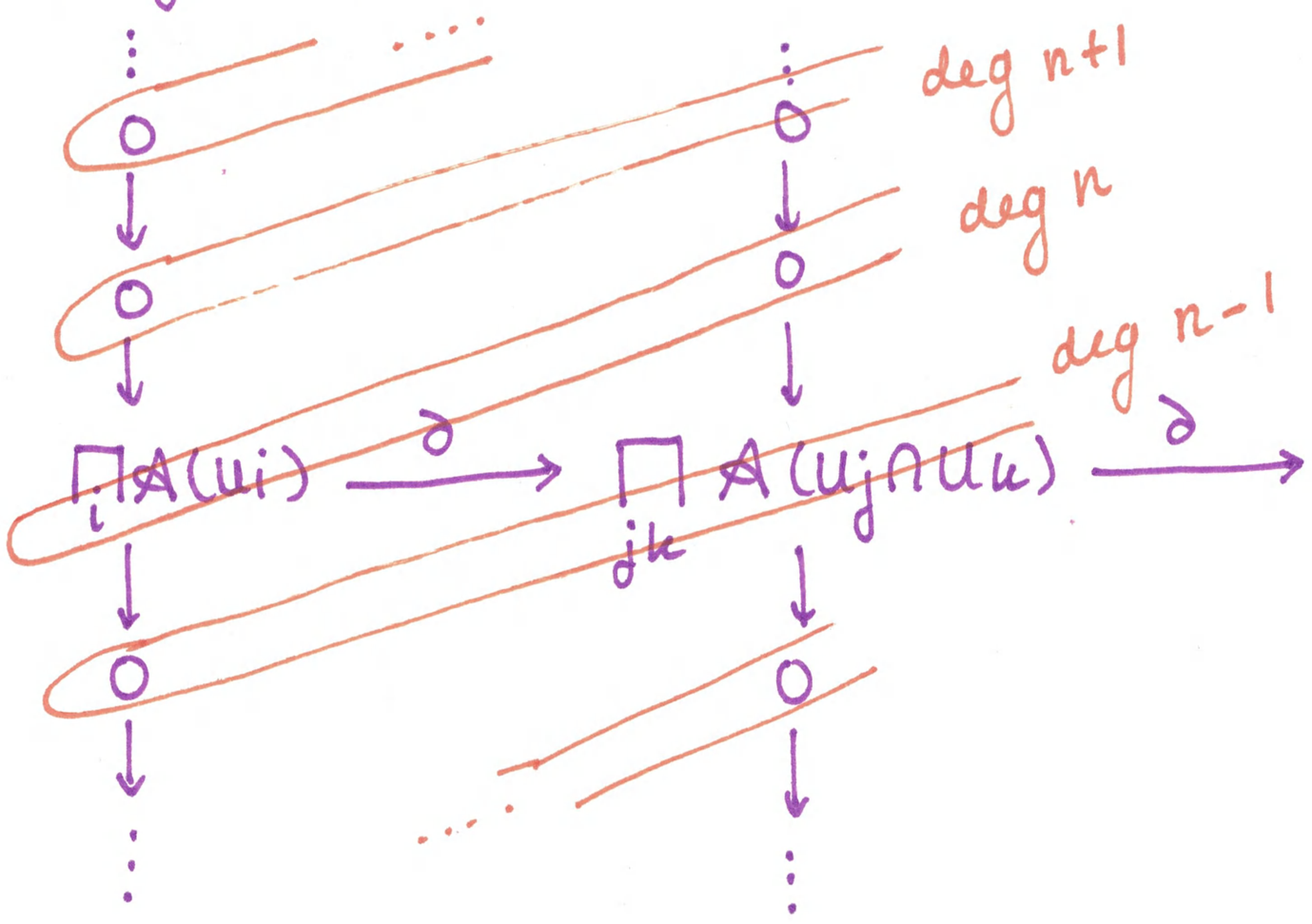
differential: alternating sum of restriction maps

$$\partial = q_0 - q_1 + q_2$$

so we have bicomplex



② Apply the product totalization, $Tot^{\sqcap}$, to the bicomplex:



so in deg n of $Tot^{\sqcap}$, we have $\prod_i \mathcal{A}(U_i)$

in deg n-1 of $Tot^{\sqcap}$, we have $\prod_{jk} \mathcal{A}(U_j \cap U_k)$ ... etc. ....

TOTALIZATION:

so, $Tot^{\sqcap} = 0 \cdots 0 \xrightarrow{\partial} \underbrace{\prod_i \mathcal{A}(U_i)}_{\deg n} \xrightarrow{\partial} \underbrace{\prod_{jk} \mathcal{A}(U_j \cap U_k)}_{\deg n-1} \xrightarrow{\partial} \cdots$ (with $\mathcal{A}(X)$ mapping down to $\prod_i \mathcal{A}(U_i)$)

Where $\partial = \partial_H + \partial_V$ horizontal and vertical differentials of bicomplex

The descent object is the Čech chain/cochain complex!

on each component, there is a restriction map from X to each U_i, hence $\mathcal{A}(X) \longrightarrow \mathcal{A}(U_i)$, $\forall i \in I$

These maps induce an isomorphism between the nth homology groups of $\prod_i \mathcal{A}(U_i)$ and $\mathcal{A}(X)$

$$H_n\left(\prod_i \mathcal{A}(U_i)\right) = \underset{\text{families of } U_i}{\text{matching}} = \underset{\text{over } X}{\text{sections}} = H_n(\mathcal{A}(X)) = \mathcal{A}(X)$$

But this is not true for all other Homology groups (H_k, $k < n$) without some additional restrictions ...

PROPOSITION 1

$\mathcal{A} \in PreSh(X, Ch)$, presheaf

For $X \in Cart$
$\mathcal{U} = \{U_i\}_{i \in I}$
cover of X

$\mathcal{A}(\mathcal{U}) = B \in Ab$

$\mathcal{A}(X)$ is an ∞-sheaf

PROPOSITION 2

$\mathcal{A} \in PreSh(X, Ch)$, presheaf

For $X \in Man$
$\mathcal{U} = \{U_i\}_{i \in I}$
cover of X

$\mathcal{A}(\mathcal{U}) = C^\infty(\mathcal{U}, \mathbb{R}^n)$, $n = dim(M)$

$\mathcal{A}(X)$ is an ∞-sheaf

PROPOSITIONS **1** and **2** above mean

(a) (presheaves)
$Man^{op} \longrightarrow Ch$

$M \longmapsto C^\infty(M, V)[n]$

$V \in Veet_{\mathbb{R}}$, $\forall n$

This is an ∞-sheaf

(b) (presheaves)
$Cart^{op} \longrightarrow Ch$

$X \longmapsto \mathcal{A}[n]$

$A \in Ab$

This is an ∞-sheaf

(c) (presheaves)
$Cart^{op} \longrightarrow Ch$

$X \longmapsto C^\infty(X, \mathcal{A})[n]$

$\mathcal{A} \in AbLieGroup$

This is an ∞-sheaf

(d) (generalization of (a))
(presheaves)
$Man^{op} \xrightarrow[\mathcal{F}]{} Ch_{\mathbb{R}}$ such that $\mathcal{F}$ is a degreewise sheaf and a module over $C^\infty(-, \mathbb{R})$

This is an ∞-sheaf

→ proof: To show these are ∞-sheaves, we need to verify the descent condition.

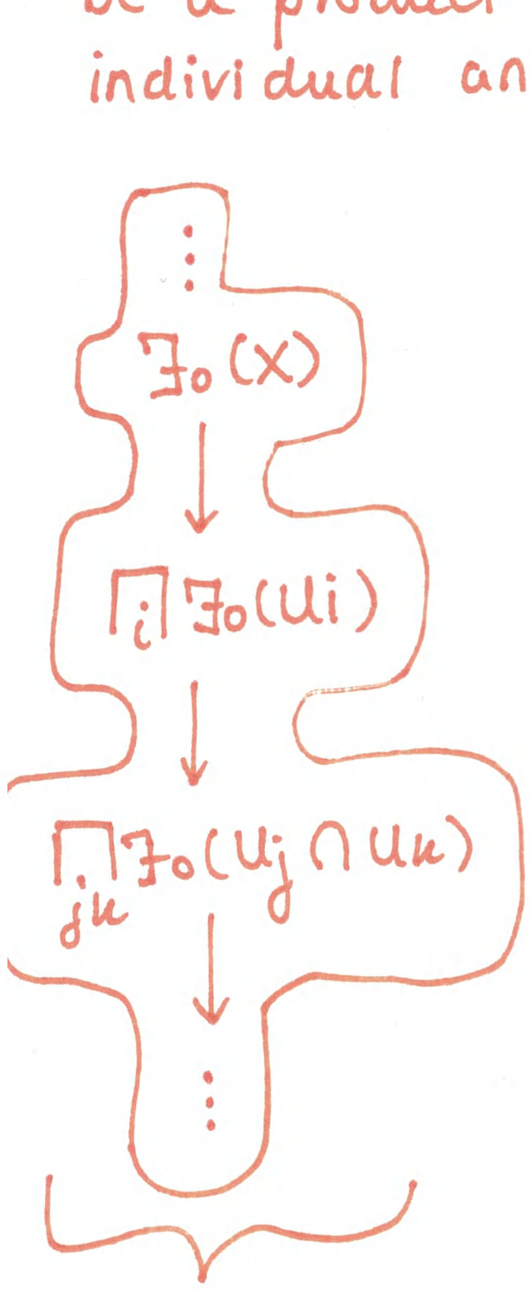
(a) Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of $X \in Man$

Choose a partition of unity $h_i : X \longrightarrow [0,1]$ subordinate to $\mathcal{U}$.

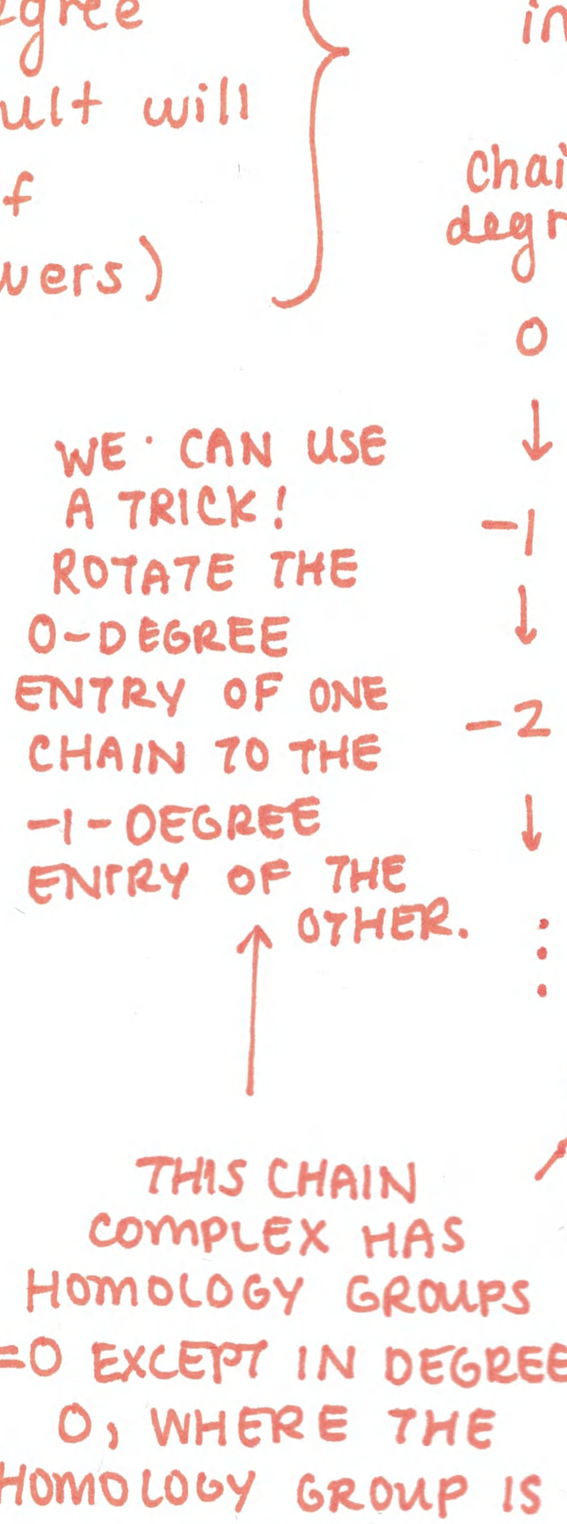
(we will multiply the sections of $\mathcal{F} : Man^{op} \longrightarrow Ch$ by h_i)

Consider (for simplicity) a single chain degree (for then the result will be a product of individual answers)

Choose $\mathcal{F}$ concentrated in degree 0 ($\mathcal{F}_0$)

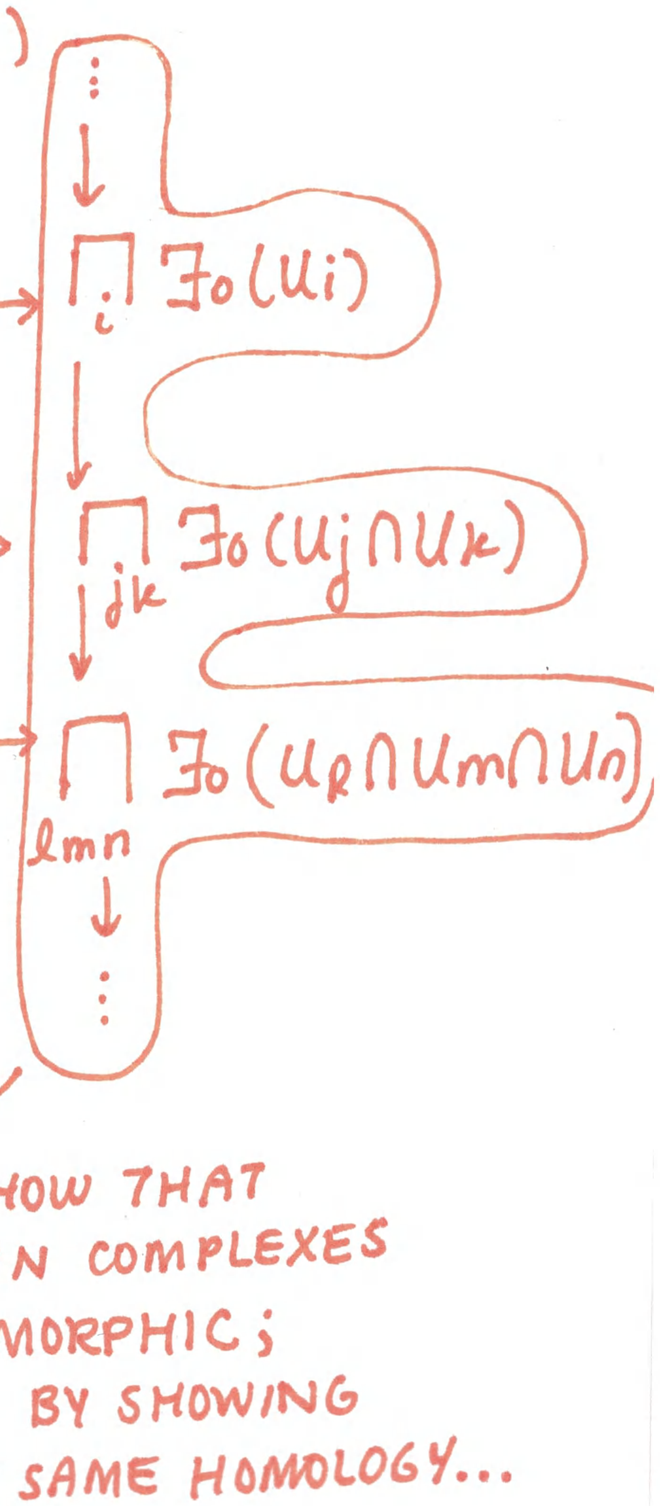


NOW, WE CAN SHOW THIS NEW CHAIN COMPLEX HAS HOMOLOGY GROUPS = 0 IN ALL DEGREES! [EQUIVALENT TO WHAT WE NEEDED ↑ TO SHOW BEFORE]



WE CAN USE A TRICK! ROTATE THE 0-DEGREE ENTRY OF ONE CHAIN TO THE -1-DEGREE ENTRY OF THE OTHER. ↑

THIS CHAIN COMPLEX HAS HOMOLOGY GROUPS = 0 EXCEPT IN DEGREE 0, WHERE THE HOMOLOGY GROUP IS $\mathcal{F}_0(X)$



WE WANT TO SHOW THAT THESE TWO CHAIN COMPLEXES ARE QUASI-ISOMORPHIC; WE CAN DO THIS BY SHOWING THEY HAVE THE SAME HOMOLOGY...

SO, WE NEED TO SHOW THAT THE OTHER CHAIN COMPLEX HAS HOMOLOGY GROUPS = 0 EXCEPT IN DEGREE 0...

CHAIN HOMOTOPIES PRESERVE HOMOLOGY

THE ZERO CHAIN HAS 0-HOMOLOGY GROUPS

If we can show our new chain is chain homotopic to the zero chain, then we will know that all homology groups = 0. →

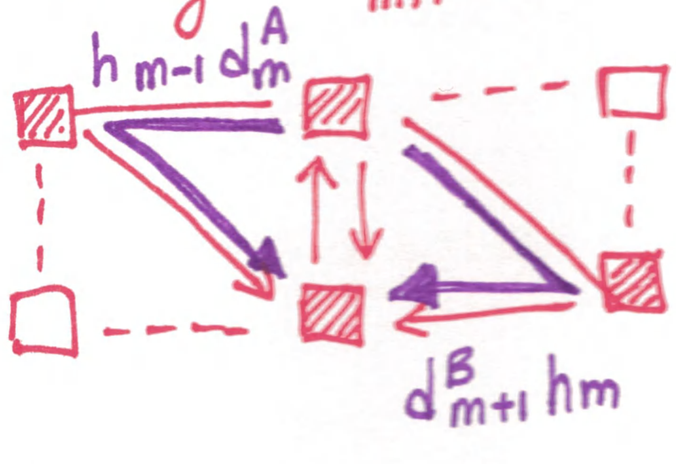
Recall, **chain homotopy** BETWEEN CHAINS "A" AND "B": (complexes)

$$\cdots \leftarrow A_{-1} \xleftarrow{d_0^A} A_0 \xleftarrow{d_1^A} A_1 \xleftarrow{d_2^A} A_2 \leftarrow \cdots$$

$$\cdots \leftarrow B_{-1} \xleftarrow{d_0^B} B_0 \xleftarrow{d_1^B} B_1 \xleftarrow{d_2^B} B_2 \leftarrow \cdots$$

with maps f_{-1}, g_{-1}; h_{-1}; f_0, g_0; h_0; f_1, g_1; h_1; f_2, g_2

define $h: f \to g$ such that, $\forall m$, $f_m - g_m = d^B_{m+1} h_m + h_{m-1} d^A_m$



In our case ... THE ZERO CHAIN COMPLEX

$$\cdots \; 0 \leftarrow 0 \leftarrow 0 \leftarrow 0 \leftarrow \cdots$$

$$\cdots \; \prod_i \mathcal{F}_0(U_i) \leftarrow \prod_i \mathcal{F}_0(U_i) \leftarrow \prod_i \mathcal{F}_0(U_i) \;\; \mathcal{F}_0(X) \leftarrow \cdots$$

$i \in I$

THE NEW CHAIN COMPLEX

and we can define the chain homotopy by

$f := \text{identity}$, $g := 0$ } NTS $\quad id - 0 = d^B_{m+1} h_m + h_{m-1} d^A_m$

$\Rightarrow id = d^B_{m+1} h_m + h_{m-1} d^A_m$

so let us define:

$$h_m : \prod_i \mathcal{F}_0(U_{i_0} \cap \cdots \cap U_{i_m}) \longrightarrow \prod_j \mathcal{F}_0(U_{j_0} \cap \cdots \cap U_{j_{m-1}})$$

where for some fixed index j, the jth component is:

$$(h_m)_j = (-1)^m \sum_{i_m \in I} h_{i_m} \cdot (x_{j, i_m}) \Big|_{\bigcap_j U_j}$$

where $\bigcap_j U_j = U_{j_0} \cap \cdots \cap U_{j_{m-1}}$

and $x \in \prod_i \mathcal{F}_0(U_{i_0} \cap \cdots \cap U_m)$

Then, we have a chain homotopy, our chain (complex) is homotopic to the zero chain (complex) it has all zero homology groups $\Rightarrow$ the original two chains (complexes) are quasi-isomorphic

→ continued...

which is to say that the presheaf $\exists$ satisfies the descent property of an ∞-sheaf (as desired).

Now, we also should verify the identity } Exercise for reader, not writer — HA!

$$id = d^B_{m+1} h_m + h_{m+1} d^A_m, \quad \forall m \geq -1$$

which uses the partition of unity defined previously)

(b) $X \in \mathrm{Cart}$, $A \in \mathrm{Ab}$ $\quad \exists(X) := A[0]$ (concentrated in degree 0)

Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a good open cover of X.

Then we need to show a quasi-isomorphism between chains:

$$\begin{array}{ccccccccc}
\cdots & \to & 0 & \to & A & \to & 0 & \to & 0 & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & 0 & \to & \prod_{i \in I} A & \to & \prod_{j,k \in I} A & \to & \prod_{\ell,m,n \in I} A & \to & \cdots
\end{array}$$

$U_j \cap U_k \neq \emptyset \quad U_\ell \cap U_m \cap U_n \neq \emptyset$

$= C^*(N(\mathcal{U}), A)$

cochains ~~from~~ on the nerve ~~to~~ A with coefficients

Recall,

$N(\mathcal{U})_p := \{ t \in I^{p+1} \mid U_{t_0} \cap \cdots \cap U_{t_p} \neq \emptyset \}$

and for a cartesian space X w/ good cover $\mathcal{U}$

$N(\mathcal{U}) \cong \mathrm{sing}(X) \cong \Delta^0$ (Nerve Theorem)

$\uparrow$ weak equivalences $\uparrow$

and $N(\mathcal{U}) \xrightarrow{\pi} \Delta^0$

$\exists$ projection maps (weak equivalence)

Thus, the map

$$\begin{array}{ccccccccc}
\cdots & \to & 0 & \to & A & \to & 0 & \to & 0 & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
C^*(\Delta^0, A) & & & & & & & & & & \\
\| \cdots & \to & 0 & \to & \prod A & \to & \prod A & \to & \prod A & \to & \cdots \\
C^*(\pi, A) & & & & & & & & & &
\end{array}$$

which is a quasi-isomorphism.

→ $\mathcal{D}(\mathcal{A}_0^u)$ is contractible (part @)
so $\mathcal{D}(\mathcal{A}_0^u) \simeq 0$

and by the 5-lemma, we can say we have a quiso between chains.

[$\mathcal{D}(\mathcal{A}_0)$ is homotopy cofiber of map $\iota : \mathcal{D}(\mathcal{L}) \longrightarrow \mathcal{D}(\mathcal{A}_0^u)$ cofibration in the inj. model structure
hence $\simeq 0$, thus $\mathcal{D}(\mathcal{A})$ must be $\simeq 0$ as well.]

Homological Algebra (Day 4 P.M.)

Sites, coverages, and Grothendieck topologies

A site is something on which you can define sheaves

DEF: A **site** is a category with a **coverage**

DEF: A **coverage** T on a category $\mathcal{C}$ (presheaves)
is an assignment of a collection of families, $T(X)$,
of morphisms in $\mathcal{C}$ w/ common codomain, X,

$$\{ f_i : U_i \longrightarrow X \mid i \in I \}$$

such that $\forall h : Y \longrightarrow X$

$$\forall \{ f_i : U_i \longrightarrow X \} \in T(X)$$

$$\exists \{ g_j : V_j \longrightarrow Y \mid j \in J \} \in T(Y)$$

that is, $\forall j \in J$, $\exists i \in I$ and a map $V_j \longrightarrow U_i$
such that the diagram commutes:

refines the pullback of f

$$\begin{array}{ccc} U_i & \xrightarrow{f_i} & X \\ \uparrow & \# & \uparrow h \\ V_j & \xrightarrow[g_j]{} & Y \end{array}$$

axiomatizes category of opens

EXAMPLE:

$\mathcal{C}$ = Topological spaces

$$T(X) = \{ \mathcal{U} \mid \mathcal{U} = \{ f_i : U_i \longrightarrow X \}_{i \in I} \text{ open subset}, \bigcup_{i \in I} U_i = X \} \longrightarrow$$

→ That is, assign to X the collection of all open covers.

Then $\forall h: Y \longrightarrow X$ (map between topological spaces)

$\forall \mathcal{U} = \{f_i: U_i \longrightarrow X\}_{i \in I}$, $\bigcup_{i \in I} U_i = X$

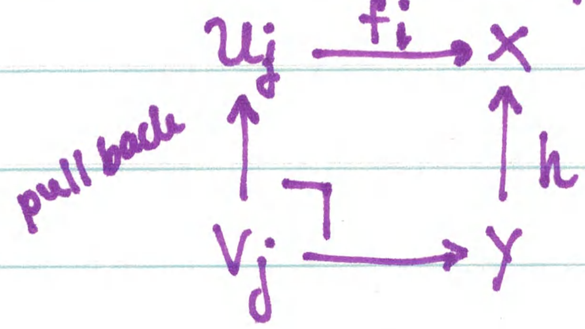
we need to find

an open cover $\mathcal{U}' = \{g_j: V_j \longrightarrow Y\}_{j \in J}$, $\bigcup V_j = Y$

Take $J = I$ ⬇

and $V_j = h^* U_j$

that is, we have a pullback square



EXAMPLE: $\mathcal{C}$ = Cartesian Spaces (Cart)

$T(X) = \{\mathcal{U} \mid \mathcal{U} = \{f_i: U_i \longrightarrow X\}_{i \in I}, \bigcup_{i \in I} U_i = X$

(U_i open subset; $U_i \in$ Cart)

↗ (good covers of X)

covering families of X

$U_{i_0} \cap \dots \cap U_{i_k} = \emptyset$ or $U_{i_0} \cap \dots \cap U_{i_k} \in$ Cart

Then $\forall h: Y \longrightarrow X$ (smooth map of cartesian spaces

$\forall \mathcal{U}$ (covering family), $\mathcal{U} \in T(X)$

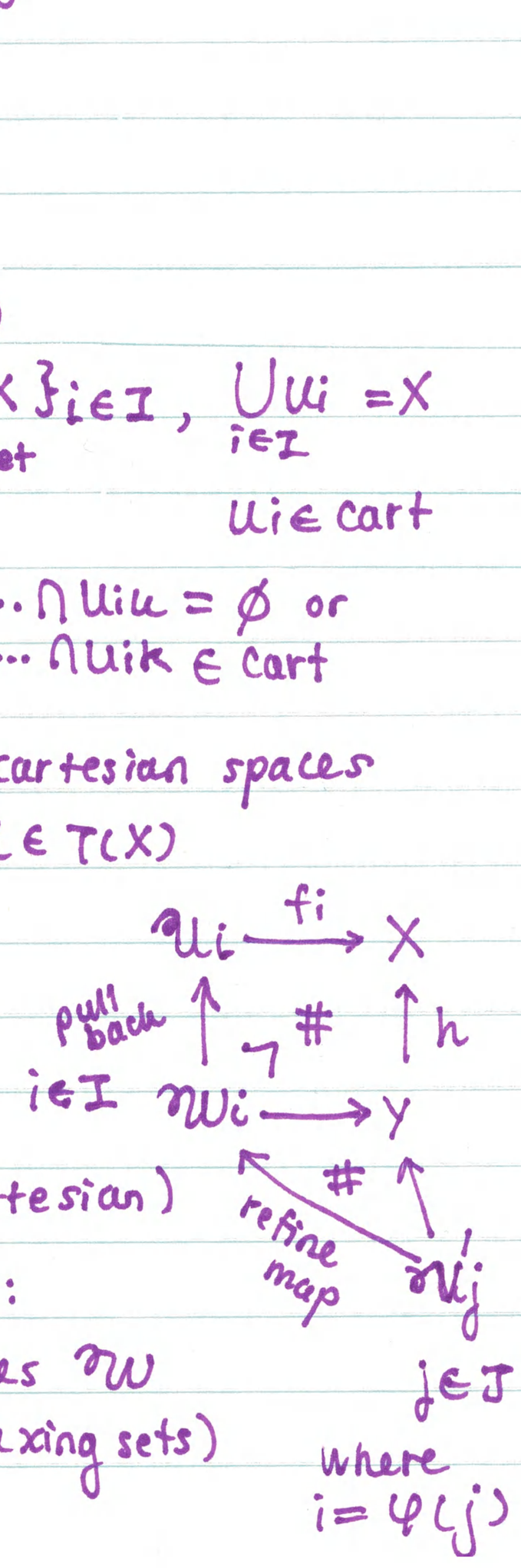
We need to find $\mathcal{U}' \in T(Y)$

Consider $\mathcal{W} = h^* \mathcal{U}$

$\mathcal{W}$ is an open cover of Y

but $\mathcal{W} \notin T(Y)$ in general

(each $\mathcal{W}_i$ is not necessarily Cartesian)



Refine $\mathcal{W}$ by a good cover:

$\mathcal{U}'$ is a good cover that refines $\mathcal{W}$

via $\varphi: J \longrightarrow I$ (map of indexing sets)

EXAMPLE: For $X \in Top$, we can construct a site

$Open(X)$

(whose underlying category is the poset of open subsets of X)

and the "coverage" is specified exactly as in Top.

This site gives sheaves (usual sense) on topological space X

EXAMPLE:

Another important site: For $X \in Top$ construct a site Top/X "slice category"

(whose underlying category is

Ob: $Y \to X$ maps

Mor: commutative diagrams between maps with same codomain

(fiberwise continuous maps))

And the "coverage" is specified as inherited from Top via the forgetful functor

$$Top/X \longrightarrow Top$$

$$Y \to X \longmapsto Y$$

(object → total space)

EXAMPLE:

Variation: For $X \in Man$

the SMALL "petit" topos $Open(X)$ ↕ very different! the BIG "gross" topos Man/X

NOTE: There are more sheaves over gros site than over petit site!

Example: $X = pt$

$Sh(open(pt), set) \simeq set$ ← very different! → $Sh(Man/pt \simeq Man, set) \simeq sh(Man, set)$

A sheaf over $Open(X) \equiv X$-indexed family of (discrete) sets.

and A sheaf over Man/X ≡ X-indexed family of smooth sets

DEF: Given $\mathcal{C}$ site, we can define

ⓐ sheaves on $\mathcal{C}$

$$Sh(\mathcal{C}, set) = \{ \mathcal{F} : \mathcal{C}^{op} \to set \mid \forall X \in \mathcal{C}, \forall \mathcal{U} \in T(X), \mathcal{F}(X) \to \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod \mathcal{F}(V) \}$$

$\mathcal{U} = \{ f_i : U_i \to X \}$, $\bigcup_{i \in I} U_i = X$

$$\mathcal{F}(X) \to \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod \mathcal{F}(V)$$

$$\begin{array}{ccc} U_i & \to & X \\ a \uparrow & \# & \uparrow \\ V & \to & U_j \\ & b & \end{array}$$

map #1 $\longrightarrow$ project to U_i, restrict to V along a

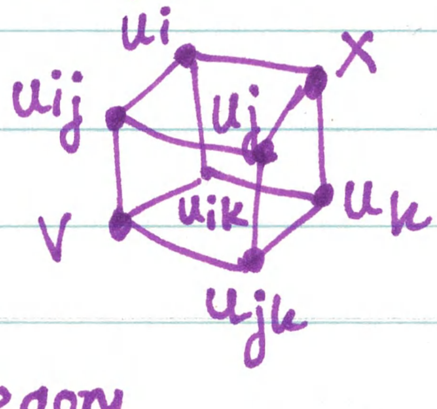
map #2 $\longrightarrow$ project to U_j, restrict to V along b

compatibility condition on coverage

ⓑ we can define ∞-sheaves similarly

$$\mathcal{F}(X) \to \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod \mathcal{F}(V) \;\substack{\rightarrow\\ \rightarrow\\ \rightarrow}\; \prod$$

$$\begin{array}{ccc} U_i & \to & X \\ a \uparrow & \# & \uparrow \\ V & \to & U_j \\ & b & \end{array}$$



DEF: A (Grothendieck) topos is a category equivalent to sheaves on a small site, $\mathcal{C}$, $Sh(\mathcal{C}, set)$

↑ small category

$\longrightarrow$

Geometric Mechanics (Day 4 P.M.)

→ How do we generalize to ~~pro-sheaves~~ ∞-topos

DEF: A sieve on $X \in \mathcal{C}$ is a family of morphisms $S \subset \bigsqcup_Y \hom(Y, X)$ with a fixed codomain such that $\forall s \in S$, $s : Y \longrightarrow X$

$\forall g : Z \longrightarrow Y$ (which need to be in S)

Then $s \circ g : Z \longrightarrow X$, $s \circ g \in S$



sift out all the "good" morphisms!

DEF: The sieve generated by a covering family $A = \{f_i : U_i \longrightarrow X\}$ is the intersection of all sieves that contain A.

$[= \{a \circ g \mid a \in A, g \text{ any morphism, where } a \circ g \text{ is defined}\}]$

DEF: A Grothendieck topology, T, on a category, $\mathcal{C}$, assigns a collection $T(X)$ of sieves on X (known as "covering sieves"), $\forall X \in \mathcal{C}$, such that $(\mathcal{C}, T)$ is a site and

saturation conditions:

- $\{f : Y \longrightarrow X\} \in T(X)$
- $\forall X \in \mathcal{C}$, $\forall R \in T(X)$
  $\forall S$: sieve on X
  if $\forall f \in R$: ~~$f \in S$~~
  $f^* S \in T(\mathrm{dom}(f))$
  then $S \in T(X)$.
  covering sieve!

Here, $f^* S = \{g \mid f \circ g \in S\}$

Recall, "coverage"

$\mathcal{U} \in T(X)$

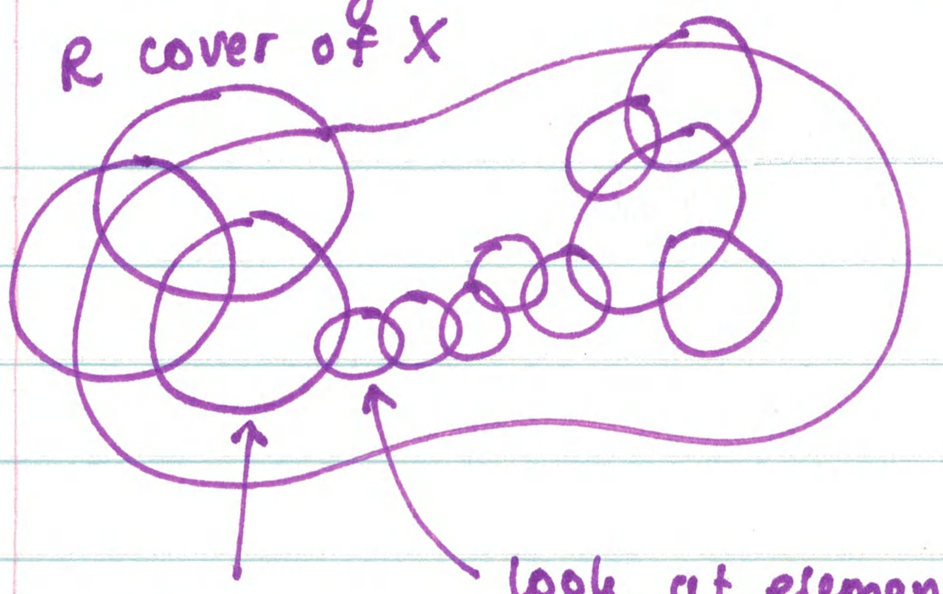
$h : Y \to X$

$h^* \mathcal{U} \supset \mathcal{U}'$ where $\mathcal{U}' \in T(Y)$

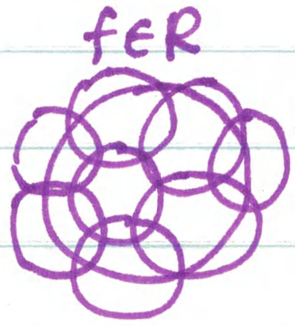
a way to pullback covers!

Geometrically

R cover of X



S is a candidate cover of X



look at elements of S that fit inside it...

the elements of S inside f∈R form an open cover of f

∀f∈R

Fix element of R

PROP: Any coverage T can be saturated to a Grothendieck topology T'

$$sh((\mathcal{C}, T'), set) \xrightarrow{\simeq} sh((\mathcal{C}, T), set)$$

equivalence of categories

(This implies that

A presheaf is a sheaf w/respect to T

iff it is a sheaf w/respect to T')

$\mathcal{F}: \mathcal{C}^{op} \longrightarrow set$

$\mathcal{F}$ is a T-sheaf $\Leftrightarrow$ $\mathcal{F}$ is a T'-sheaf

(and there are similar statements about ∞-sheaves)

The category of sheaves $\subset Fun(\mathcal{C}^{op}, set)$

uniquely determines a Grothendieck topology T on $\mathcal{C}$.

$T \in$ Grothendieck topologies on $\mathcal{C}$: $GT(\mathcal{C})$

$GT(\mathcal{C}) \longrightarrow$ full subcategories of $Fun(\mathcal{C}^{op}, set)$ (presheaves) that are toposes.

$T \longmapsto sh((\mathcal{C}, T), set)$

Grothendieck Topology is like "topology on a set"
Coverage is like "base of a topology"

Resource: "sheaves in Geometry and Logic" (book)

Saturation property allows us to get arbitrary open covers from good covers:

EXAMPLE: Cart, with a (T) coverage of good covers

$\forall X \in Cart$, $\forall U$ open cover of X (NOT good)

Claim: $\mathcal{U} \in GT(C)$ generated by T

1. $\mathcal{U}$ is enhanced to a sieve by ~~[illegible]~~ replacing $\mathcal{U}$ by the sieve generated by $\mathcal{U}$, U_S (by the intersection of all sieves containing $\mathcal{U}$)

2. Any open cover has a refinement by a good cover, so $\mathcal{U}$ has a refinement V where V is a good cover.

3. V is a subset of U_S (since all subsets of elements of $\mathcal{U}$ have been added to $\mathcal{U}$)

4. Take $R = V$ in saturation properties
   $S = U_S$
   Then $\forall f \in R$, ~~[illegible]~~ $f^*S \in T(dom(f))$

   because $\mathcal{U}$ covers X $\left(\bigcup_{i \in I} U_i = X\right)$, we have, $\bigcup_{Z \in f^*S} dom(Z) = dom(f)$

and note that any cover can be refined by a good cover ... →

→ (5.) f^*S is an open cover of $dom(f)$

f^*S can be refined by a good cover W, where W is a subcover of f^*S because f^*S is a sieve.

⇒ f^*S is a covering family of ~~dom(f)~~ the domain of f

$f^*S \in T(dom(f))$

Moral: Canonical Groth top has generators with which you can work to make computations easier

THEOREM The following coverage generates the Grothendieck topology on Man:

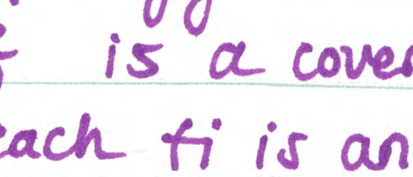
$A = \{ f_i : U_i \longrightarrow X \}$ is a covering family if each f_i is an open inclusion, and

card $(A) = 2$ or $U_i \times_X U_j = \emptyset$ if $i \neq j$

intersection

Proof use saturation conditions to show that you can eventually add all open covers from these very limited covers...

① Add covers of the form where only consecutive open sets intersect:



"zig zag cover"

U_i, $i \geq 0$, $U_i \cap U_j = \emptyset$ if $|i-j| > 1$

union of all even covering elements

union of all odd covering elements

② Add families $U_0 \subset U_1 \subset U_2 \subset \cdots$

~~(see picture on board! on phone)~~

Homological Algebra (Day 5 P.M.)

Man w/open covers generated by } From Geometric Mechanics (Day 4 P.M.)

- covers w/ 2 elements
- disjoint covers

But Man can be replaced by: ← paracompact Hausdorff Topological spaces w/open covers

← Topological spaces w/numerable open covers (open covers that admit a sub partition of unity)

These are called "numerable sites" ↗

Theorem (Pavlov) suppose we have a pre-sheaf of cochain complexes

$$\mathcal{F} : Man^{op} \longrightarrow coCh$$

such that $\mathcal{F}_n : Man^{op} \longrightarrow Ab$

$\forall n \in \mathbb{Z}$ (one particular degree)

is a sheaf of abelian groups

and $\forall M \in Man$, $\forall U, V \subset M$, $U \cup V = M$ (open cover)

the map

$$F_n(U) \oplus F_n(V) \longrightarrow F_n(U \cap V)$$

$$\underset{\text{section}}{U}, \underset{\text{section}}{V} \longmapsto U|_{U\cap V} - V|_{U\cap V}$$

is surjective — difference of their restriction to intersection

Then $\mathcal{F}$ is an ∞-sheaf!

Thus we can deduce that $\mathcal{F}$ is an ∞-sheaf from looking at individual degrees!

Proof **Case 1:** disjoint, open covers

(intersections are empty, so ∞-sheaf equalizer fork collapses to a single stage...)

$\forall M$, $\forall \{U_i\}_{i\in I}$ $\bigcup_{i\in I} U_i = M$

so

$$\mathcal{F}_n(M) \xrightarrow{\simeq} \prod_{i\in I} \mathcal{F}_n(U_i)$$ is a quasi-isomorphism (same homology groups! →)

$$\mathcal{F}_n(M) \xrightarrow{\simeq} \prod_{i\in I}{}^{n} \mathcal{F}(U_i) \quad = \quad \prod_{i\in I} \mathcal{F}_n(U_i)$$

isomorphism $\simeq$ (by the sheaf property of $\mathcal{F}_n$, $\forall_n$)

(since products are already computed degreewise!)

Case 2: covers w/ 2 elements

$\forall M,\ \forall U, V \subset M,\ U \cup V = M$

(only two elements, so there are intersections, but no triple intersections so, there are only two stages)

$$\begin{array}{ccc} \mathcal{F}(M) & \longrightarrow & \mathcal{F}(U) \\ \downarrow & \lrcorner_h & \downarrow \\ \mathcal{F}(V) & \longrightarrow & \mathcal{F}(U\cap V) \end{array}$$

homotopy pullback

$\Updownarrow$

short exact sequence formulation

$$\mathcal{F}(M) \longrightarrow \mathcal{F}(U) \times^h \mathcal{F}(V) \underset{0}{\overset{\hat{u}}{\rightrightarrows}} \mathcal{F}(U\cap V)$$

is a homotopy equalizer fork

$\Updownarrow$

$$\mathcal{F}(M) \longrightarrow \mathcal{F}(U) \times^h \mathcal{F}(V) \xrightarrow{\hat{u}-\hat{v}} \mathcal{F}(U\cap V)$$

is a homotopy fiber sequence

(Note: The homotopy equalizer of $\underset{\hat{v}}{\overset{\hat{u}}{\rightrightarrows}}$ is the same as the homotopy equalizer of $\underset{0}{\overset{\hat{u}-\hat{v}}{\rightrightarrows}}$)

The theorem condition that

$\pi_0 F_n(U) \oplus F_n(V) \longrightarrow F_n(U\cap V)$ is surjective

means that the map $\hat{u} - \hat{v}$ is surjective!

(For it is the same map)

Recall, degreewise surjections are fibrations in the projective model structure!

so $\Updownarrow$ $\mathcal{F}(M) \longrightarrow \mathcal{F}(U) \times \mathcal{F}(V) \xrightarrow{\hat{u}-\hat{v}} \mathcal{F}(U\cap V)$

is an exact sequence! (fiber sequence)

Fix n, $\quad \mathcal{F}_n(M) \longrightarrow \mathcal{F}_n(U) \times \mathcal{F}_n(V) \xrightarrow{\hat{u}-\hat{v}} \mathcal{F}(U \cap V)$

$\cong$ isomorphism by the sheaf property of $\mathcal{F}_n$

$\mathcal{F}_n(M) \longrightarrow \ker(\hat{u}-\hat{v})$

∎

Now, we can use this Theorem to show that many well-known ∞-sheaves are, indeed, ∞-sheaves.

THEOREM (Weil)

$\Omega : \mathrm{Man}^{op} \longrightarrow \mathrm{coCh}$ (differential forms)

$M \longmapsto \Omega(M)$

Ω is an ∞-sheaf

Proof: In any fixed-degree, differential n-forms form a sheaf of abelian groups (in the usual sense)

$\Omega^n : \mathrm{Man}^{op} \longrightarrow \mathrm{Ab}$

(smooth sections of a vector bundle form a sheaf)

$\forall M$, $\forall U, V \subset M$, $U \cup V = M$

The map $\Omega^n U \oplus \Omega^n V \twoheadrightarrow \Omega^n(U \cup V)$ is surjective

To show this surjectivity, choose partition of unity, and $w \in \Omega^n(U \cup V)$

Then $(f \circ w, (1-f) \circ w) \longmapsto w$

extend by 0 to U and $U \supset U \cap V$

$f : M \longrightarrow \mathbb{R}_{\geq 0}$

$1-f : M \longrightarrow \mathbb{R}_{\geq 0}$

partition of 1 such that subordinate to $\{U, V\}$

$\mathrm{supp} f \subset U$

$\mathrm{supp}(1-f) \subset V$

will be smooth because supported on intersections! ∎

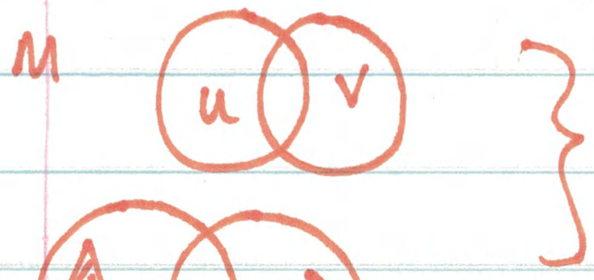
→ Geometric Mechanics (Day 5 P.M.)

Theorem $A \in Ab$

$C^*: Man^{op} \longrightarrow coCh$

$M \longmapsto C^*(M, A)$ is an ∞-sheaf.

Proof: $C^n(-, A)$ is not a sheaf



} so, we can sheafify!



assigned a value in A

assigned value in A

but



cannot assign a value in A

sheafification : $C^n(-, A) \longrightarrow C^n_{local}(-, A)$

assigns values in A only to sufficiently small simplices, those that are subordinate to some open cover.

Warner §5.32

(Exercise...)

Using the chain Whitehead Theorem:

$$K \hookrightarrow C(-, A) \twoheadrightarrow C_{local}(-, A)$$

K ↪ kernel; $C \twoheadrightarrow C_{local}$ degreewise surjective

K is acyclic, i.e., $H^n(K) = 0$, $\forall n$

$C_{local}(-, A)$: elements here are equivalence classes

Recall, $C^n_{local}(-, A) = \{$ ($\mathcal{U}$ open cover of M) and $c: Sing\mathcal{U} \to A$ $\}$ / $\mathcal{V}$ refines $\mathcal{U}$ $(\mathcal{U}, c) \sim (\mathcal{V}, c|_{sing\mathcal{V}})$

$= \{ \sigma: \Delta^n \to M$ continuous that factor through U_i for some $i \}$

and the map

$C(-, A) \to C_{local}(-, A)$

$c \longmapsto (\{M\}, c^{\{M\}})$

→

(Thus, the map is degreewise surjective because we can extend ~~[illegible]~~ c by 0)

The map is a projective fibration (degree wise surjection) $\Rightarrow$ The map is a quasi-isomorphism

$\Rightarrow$ so they are objectwise quasi-isomorphic, hence,

$C(-, A)$ ∞-sheaf $\Leftrightarrow$ $C_{local}(-, A)$ ∞-sheaf

$\uparrow$

Here, each degree is a sheaf of abelian groups

$\forall M$, $\forall U, V \subset M$, $U \cup V = M$

$$C^n_{local}(U) \oplus C^n_{local}(V) \longrightarrow C^n_{local}(U \cap V)$$

is surjective

To see this, choose some $c \in C^n_{local}(U \cap V)$

then

$c : Sing\, W \to A$

W open cover of $U \cap V$

preimage

$(c', 0) \longmapsto c$

where the map is extension of c by 0 to Sing U

$c' \in C^n_{local}(U)$

$c' = (\{U\}$, extend c using 0 to all other elements) ▨

Theorem Alexander-spainer cochains ~~[illegible]~~

$As^*(-, A) : Man^{op} \longrightarrow$ ~~[illegible]~~ coCh

is an ∞-sheaf

Alexander-spainer...

(Recall, replace simplex by vertices, so △ gets replaced by 3-tuple)

yields an analog of an n-chain; then we sheafify by identifying those which correspond on sufficiently small --- →

proof: $AS^n(-, A) :=$ sheafification $\left(M \longmapsto (n \longmapsto \hom(M^{n+1}, A))\right)$

$AS^n(U, A) \oplus AS^n(V, A) \twoheadrightarrow AS^n(U \cap V, A)$ ▨

$\sim$ maps of sets

corollary On the site of smooth manifolds
(or any paracompact, Hausdorff topological spaces)

$$\text{Man}, \quad \Omega \overset{\text{quiso}}{\simeq} C^*(-, \mathbb{R}) \simeq AS^*(-, \mathbb{R}) \simeq \mathbb{R}_\delta[0]$$

$C^*(-,\mathbb{R})$: singular cochains w/ real coeff

$AS^*(-,\mathbb{R})$: Alexander spanier cochains w/ real coeff

$\mathbb{R}_\delta[0]$: discrete (constant sheaf) in degree 0

Ω, $C^*(-,\mathbb{R})$, $AS^*(-,\mathbb{R})$ are quasi-isomorphic
(objectwise - weakly equivalent)
∞ - sheaves

Real #s w/ discrete topology!

and $\mathbb{R}_\delta[0]$ is only a presheaf
(so $\simeq$ is a local weak equivalence only)

and recall $C^*(-, \mathbb{R}) \simeq C^*_{C^\infty}(-, \mathbb{R})$

(continuous) (smooth maps and continuous)

and we can write equivalences between homology:

$$H^n_{dR}(M) \cong H^n_{sing}(M, \mathbb{R}) \cong H^n_{As}(M, \mathbb{R}) \cong H^n_{sh}(M, \mathbb{R}_\delta)$$

$$\| \text{s}$$

$$\check{H}^n(M, \mathbb{R}_\delta)$$

Čech cohomology

Proof!

$H^n_{dR}(M) \cong H^n_{sing}(M, \mathbb{R})$

Map #1:

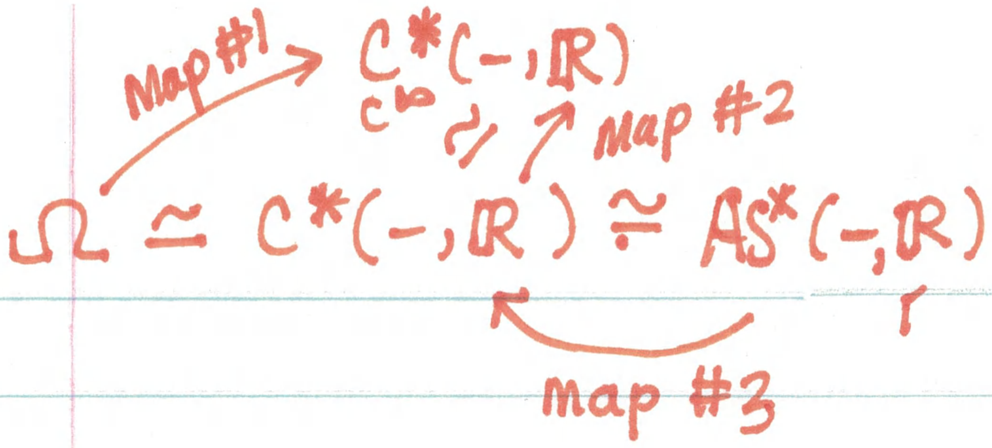
$\Omega \xrightarrow{\int} C^*_{C^\infty}(-, \mathbb{R})$

integration

$\omega \longmapsto ((\sigma : \Delta^n \to M) \longmapsto \int \sigma^* \omega \in \mathbb{R}$ → continued...

$\omega \in \Omega^n(M)$

$\sigma^* \omega$: diff n-form on Δ^n



Map #2: restriction

Map #3: evaluate alexander spainer cochain on vertices (restriction maps) of singular simplex

show Map #1, Map #2, Map #3 are quasi-isomorphisms

we can restrict to Cart$^\infty$ (Cart $\subset$ Man, since (∞-sheaves on Man equivalent to ∞-sheaves on Cart)

so, we can substitute
$- = \mathbb{R}^a$ for some $a \geq 0$

$$\Omega(\mathbb{R}^a) \xrightarrow[\text{Map \#1}]{} C^*_{C^\infty}(\mathbb{R}^a,\mathbb{R}) \xleftarrow[\text{Map \#2}]{} C^*(\mathbb{R}^a,\mathbb{R}) \xleftarrow[\text{Map \#3}]{} AS^*(\mathbb{R}^a,\mathbb{R})$$

Reeall, Poincaré lemma
$\mathbb{R}_s[0] \simeq \Omega(\mathbb{R}^a)$

computing cohomology in the normal way, since

$$\begin{cases} C^*(\mathbb{R}^a,\mathbb{R}) \simeq \mathbb{R}_s[0] \\ AS^*(\mathbb{R}^a,\mathbb{R}) \simeq \mathbb{R}_s[0] \end{cases}$$

$\mathbb{R}^a$ is contractible to a point, it has the same cohomology as a point.

so all maps induce isomorphisms in 0-cohomology. ▨

Homological Algebra (Day 6 P.M.)

Ref: Dugger "A primer on homotopy colimits"

Revisiting Homotopy (co)limits & actually calculating homotopy limits/colimits

$$\mathcal{C}^I \underset{\text{const}}{\overset{\text{colim}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{C} \qquad \mathcal{C} \underset{\text{lim}}{\overset{\text{const}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{C}^I$$

where $\mathcal{C}^I = \text{Fun}(I, \mathcal{C})$

$\text{hocolim} = \mathbb{L}\,\text{colim}$ (cofibrantly replace diagram)

$\text{holim} = \mathbb{R}\,\text{lim}$ (fibrantly replace diagram)

Homotopy (co)ends

DEF Given a functor,

$$F: I^{op} \times I \longrightarrow \mathcal{C}$$

(where I : indexing category)

definition: The "end" of F is an object in the category $\mathcal{C}$

object: $\int_I F = \int_{i \in I} F(i,i) \in \mathcal{C}$

together w/ projection maps $\int_{i \in I} F(i,i) \xrightarrow{\pi_i} F(i,i)$, $\forall i \in I$

commutativity condition: such that, $\forall h: i \to j$ (a morphism in I)

$$\begin{array}{ccc} \int_{i \in I} F(i,i) & \xrightarrow{\pi_i} & F(i,i) \\ \downarrow{\scriptstyle \pi_j} & \# & \downarrow{\scriptstyle F(i,h)} \\ F(j,j) & \xrightarrow[F(h,j)]{} & F(i,j) \end{array}$$

Ref: Fosco Leregian "Coend Calculus"

universal property: and the following universal property holds:

if $A \in \mathcal{C}$, $p_i: A \to \exists(i,i)$ (w/ the commutative diagram above)

then, $\exists!$ map: $A \longrightarrow \int_I F$ such that it is compatible w/ projection maps

"wedge" (like one):

$$\begin{array}{ccc} A & \xrightarrow{p_i} & F(i,i) \\ \downarrow & \# & \nearrow{\scriptstyle \pi_i} \\ \int_I F & & \end{array}$$

ends are special cases of limits, coends are special cases of colimits. In fact, ends can be defined using limits.

→ The coend is dually defined!

"coend" $\int^I F = \int_{i \in I} F(i,i)$

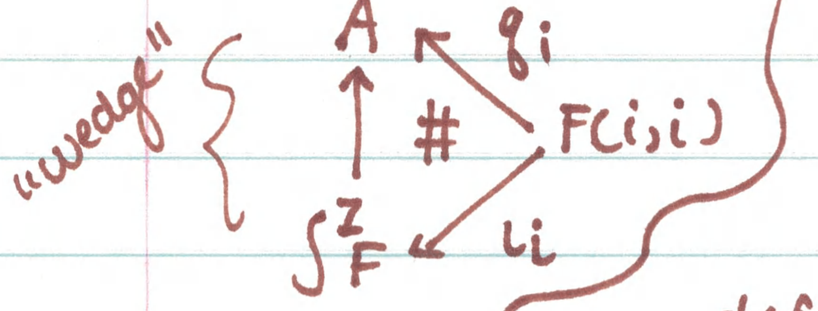
together w/ injection maps $\int^I F \xleftarrow{Li} F(i,i)$

and diagram

$$\begin{array}{ccc} \int^I F & \xleftarrow{Li} & F(i,i) \\ {\scriptstyle Lj}\uparrow & & \uparrow{\scriptstyle F(h,i)} \\ F(j,j) & \xleftarrow{} & F(j,i) \end{array} \qquad F(j,h)$$

and ∃! map $q_i : A \leftarrow \int^I F$

such that



EXAMPLE

$G, H : I \longrightarrow \mathcal{D}$

consider a functor

$F : I^{op} \times I \longrightarrow Set$

define $F(i,j) := hom(G(i), H(j))$

The "end of F" is well-defined:

$\int_I F \in \mathcal{C} = Set$

The end is a set

and we have projection maps $\int_I F \xrightarrow{\pi i} hom(G(i), H(i))$ $\forall i \in I$

w/ commutativity condition:

$\forall h : i \longrightarrow j$ (morphism in I)

$$\begin{array}{ccc} \int_I F & \xrightarrow{\pi i} & hom(G(i), H(i)) \\ {\scriptstyle \pi j}\downarrow & & \downarrow{\scriptstyle \text{define!}} \\ hom(G(j), H(j)) & \longrightarrow & hom(G(i), H(j)) \end{array}$$

define!

choose $t \in \int_I F$. Then $\pi i(t) \in hom(G(i), H(i))$

$\pi i(t) : G(i) \longrightarrow H(i)$

we use diagram in $\mathcal{D}$ to define map

$$\begin{array}{ccc} G(i) & \xrightarrow{\pi i(t)} & H(i) \\ {\scriptstyle G(h)}\downarrow & & \downarrow{\scriptstyle H(h)} \\ G(j) & \xrightarrow[\pi j(t)]{} & H(j) \end{array}$$

so, we define →

$$\begin{array}{ccc} hom(G(i), H(i)) & & \\ \downarrow{\scriptstyle hom(G(i), H(h))} & & \\ hom(G(i), H(j)) & & \end{array}$$

so the map $\hom(G(i), H(h)) : \hom(G(i), H(i)) \longrightarrow \hom(G(i), H(j))$

takes an element $\pi_i(t) \longmapsto H(h) \circ \pi_i(t)$

$\pi_i(t) : G(i) \longrightarrow H(i)$

and composes it with the map $H(h)$

$H(h) : H(i) \longrightarrow H(j)$

so we get a map

$H(h) \circ \pi_i(t) : G(i) \longrightarrow H(j)$

an element of $\hom(G(i), H(j))$

so we have a commutative diagram (a diagram of sets):

$$\begin{array}{ccc} \int_I F & \xrightarrow{\pi_i} & \hom(G(i), H(i)) \\ \downarrow{\scriptstyle \pi_j} & \# & \downarrow \\ \hom(G(j), H(j)) & \longrightarrow & \hom(G(i), H(j)) \\ & & \hom(G(h), H(j')) \end{array}$$

we defined this map as above

we defined this map as above

Diagram in $\mathcal{C} = \text{Set}$

natural transformations

NOTE:

t is a natural transformation $G \Rightarrow H$

By the Universal property of ends, we see that the "end" is precisely the set of natural transformations

(we have specified a map $A \longrightarrow \hom(G, H)$)
$\qquad\qquad\qquad

Compare to ordinary limits:

In an ordinary limit we have projection maps:

$$\lim \longrightarrow F(i,j), \quad \forall i,j \in I$$

In an end, we have projection maps

$$\int_I \longrightarrow F(i,j), \quad \forall h : i \longrightarrow j$$

Example (continued...)

"wedge" is a map $A \longrightarrow \hom_{Fun(I,\mathcal{D})}(G,H)$

and a morphism of wedges

$$\begin{array}{ccc} A & \longrightarrow & \hom(G,H) \\ \downarrow & \nearrow_{\#} & \\ A' & & \end{array}$$

equivalent to...

The category of wedges over $F \equiv$ slice category of the set

$$\begin{array}{ccc} A & \longrightarrow & \hom(G,H) \\ \downarrow & \nearrow_{\#} & \\ A' & & \end{array}$$

And the end is the terminal object in the category of wedges

So here, slice category

$Set/\hom(G,H)$

and the terminal object in this slice category is the set

$\hom(G,H)$

Recall

slice category

For $d \in \mathcal{D}$

Then $\mathcal{D}/d$ defined

Objects: $e \longrightarrow d$

Morphisms:

$$\begin{array}{ccc} e & \longrightarrow & d \\ \downarrow & \nearrow_{\#} & \\ e' & & \end{array}$$

(objects are morphisms and morphisms are commutative diagrams (triangles))

terminal object in slice category $\mathcal{D}/d$ is $id_d : d \longrightarrow d$

$$\begin{array}{ccc} e & \longrightarrow & d \\ \downarrow & \nearrow_{\#} & \\ d & \nearrow_{id} & \end{array}$$

So, the "end" is defined.

$$\int_I F = \hom(G,H)$$

set of natural transformations from G to H. ▨

The immediate benefit of reformulating natural transformations as an end is that we can generalize to an enriched setting:

Example: $G, H: I \longrightarrow \mathcal{D}$

$\mathcal{D}$: closed category, more generally, V-enriched category, V: monoidal category
(that is, there is a functor $\mathrm{Hom}: \mathcal{D}^{op} \times \mathcal{D} \longrightarrow V$)

Now, define the functor we use in def of end:

$F: I^{op} \times I \longrightarrow V$

$F(i,j) = \mathrm{Hom}(G(i), H(j))$

Then $\int_I F$ = the enriched object of natural transformations $G \Rightarrow H$
(show commutativity and universal property)

Example:

$G: I \longrightarrow \mathcal{C}$

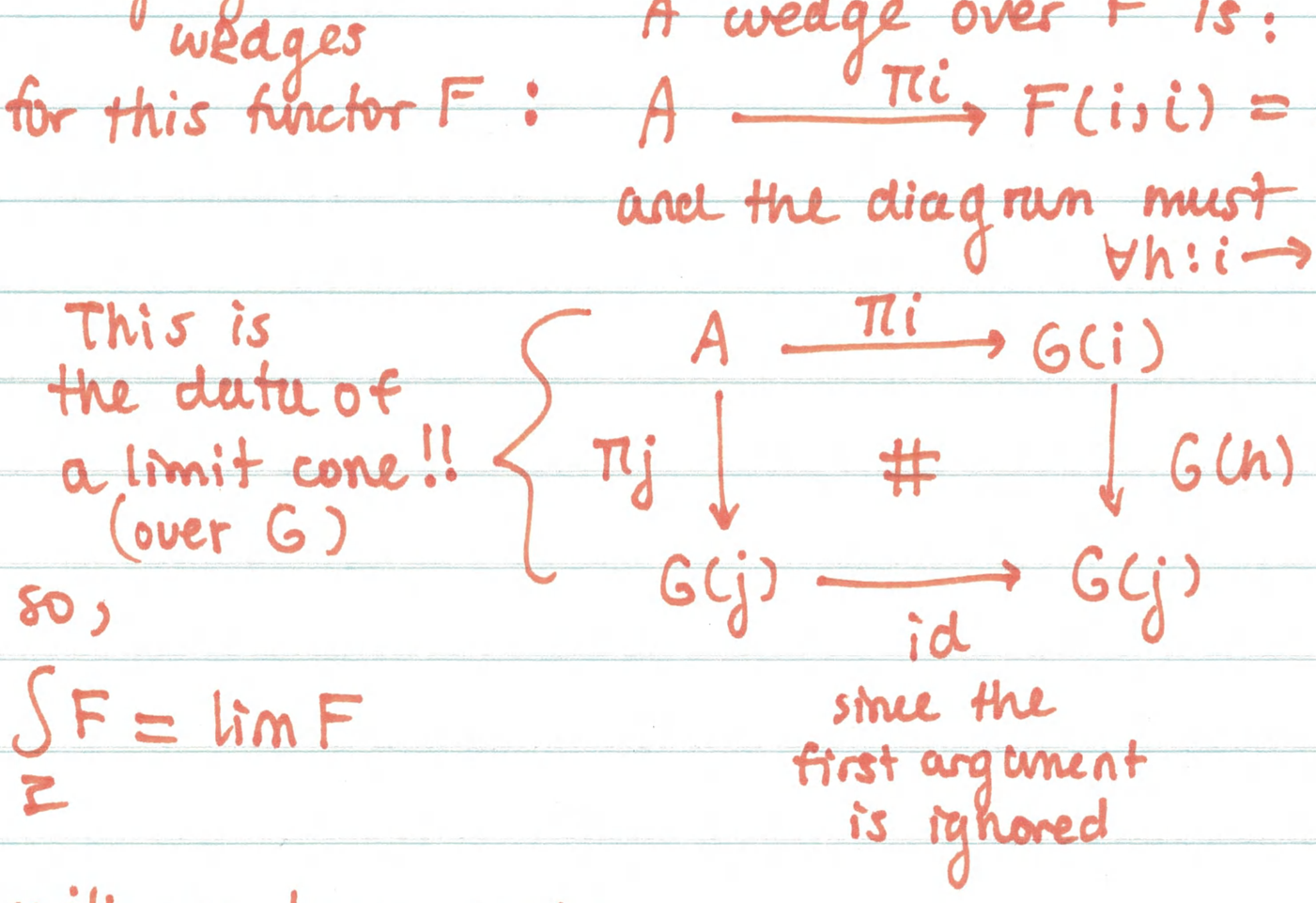
Let $F: I^{op} \times I \xrightarrow{\text{projection map}} I \longrightarrow G$, $F(i,j) = G(j)$ depends only on one variable

We need to identify the category of wedges for this functor F:

A wedge over F is:

$A \xrightarrow{\pi_i} F(i,i) = G(i)$

and the diagram must commute: $\forall h: i \to j$



This is the data of a limit cone!! (over G)

so,

$\int_I F = \lim F$

We will see how ends can be expressed as limits in general.

→ **PROPOSITION**

$$\int_I F \longrightarrow \prod_{i\in I} F(i,i) \rightrightarrows \prod_{h:j\to k} F(j,k)$$

the end is an equalizer of the two maps

Fix h, define map into a product

$h: j \longrightarrow k$

$$\prod_{i\in I} F(i,i) \xrightarrow{\text{project onto } i=j} F(j,j) \xrightarrow{\text{use } h \text{ in 2nd argument}} F(j,k)$$

$$\prod_{i\in I} F(i,i) \xrightarrow{\text{project onto } i=k} F(k,k) \xrightarrow{\text{use functoriality in first argument}} F(j,k)$$

↓↓ How the two maps work

use h in first argument

Geometric Mechanics (Day 6 P.M.)

PROPOSITION The "end" is a right adjoint functor to the "coproduct over hom"

For $I \in \mathrm{Cat}$

$$\mathcal{C} \underset{\int_I}{\overset{\mathrm{Hom}_I}{\rightleftarrows}} \mathcal{C}^{I^{op}\times I}$$

The left adjoint functor

$$\bigsqcup : \mathcal{C} \longrightarrow \mathcal{C}^{I^{op}\times I}$$

defined!

$\hom_I$ bimodule

$$c \longmapsto \left((i,j) \longmapsto \bigsqcup_{\hom_I(i,j)} c \right)$$

Homotopy Ends

The above adjunction is a Quillen adjunction when the right side is equipped w/ the correct model structure!

THEOREM
(Arkhipov, Orsted, 2018)

The adjunction

$$\mathcal{C} \underset{\int_I}{\overset{\bigsqcup_{\hom_I}}{\rightleftarrows}} \mathcal{C}^{I^{op}\times I}$$

is a Quillen adjunction if $\mathcal{C}$ is a model category and if the right side →

$\longrightarrow$ $\mathcal{C}^{I^{op}\times I}$ is equipped w/ one of 3 model structures:

NOTE:

$$\mathcal{C}^{I^{op}\times I} = (\mathcal{C}^{Z^{op}})^{I} = (\mathcal{C}^{Z})^{I^{op}}$$

$$(\mathcal{C}^{I^{op}}_{Reedy})^{Z}_{Reedy} = \mathcal{C}^{I^{op}\times I}_{Reedy}$$

(1) $(\mathcal{C}^{I^{op}}_{proj})^{I}_{inj}$ (2) $(\mathcal{C}^{Z}_{proj})^{Z^{op}}_{inj}$ (3) $(\mathcal{C}^{I}_{Reedy})^{I^{op}}_{Reedy}$

NEW! The Reedy Model structure! $\left\{ \mathcal{C}^{I^{op}\times I}_{Reedy} \right.$

The Reedy Model structure is somewhere between projective and injective model structures, and only exists for certain categories, I,

"Reedy categories"

$$\mathcal{C}^{Z}_{proj} \longrightarrow \mathcal{C}^{I}_{Reedy} \longrightarrow \mathcal{C}^{Z}_{inj}$$

Def A Reedy category is a category I equipped with a map

$$degree: Ob(I) \longrightarrow \alpha$$

where α is an ordinal number set (thought of as $\mathbb{N}$, but more technical)

$I^{+} \subset I$, $I^{-} \subset I$ } subcategories

$$Ob(I^{+}) = Ob(I^{-}) = Ob(I)$$

$f: i \longrightarrow j$

$f \neq id$, $f \in I^{+} \Rightarrow deg(i) < deg(j)$

$f \neq id$, $f \in I^{-} \Rightarrow deg(i) > deg(j)$ $\longrightarrow$

Factorization Property

$\forall f: i \longrightarrow j$

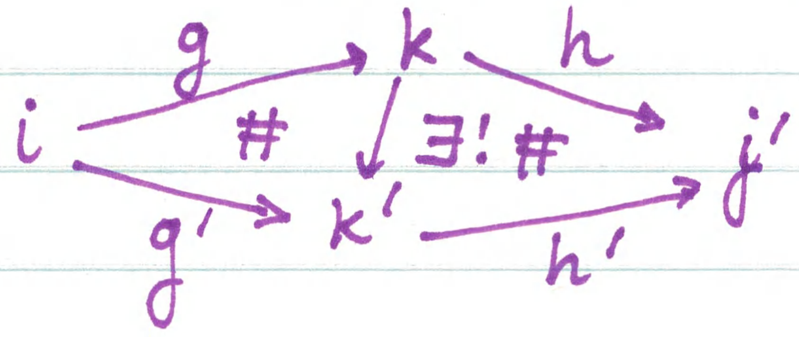
$\exists g: i \longrightarrow k$, $g \in I^-$

$\exists h: k \longrightarrow j$, $h \in I^+$

$$f = hg$$

(g,h) are unique up to an unique isomorphism

Given (g,h), (g',h')



Example

$I = \Delta$

degree $([n]) = n$

I^+ = injective maps of simplices

I^- = surjective maps of simplices

Example/Prop: If I is Reedy then I^{op} is Reedy

(I Reedy $\Leftrightarrow$ I^{op} Reedy)

Ex Δ^{op} Reedy

since Δ is Reedy, simply exchange I^+ and I^- to see Δ^{op} is Reedy

Def $\mathcal{C}$: model category

$I \in$ cat, Reedy

The Reedy Model structure on $\mathcal{C}^I$ defined:

- weak equivalences = objectwise
- cofibrations
  acyclic cofibrations

$t: F \longrightarrow G$ is a/an (acyclic) cofibration if the relative latching map

$\forall n \in I$ $F_n \bigsqcup_{L_nF} L_nG \longrightarrow G_n$ is a/an (acyclic) cofibration in $\mathcal{C}$.

→ (acyclic) fibration
if the relative matching map

$\forall n \in I$, $F_n \longrightarrow G_n \underset{M_n G}{\times} M_n F$

is an (acyclic) fibration in $\mathcal{C}$

DEF $I \in \mathrm{ReedyCat}$

$n \in I$, $F \in \mathcal{C}^I$, $\mathcal{C} \in \mathrm{CAT}$

$L_n F \longrightarrow F_n \longrightarrow M_n F$

latching object — latching map — matching map — matching object

Given functor, F,

$L_n \longrightarrow \mathrm{eval}_n \longrightarrow M_n$

are all functors $\mathrm{Fun}(\mathcal{C}^I, \mathcal{C})$

$L_n F = \mathrm{colim}\, F_m$ and $L_n F \longrightarrow F_n$

Ob: $m \to n$, $\neq \mathrm{id}$, $\in I^+$; all positive arrows landing in n

Mor: $m \to m' \# \to n$

$L_n F = \mathrm{colim} F_m$
we can define a map out of a colimit in the usual way

and, similarly, for matching object

$F_n \longrightarrow M_n F = \lim F_m$

Ob: $n \to m$, $\in I^-$, $\neq \mathrm{id}$

Mor: $n \to m \# \to m'$

Example $X : \Delta^{op} \longrightarrow \mathrm{Set}$

Δ^{op} Reedy

latching object

$L_n X = \mathrm{colim}_{[n] \to [m] \neq \mathrm{id}} X_m$

In Δ^{op}, I^+ = surjective maps
I^- = injective maps

(Δ I^+ = inj, I^- = surj)

and $L_n X \longrightarrow X_n$

so, latching object = the set of degenerate simplices in X_n

latching map = inclusion map into X_n

→ For $[n] \twoheadrightarrow [m]$

degenerate across all vertices } degeneracy maps

For the matching map, $X_n \longrightarrow M_n X = \lim_{[m] \twoheadrightarrow [n] \neq i} X_m$

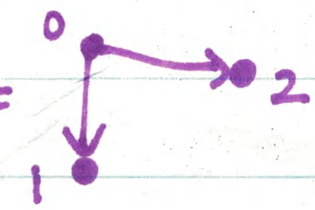
and by Yoneda $X_n = \hom(\Delta^n, X)$ face maps! { picks a face!

so $M_n X = \hom(\partial\Delta^n, X)$

so the matching map $\hom(\Delta^n, X) \longrightarrow \hom(\partial\Delta^n, X)$

takes an n-simplex and computes its boundary

so latching map includes degenerate simplices and matching map computes boundaries!

Example $I =$ 

$Ob = \{0, 1, 2\}$

$Mor = \{id_0, id_1, id_2, 0 \to 1, 0 \to 2\}$

Reedy category: degree function sends $0 \longmapsto 0$

$1 \longmapsto 1$

$2 \longmapsto 1$

$$\begin{cases} I^+ = I \\ I^- = \{id\} \end{cases}$$

or

degree function sends $0 \longmapsto 1$

$1 \longmapsto 0$

$2 \longmapsto 0$

$$\begin{cases} I^+ = \{id\} \\ I^- = I \end{cases}$$

and then we have two different ways to compute homotopy pushouts!

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \\ C & & \end{array}$$

way #1 cofibrantly replace

$$\begin{array}{ccc} A & \rightarrowtail & B \\ \downarrow & & \\ C & & \end{array}$$

way #2 glue cylinder between

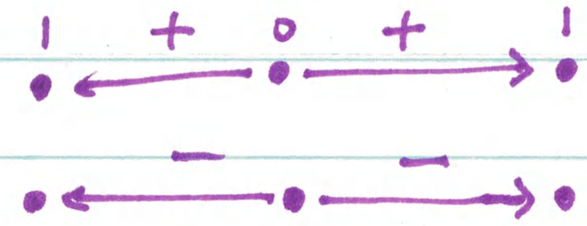
Homological Algebra (Day 7 P.M.)

Ex $I =$ 

belong to $\bar{I}$

indexing category

homotopy pushouts belongs to I^+

degree 

Three Reedy category structures

Def $\mathcal{C}$ is a $\mathcal{V}$-enriched category ($\mathcal{V}$: symmetric monoidal category)

Recall $\mathcal{V}$-enriched: $\forall c, c' \in \mathcal{C}$ we can specify $\hom_{\mathcal{V}}(c,c') \in \mathcal{V}$ (target category)

(a) tensoring $\otimes : \mathcal{V} \times \mathcal{C} \longrightarrow \mathcal{C}$

$v, c \longmapsto v \otimes c$

tensor aka "copower"

universal property

morphisms $v \otimes c \longrightarrow c'$ in bijection with morphisms $v \longrightarrow \hom_{\mathcal{V}}(c,c')$

$\Rightarrow$ $\{ v \otimes c \longrightarrow c' \}$ $\cong$ $\{ v \longrightarrow \hom_{\mathcal{V}}(c,c') \}$

and $(v_1 \hat{\otimes} v_2) \otimes c = v_1 \otimes (v_2 \otimes c)$

associative property holds; $\hat{\otimes}$: monoidal product in $\mathcal{V}$; $\otimes$: tensor product; $\otimes$: tensor product

since $\{ (v_1 \hat{\otimes} v_2) \otimes c \longrightarrow c' \}$

$\cong$

$\{ v_1 \hat{\otimes} v_2 \longrightarrow \hom_{\mathcal{V}}(c,c') \}$ $\rightleftharpoons$ (use lemma) $\{ v_1 \longrightarrow \mathrm{Hom}_{\mathcal{V}}(v_2, \hom_{\mathcal{V}}(c,c')) \}$

$\cong$

$\{ v_1 \longrightarrow \hom_{\mathcal{V}}(v_2 \otimes c, c') \}$ $\cong$

$\cong$

$\{ v_1 \otimes (v_2 \otimes c) \longrightarrow c' \}$ $\longrightarrow$

$\mathcal{V}$ is a closed, symmetric monoidal category w/ $\mathrm{Hom}_{\mathcal{V}}$ (internal hom) and tensor product $\hat{\otimes}$

→ **Lemma:**

Note that we need this eq. in previous result

$\{U_1 \longrightarrow \hom_V(U_2 \otimes C, C')\}$

$\cong$

$\{U_1 \longrightarrow \mathrm{Hom}_V(U_2, \hom_V(C, C'))\}$

$\otimes$, $\hom_V$ } tensor-hom in $\mathcal{C}$

$\hat{\otimes}$, Hom_V } tensor-hom in V

maybe change Hom_V to $\hat{\hom}_V$

Lemma so, $\hom_V(U_2 \otimes C, C') \cong \mathrm{Hom}_V(U_2, \hom_V(C, C'))$

(easily proved, too formal for Dr. Pavlov to want to prove)

DEF continued... (b) powering [redacted] "cotensor" aka: power

$(-)^{(-)} : V^{op} \times \mathcal{C} \longrightarrow \mathcal{C}$

$U, C \longmapsto C^U$

$\| $

$\hom_V(U, C)$

Universal property:

[redacted]

$\{C' \longrightarrow C^U\}$

$\cong$

$\{V \longrightarrow \hom_V(C', C)\}$

Ex $V = Ab$ $\mathcal{C} = Ch$

(a) $(U \otimes C)_n := U \otimes C_n$

$(C^U)_n := (C_n)^U$

(b) $V = sSet$

$\mathcal{C} = sPSh$

$\otimes$:= objectwise tensoring

power := objectwise internal hom

(this was defined previously)

Def (a) Weighted colimit

Given $\mathcal{D} : I \to \mathcal{C}$

(diagram in category $\mathcal{C}$)

new input: weight, [redacted] $W : I^{op} \longrightarrow V$

Then, $\operatorname{colim}^W \mathcal{D} := \operatorname{coend}(I^{op} \times I \longrightarrow \mathcal{C})$

$i, i \longmapsto W(i) \otimes \mathcal{D}(i)$

→ (b) Weighted Limit $\mathcal{D}: I \to \mathcal{C}$ (Day 7.P.M. Hom Algebra)

NOTE not I^{op} here!

$W: I \to V$

$\lim_W \mathcal{D} = \text{end}\left(I^{op} \times I \to \mathcal{C} \right)$

$i, j \mapsto \mathcal{D}(j)^{W(i)}$

Ex $W(i) = \mathbb{1}_V$ (monoidal unit in V) ← special case!

LIMIT Then, NOTE: tensoring/powering by monoidal unit in V doesn't change value

$\lim_W \mathcal{D} = \text{end}(I^{op} \times I \to \mathcal{C})$

so that, $i, j \mapsto \mathcal{D}(j)^{\mathbb{1}_V}$ is unit w.r.t. $\overrightarrow{\otimes}$

$\lim_W \mathcal{D} = \lim \mathcal{D}$ ⇒ $\mathbb{1}_V$ is unit w.r.t. $\otimes$ and power tensor

and COLIMIT

$\text{colim}_W \mathcal{D} = \text{colim}\, \mathcal{D}$ } when $W(i)$ is the monoidal unit, weighted limit/colimit coincides w/ regular limit/colimit

Think of weighted limits/colimits of generalization of natural transformations in the enriched setting

previously $\mathcal{D}: I \to \mathcal{C}$, $W: I^{op} \to \mathcal{C}$ } same target category

Now, $\mathcal{D}: I \to \mathcal{C}$, $W: I^{op} \to V$ different target categories

Recall, $\exists: I^{op} \times I \to \mathcal{C}$ Info to added to Day 6 P.M. Homological Algebra

$\int_I F \to \prod_{i \in I} F(i,i) \rightrightarrows \prod_{h: j \to k} F(j,k)$

end of F

There is a projection map $\forall i$, $\int_I F \xrightarrow{\pi_i} F(i,i)$

these two maps use h

Ex $|-|: sSet \to Top$ geometric realization

takes combinatorial simplex and gives topological simplex

$|X| = \text{colim}_{\Delta} X$ where $\Delta: \Delta \to Top$

~~∞-GROUPOIDS IN SYNTHETIC~~ Geometric Mechanics ~~PRESHEAVES~~

~~Riemannian Metric~~ (Day 7. P.M.)

~~Differentiable~~

Example (continued from Homological Algebra)

$|-| : sSet \longrightarrow Top$

$X \in sSet$

$* \begin{cases} X : \Delta^{op} \to Set \\ \Delta : \Delta \to Top \end{cases}$, [Recall, set included in category of topological spaces as discrete topological space. $Set \xrightarrow{dis} Top$]

$[n] \longmapsto \Delta^n = \{ x \in \mathbb{R}^{n+1}_{\geq 0} \mid \sum_i x_i = 1 \} \in Top$

Then, $|X| := colim_{\Delta} (dis \circ X)$

Here

$* \begin{cases} X : \Delta^{op} \to Set \xrightarrow{dis} Top & (\mathcal{D} : I \to \mathcal{C}) \text{ diagram} \\ \Delta : \Delta \to Top & (W : I \to V) \text{ weight} \end{cases}$

Geometric Realization of X

$$\coprod_{k} \Delta^k \times X_k \underset{\text{comp \#2}}{\overset{\text{comp \#1}}{\rightrightarrows}} \coprod_{n \in \Delta^{op}} \Delta^n \times X_n \longrightarrow |X|$$

$[k] \xrightarrow{h} [\ell]$ map of simplices

using h: $\Delta^\ell \times X_\ell$ → comp #1 → $\Delta^k \times X_k$

comp #2: $\Delta^\ell \times X_\ell$ → $\coprod_{n \in \Delta^{op}} \Delta^n \times X_n$

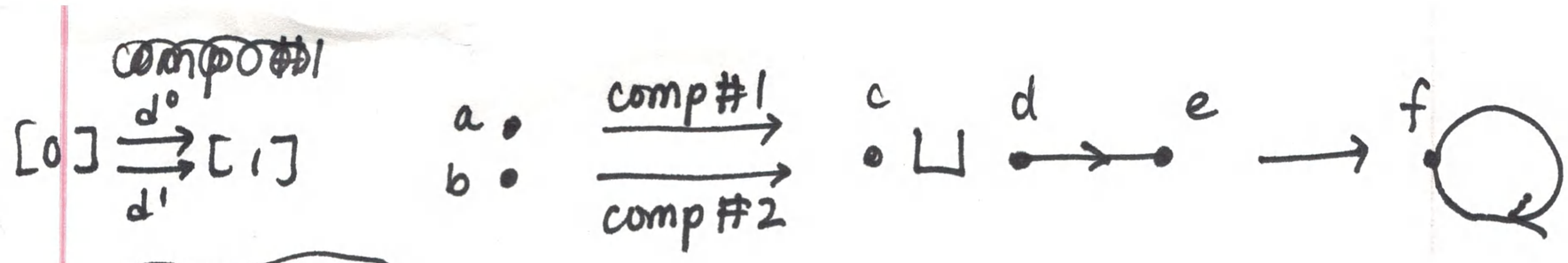
using simplicial structure map $[\ell] \to [k]$ of X_ℓ: $\Delta^k \times X_k$ → comp #1 → $\coprod_{n \in \Delta^{op}} \Delta^n \times X_n$

$\coprod_{n \in \Delta^{op}} \Delta^n \times X_n \longrightarrow |X|$ coequalizer of $\underline{\underline{Ex}}$ $k = \ell - 1$ (k is face of ℓ)

comp #1 Δ^k embeds into larger simplex Δ^ℓ

comp #2 makes face separate simplex then includes into larger

glue face as separate simplex to a larger simplex as a face



How to compute homotopy limits/colimits in a Reedy category

comp #1 maps $a \mapsto c$, $b \mapsto c$

comp #2 maps $a \mapsto e$, $b \mapsto d$

so coequalizer identifies images of comp #1 and comp #2

$c \mapsto e$, $c \mapsto d$ $\Rightarrow$ hence $e = d = f$

Recall,

$I \in Cat$

$\mathcal{C} \in$ Model CAT

$$\mathcal{C}^{I^{op} \times I} \underset{\hom(-,-)}{\overset{\int^I}{\rightleftarrows}} \mathcal{C}$$

left-derived $\int^I$ computes homotopy coends, in particular, homotopy weighted colimits and homotopy colimits in general

[right-derived "$\mathbb{R}\int^{hom}$, ends"]

Recall, there are 3 model structures on $\mathcal{C}^{I^{op} \times I}$

(i) Proj - Inj
(ii) Inj - Proj
(iii) Reedy - Reedy

In particular, COROLLARY

Homotopy weighted colimits can be computed by deriving:

input: $W: I^{op} \to V$ weight, $D: I \to \mathcal{C}$

so,

$$Fun(I^{op}, V) \times Fun(I, \mathcal{C}) \to \mathcal{C}$$

W $\quad$ D

$Fun(I^{op} \times I, \mathcal{C})$ $\xrightarrow{\int^I}$ $\mathcal{C}$, take end

$i, j \mapsto W(i) \otimes D(j)$

(iii) is usually better because the formulas for fibrant/cofibrant replacement are easy and practical

(i) Projective — Injective
(ii) Injective — Projective
(iii') Reedy — Reedy

so, we will use (iii) model structure $\to$

Ex Reedy-Reedy Model structure

(a) $a \xleftarrow{\text{inA}} b \xrightarrow{\text{inA}} c$ $= I$ (Reedy category, all in I^- and I^+ is identity morphisms)

degrees: 0 — 1 — 0

What it means to be cofibrant in Reedy model structure:

$I = I^-$

$I = \{id_{(-)}\}$

"Reedy cofibrant"

$\mathcal{D} : I \to \mathcal{C}$

latching object of $\mathcal{D}$ $\quad L_a\mathcal{D} = \operatorname{colim}_{a' \to a,\ \neq id} \mathcal{D}(a') = \emptyset$ (initial object; empty: all morphisms I^+ are identity)

same for other latching objects: $L_b\mathcal{D} = \emptyset$, $L_c\mathcal{D} = \emptyset$

(What does it mean for $\mathcal{D}$ to be Reedy cofibr.)

$\mathcal{D}$ is "Reedy cofibrant" if $L_{(-)}\mathcal{D} \xrightarrow{l_{(-)}} \mathcal{D}(-)$ is cofibrant (is a cofibration)

"latching map"

$\emptyset \rightarrowtail \mathcal{D}(a)$

$\emptyset \rightarrowtail \mathcal{D}(b)$ $\Rightarrow$ $\mathcal{D}(a)$, $\mathcal{D}(b)$ and $\mathcal{D}(c)$ are cofibrant

$\emptyset \rightarrowtail \mathcal{D}(c)$

latching maps

Lemma

when $I = I^-$, the Reedy Model structure coincides with Injective model structure!

When $I = I^+$, the Reedy Model structure coincides with projective model structure!

In this special case To cofibrantly resolve $\mathcal{D}$, compose $\mathcal{D}$ w/ cofibrant replacement functor for $\mathcal{C}$

so, if Q is a cofibrant replacement functor for $\mathcal{C}$

Then $Q \circ \mathcal{D}$ is cofibrant replacement for $\mathcal{D}$

(What does it mean for W to be "Reedy cofibrant"?)

$W : I^{op} \to V$, W is Reedy cofibrant if latching maps are cofibrations

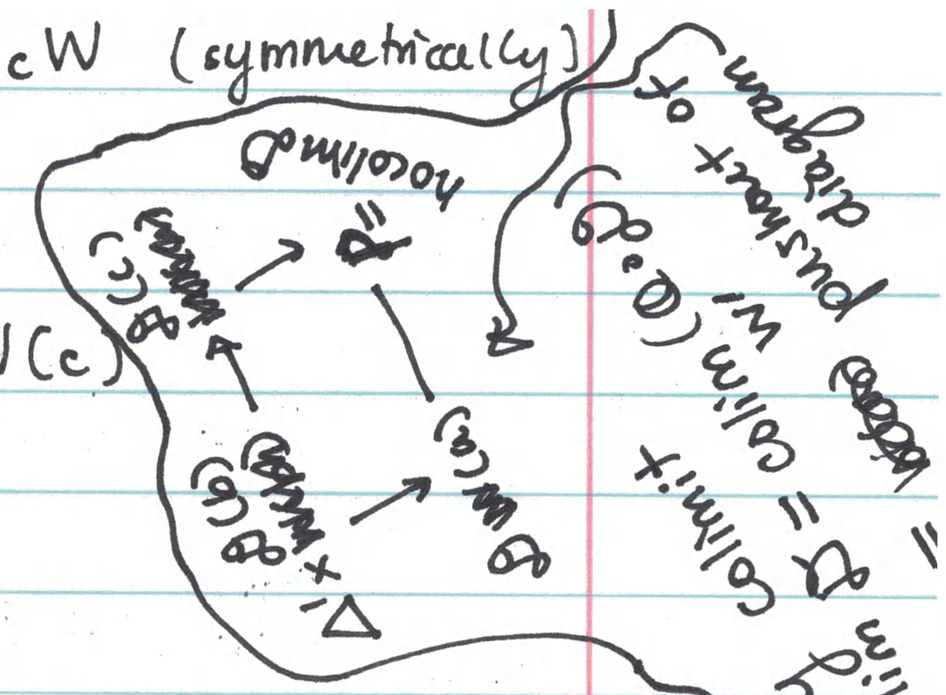
$W(a) \xleftarrow{+} W(b) \xrightarrow{+} W(c)$

so, compute the latching maps → now both maps are positive since we're in I^{op} →

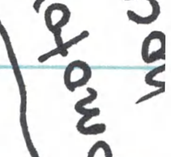
$\to L_a W = \operatorname{colim}_{\substack{a' \to a \\ + \\ \neq id}} (W(a')) = \emptyset = L_c W$ (symmetrically)

$L_b W = \operatorname{colim}_{\substack{b' \to b \\ + \\ \neq id}} W(b') = W(a) \sqcup W(c)$

there are two positive arrows, two objects, and no morphisms (non-id. morphisms)



we need this map to be a cofibration



latching map $\ell_b : W(a) \sqcup W(c) \longrightarrow W(b)$

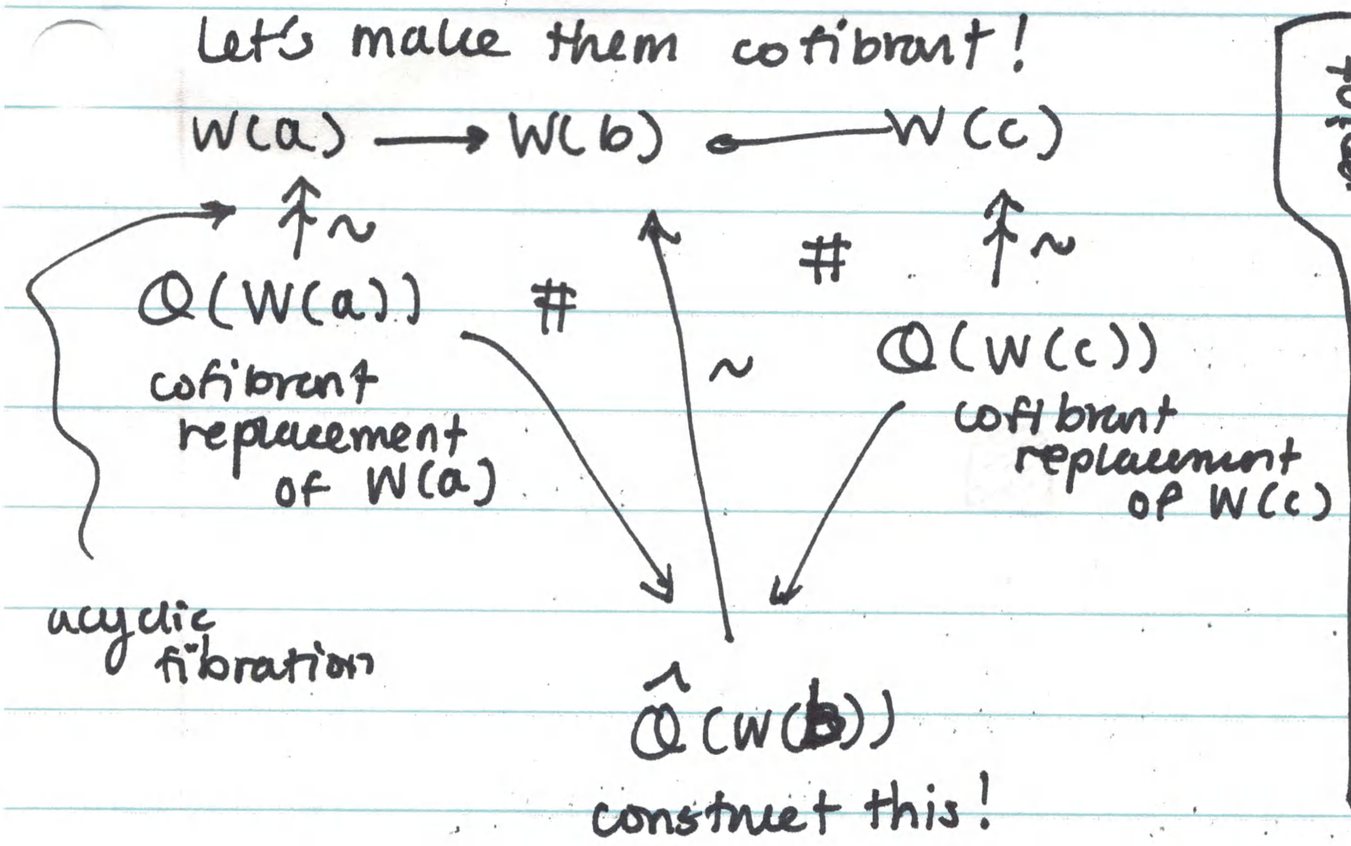
$\ell_a : \emptyset \rightarrowtail W(a)$

$\ell_c : \emptyset \rightarrowtail W(c)$

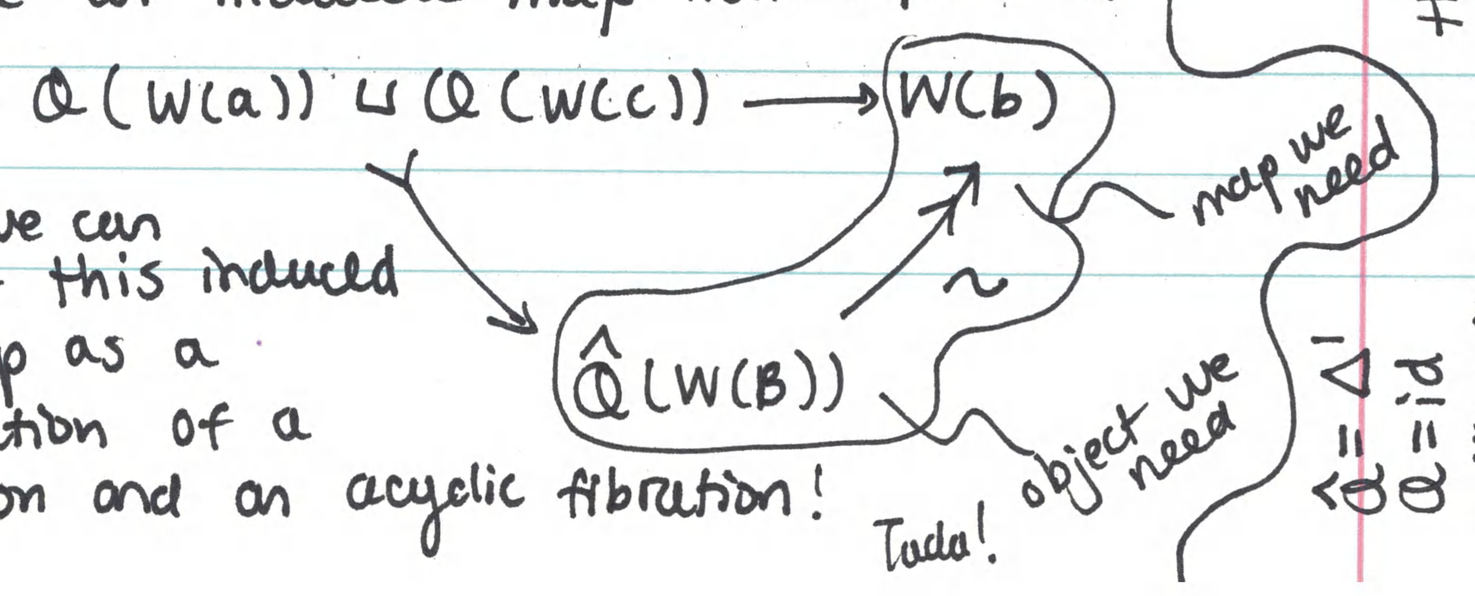
} we need $W(a), W(c)$ to be cofibrant

$\Rightarrow$ W is Reedy cofibrant iff $W(a), W(c)$ are cofibrant and $W(a) \sqcup W(c) \rightarrowtail W(b)$ is a cofibration!

Let's make them cofibrant!



We have an induced map from coproduct



And we can factor this induced map as a factorization of a cofibration and an acyclic fibration!

Tada!

point = * = Δ[0]

Ex = sSet

W(a) = W(c) = W(b) = Δ[1]

W(a) ∐ W(c) → W(b)

* ∐ * → *

the map is NOT cofibrant

* ∐ * → * ≃ Δ[1]

Q = id

cofibrant replace

* ≃ * ≃ * ≃ Δ[1]

Homological Algebra (Dan Grady) 02/11

Man
$\mathcal{I} \in sh(Man)$

DEF

A sheaf $I \in sh(Man)$ is called "injective" if $\forall$ monomorphisms

$m: \mathcal{F} \longrightarrow G$

(between sheaves) and $\forall$ maps $g: \mathcal{F} \longrightarrow I$

we can extend to h

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{m} & G \\ {\scriptstyle g}\downarrow & \# & \swarrow {\scriptstyle h} \\ I & & \end{array}$$

such that diagram commutes

(this is like saying "I" is fibrant, and the def of projective is similar)

Def $\mathcal{F}$ is a presheaf

$\mathcal{F} \in Psh(Man)$

$\mathcal{F}$ is a sheaf if the value

~~$\mathcal{F}(M) \in eq\{\prod_\alpha \mathcal{F}(U_\alpha) \rightrightarrows \prod \mathcal{F}(U_\alpha)\}$~~

$$\mathcal{F}(M) = eq\left\{\prod_\alpha \mathcal{F}(U_\alpha) \rightrightarrows \prod_\alpha \mathcal{F}(U_\alpha)\right\}$$

equalizer (limit over the diagram w/ two parallel arrows)

where $\{U_\alpha\}$ covers M

(1) $f, g \in \mathcal{F}(M)$

$f|_{U_\alpha} = g|_{U_\alpha}, \forall \alpha$

$\Rightarrow f = g$

(2) $f_\alpha, g_\beta \in \prod_\alpha \mathcal{F}(U_\alpha)$

$f_{\alpha\beta} = g_{\alpha\beta}$

$\Rightarrow \exists f$ such that $f|_{U_\alpha}$

DEF An injective resolution of a sheaf $\mathcal{F}$ is an exact sequence of sheaves such that

$$\mathcal{F} \hookrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

each of the I^k are injective

NOTE: This sequence must be exact in sheaves, but not necessarily exact in presheaves

(that is, will split locally, but maybe not globally)

→ Think in terms of chain complexes

$$\begin{array}{ccc} \uparrow & & \uparrow \\ 0 & \to & I' \\ \uparrow & & \uparrow \\ \mathcal{F} & \hookrightarrow & I^0 \end{array}$$

{ we have a diagram like this

This is a fibrant replacement (and we know how to work with the global sections)

DEF The sheaf cohomology of $\mathcal{F}$ is

$$H^p(M;\mathcal{F}) = H^p(R\Gamma(M,\mathcal{F})) = H^p\Gamma(M;I^\bullet)$$

PROP ~~If this exact sequence below is exact (a resolution)~~

If $\mathcal{F} \hookrightarrow \underbrace{A^0}_{\text{sheaf}} \longrightarrow A^1 \longrightarrow \cdots$

is a resolution (i.e. exact sequence) of acyclic sheaves such that

$$H^p(M, A^k) = 0, \quad \forall k$$

then $H^p(M,\mathcal{F}) = H^p\Gamma(M, A^\bullet)$

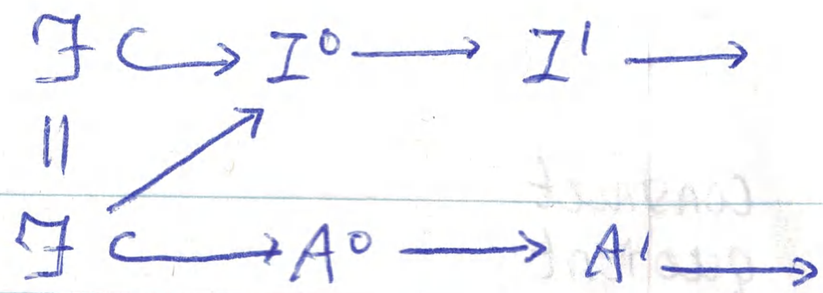
Proof We know that injective resolutions always exist (assume) proof due to Grothiedieck

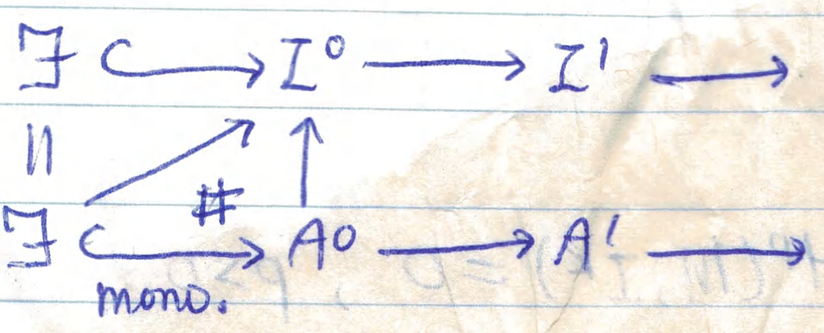
So, take

injective resolution {

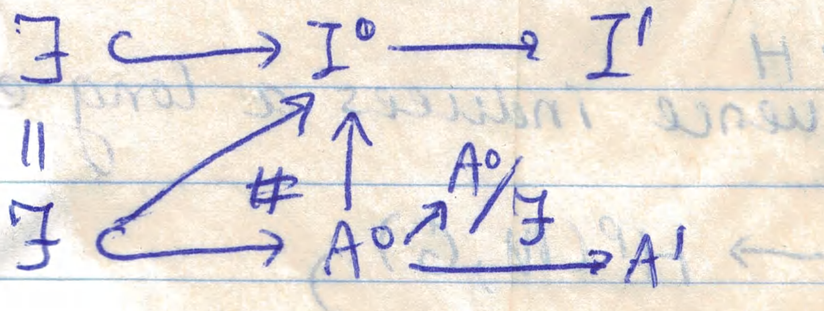
$$\begin{array}{ccccccc} \mathcal{F} & \hookrightarrow & I^0 & \longrightarrow & I' & \longrightarrow & \cdots \\ \| & & & & & & \\ \mathcal{F} & \hookrightarrow & A^0 & \longrightarrow & A^1 & \longrightarrow & \cdots \end{array}$$

construct a map between these two ---

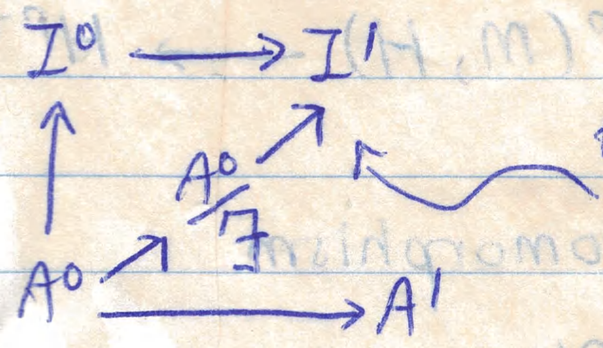




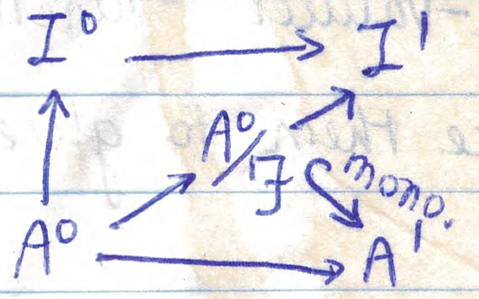
use prop to complete this triangle



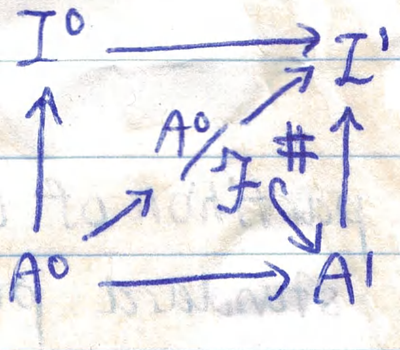
construct quotient and map to it



use exactness to argue that this map is well defined



$A^0/\mathcal{F} \hookrightarrow A'$ is a monomorphism because of exactness (1st ISOMORPHISM THM)



use prop to complete this triangle

Continue this procedure...

This induces a map

$$f : \Gamma(M, A^\bullet) \longrightarrow \Gamma(M, I^\bullet)$$

$$\to \quad \mathcal{F} = \mathcal{F}$$

$$\text{exact} \begin{cases} A^0 \to I^0 \to Q^0 \\ \downarrow \quad \downarrow \quad \downarrow \\ A^1 \to I^1 \to Q^1 \end{cases}$$

construct quotient maps

exact

I^k are injective, hence $H^p(M, I^k) = 0$, $p > 0$

A^k are acyclic, hence $H^p(M, A^k) = 0$

$F \to G \to H$ and the short exact sequence induces a long exact seq

$$H^p(M, \mathcal{F}) \to H^p(M, G) \to H^p(M, H) \to H^{p+1}(M, \mathcal{F}) \to \cdots$$

PROP We have an isomorphism

$\underline{\mathbb{R}}$ $\cong$ locally constant $\mathbb{R}$-valued function

(force them to glue together)

$$H^p(M, \underline{\mathbb{R}}) \cong H^p_{dR}(M)$$

Proof Each Ω^k (k-forms) are acyclic because of partition of unity (assume true, look up standard proof)

Then we can compute via acyclic resolution

$$\underline{\mathbb{R}} \hookrightarrow \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \cdots$$

take derivative

view as 0-forms locally constant $\mathbb{R}$-val fcn's

← This sequence is exact in sheaves (locally) not necessarily globally

so by previous prop:

$$H^p(M, \underline{\mathbb{R}}) \cong H^p_{dR}(M)$$

DEF (Recall sheaves of chain complexes)

Let $\mathcal{F}^\bullet$ be a sheaf of chain complexes

$$\cdots \to \mathcal{F}^k \xrightarrow{d} \mathcal{F}^{k-1} \xrightarrow{d} \mathcal{F}^{k-2} \to \cdots \to \mathcal{F}^0 \to \cdots$$

The sheaf cohomology of $\mathcal{F}^\bullet$ is (generalizing previous result)

assume bounded below (i.e. deg eventually vanishes)

$$H^p(M, \mathcal{F}^\bullet) := H^p R\Gamma(M, \mathcal{F}^\bullet)$$

($\Gamma(M,-) = \text{eval}_M$; $R\Gamma$ ↑ derived)

$$= H^p \Gamma(M, I^\bullet)$$

fibrant replacement then take ordinary cohomology

injective resolution we have $\mathcal{F}^\bullet \to I^\bullet$ with each I^k injective, that is, just fibrant replacement

EXAMPLE Consider the sheaf of complexes

$$\cdots \to 0 \to 0 \to C^\infty(-, \mathbb{C}^*) \to 0 \to \cdots$$

($C^\infty(-, \mathbb{C}^*)$: units of $\mathbb{C}$, $\mathbb{C}\setminus\{0\}$ invertible elements of $\mathbb{C}$; degree 1)

PROP (smooth functions into a manifold are a sheaf)

For any manifold, M smooth functions into M, $C^\infty(-, M)$ is a sheaf!

Proof Exercise, verify restriction to intersections (already locally smooth) →

→ so, since $\mathbb{C}^X$ is a manifold — Example continued

↑ open subset of $\mathbb{C}$ (manifold)
open subsets are manifolds
because they are the union of open balls

so by proposition,
$C^\infty(-, \mathbb{C}^X)$ is a sheaf

Denote: $\mathbb{C}^X[1] := (\to \cdot \to 0 \to 0 \to \underset{\uparrow}{C^\infty(-, \mathbb{C}^X)} \to 0 \to \cdots)$
sheaf of chain complexes

Example continued

Čech cohomology

- let M be a manifold
- let $\{U_\alpha\}$ be an open cover of M

DEF The Čech double complex of a sheaf of chain complexes $A^\bullet$ is

degree n bicomplex $\left\{ \check{C}^\bullet(\{U_\alpha\}, A^\bullet) := \prod_{p,q} \prod_{\alpha_0 \cdots \alpha_p} A^q(U_{\alpha_0 \cdots \alpha_p}) \right.$

↑ direction of intersections; ↑ coeff direction; ↑ take the p-fold intersection $U_{\alpha_0} \cap U_{\alpha_1} \cap \cdots \cap U_{\alpha_p}$

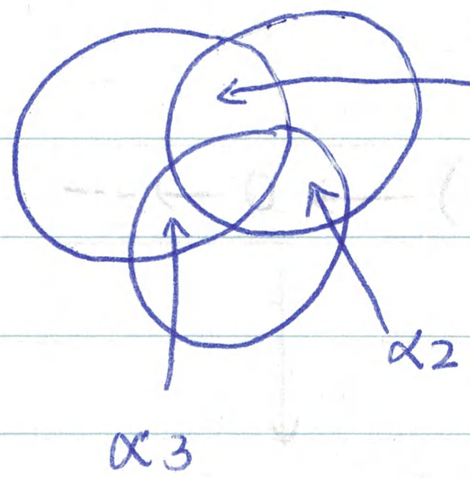
Think of it as having two directions

$$\begin{array}{ccc} \prod_{\alpha_0 \cdots \alpha_p} A^q(U_{\alpha_0 \cdots \alpha_p}) & \xrightarrow{d} & \prod_{\alpha_0 \cdots \alpha_{p+1}} A^{q-1}(U_{\alpha_0 \cdots \alpha_{p+1}}) \\ \downarrow \delta & & \downarrow \delta \text{ restrict to intersections} \\ \prod A^q(U_{\alpha_0 \cdots \alpha_p}) & \xrightarrow{d} & \prod A^{q-1}(U_{\alpha_0 \cdots \alpha_{p+1}}) \end{array}$$

arrow going down because A is chain complex but A could also be cochain complex, arrows go up...

where $\delta(\alpha_q)$ = restrict to intersection

restrict to intersections



Take $(\alpha_1, \alpha_2, \alpha_3)$

Then $\delta((\alpha_1, \alpha_2, \alpha_3))$

$= \alpha_1 - \alpha_2 + \alpha_3$

restrict and take alternating sum

DEF The "totalization" of $\check{C}(\{U_\alpha\}, A^\circ)_n$

$$= \bigoplus_{p+q=n} \prod_{\alpha_0 \cdots \alpha_p} A^q(U_{\alpha_0 \cdots \alpha_p})$$

A picture of the totalization of the Čech double complex

$$\cdots \longrightarrow \prod_{\alpha\beta} A^1(U_{\alpha\beta}) \xrightarrow{d} \boxed{\prod_{\alpha} A^0(U_\alpha)} \ni x_0$$

$$\downarrow \delta \qquad\qquad \downarrow \delta$$

$$\cdots \longrightarrow \boxed{\prod_{\alpha\beta} A^1(U_{\alpha\beta})} \xrightarrow{d} \prod_{\alpha\beta} A^0(U_{\alpha\beta}) \ni D(x_0)$$

$\ni x_1$ — first coord

The differential $D = d + (-1)^{deg+1}\delta$

so for degree 0, $(x_0, x_1, \ldots, x_n)$

x = (def on open sets, def on intersections)

$$D(x) = (\underbrace{\delta x_0 - d x_1}_{\text{first coord}}, \underbrace{\delta x_1 + d x_2}_{\text{2nd coord}}, \ldots)$$

sign wrong? "up to an action of $\mathbb{Z}/2$"

→

Totalization

$Tot(\check{C}^\bullet(\{U_\alpha\}, \mathbb{C}^\times[1]))$

$= \cdots \to 0 \to 0 \to \prod_\alpha C^\infty(U_\alpha, \mathbb{C}^\times) \to 0 \to \cdots$

$\downarrow \quad \downarrow \quad \downarrow \delta$ degree 1

$\cdots \to 0 \to 0 \to \prod_{\alpha\beta} C^\infty(U_{\alpha\beta}, \mathbb{C}^\times) \to 0 \to \cdots$

choose an element in Tot degree 0

$\to 0 \to \prod_\alpha C^\infty(U_\alpha, \mathbb{C}^\times) \to 0 \to$

$\to 0 \to \prod_{\alpha\beta} C^\infty(U_{\alpha\beta}, \mathbb{C}^\times) \to 0 \to$

$\to 0 \to \prod_{\alpha\beta\gamma} C^\infty(U_{\alpha\beta\gamma}, \mathbb{C}^\times) \to 0 \to$

$\cdots$

so an element in $Tot^0(\check{C}(\{U_\alpha\}, \mathbb{C}^\times[1]))$

is just an element in the intersection

$$(g_{\alpha\beta}) \in C^\infty(U_{\alpha\beta}, \mathbb{C}^\times)$$

since all other components in that diagonal are 0

and the differential

$$0 = D((g_{\alpha\beta})) = \delta(g_{\alpha\beta}) = 0$$

$\Rightarrow$ $(g_{\alpha\beta})$ is a Čech cocycle in $\check{C}(\{U_\alpha\}, \mathbb{C}^\times)$ of degree 1

(To Be continued...)

$g \in Tot^0(\check{C}(\{U_\alpha\}, \mathbb{C}^\times[1])$

$g = (g_{\alpha\beta})$ $g_{\alpha\beta} : C^\infty(U_{\alpha\beta}, \mathbb{C}^\times)$

$D(g) = 0 \iff \delta(g_{\alpha\beta}) = 0$

$\Rightarrow (g_{\alpha\beta}) \in \check{C}(\{U_\alpha\}, \mathbb{C}^\times)'$

so, $H^0(Tot^\bullet(\check{C}(\{U_\alpha\}, \mathbb{C}^\times[1]))) \cong \check{H}'(M, \mathbb{C}^\times)$

↑ Čech cohomology

THEOREM (Leray)

If $\{U_\alpha\}$ is a cover of M such that $H^p(U_{\alpha_0 \cdots \alpha_k}, \mathcal{F}^\bullet) = 0$

then

$$H^p(M, \mathcal{F}^\bullet) \cong \check{H}^p(M, \mathcal{F}^\bullet) \cong H^p(Tot^\bullet(\check{C}(\{U_\alpha\}, \mathcal{F}^\bullet)))$$

EXAMPLE Take $\mathcal{F}^\bullet = \Omega^\bullet$ (de Rham complex)

Poincare Lemma $\Rightarrow$ taking $\{U_\alpha\}$ (a good open cover) gives

((intersections are contractable))

$$H^p_{dR}(M) = H^p(M, \Omega^\bullet) \cong \check{H}(M, \Omega^\bullet)$$

PROP We have an isomorphism between

$$Line(M) \cong \check{H}'(M, \mathbb{C}^\times)$$

isomorphism classes of line bundles on M (defined below) →

→

Proof

The group structure of $\mathrm{Line}(M)$ is $\mathcal{L} \otimes \mathcal{L}'$ (tensor product of line bundles)

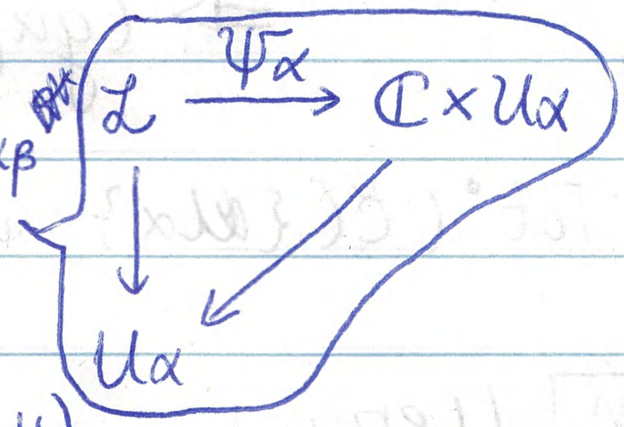
↓

define $\varphi : \mathrm{Line}_{\mathbb{C}}(M) \longrightarrow \check{H}^1(M, \mathbb{C}^\times)$

complex line bundles

by $(\mathcal{L} \to M) \longmapsto g_{\alpha\beta}$

where $\psi_\beta \circ \psi_\alpha^{-1} : \mathbb{C} \times U_{\alpha\beta} \to \mathbb{C} \times U_{\alpha\beta}$ is the transition function that maps $(z, u) \longmapsto (g_{\alpha\beta} \cdot z, u)$



(Check that it is well-defined)

↑ cannot depend on choice of trivialization $\mathcal{L} \to M$

(show that same trivializations are cohomologous in $\check{H}^1(M, \mathbb{C}^\times)$ same cohomology)

We want to show φ is a homomorphism

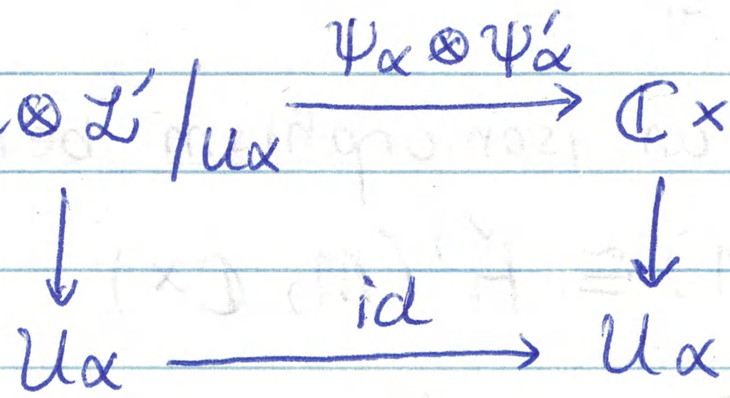
Choose $\{\psi_\alpha, U_\alpha\}$ for $\mathcal{L}$

and $\{\psi'_\alpha, U_\alpha\}$ for $\mathcal{L}'$

$\Rightarrow$ $g_{\alpha\beta} \otimes g'_{\alpha\beta}$ are tran. funcs for $\mathcal{L} \otimes \mathcal{L}'$

transition functions for $\mathcal{L}$; tran funcs for $\mathcal{L}'$

written in



so φ is a homomorphism.

To show φ is an isomorphism:

φ injective we have $\varphi(\mathcal{L}) = 0$

$\Rightarrow g_{\alpha\beta} = \delta f_\alpha$ for some $(f_\alpha) \in \check{C}(\{U_\alpha\}, \mathbb{C}^\times)$ nonvanishing $\mathbb{C}$-valued functions on open sets

$\Rightarrow \mathcal{L}$ is trivial

$\Rightarrow \varphi$ injective

(i.e. frame admits global section)

NOTE: $g_{\alpha\beta} \otimes g'_{\alpha\beta}$ is actually just $g_{\alpha\beta} \cdot g'_{\alpha\beta}$ since $\mathbb{C} \underset{\mathbb{C}}{\otimes} \mathbb{C} \cong \mathbb{C}$

φ surjective define

$$\mathcal{L} = \mathrm{coeq}\left\{ \coprod_\alpha \mathbb{C} \times U_\alpha \underset{\iota_{\alpha\beta}}{\overset{g_{\alpha\beta}}{\leftleftarrows}} \coprod_{\alpha\beta} \mathbb{C} \times U_{\alpha\beta} \right\}$$

where $\iota_{\alpha\beta} : U_{\alpha\beta} \hookrightarrow U_\alpha$

So there is an isomorphism between line bundles and $\check{H}^1(M, \mathbb{C}^\times)$

$\Rightarrow$ a 1 gerbe is a 2-line bundle and a 0 gerbe is a line bundle!

DEF An affine connection

$$\nabla : \Gamma(E) \longrightarrow \Gamma(E \otimes T^*M)$$ such that satisfies Leibniz rule

vector bundle $E \xrightarrow{\pi} M$

Γ section such that composition is id map

∇ has two arguments

$\nabla_{(-)}(-)$ for section $s \in \Gamma(E)$

$\nabla_{(-)} s \in \Omega^1(M, \mathbb{C})$ a 1-form

for $X \in \Gamma(TM)$

$\nabla_X(-) : \Gamma(E) \longrightarrow \Gamma(E)$ a map between sections

Leibniz rule $\nabla(\sigma \cdot f) = \nabla(\sigma) \otimes f + \sigma \otimes df$

$\sigma \in \Gamma(E)$

df: usual exterior derivative, 1-form

[evaluated on X

$\nabla_X(\sigma \cdot f) = \nabla_X(\sigma) \cdot f + \sigma \cdot df(X)$]

$\nabla_X(\sigma)$: section; f: smooth function; σ: section; $df(X)$: smooth function

these can be multiplied

LEMMA ~~Let $\{U_\alpha\}$ be an open cover, let $\{\sigma_\alpha\}$ be local sections, fix $\mathcal{L}|_{U_\alpha} \cong \mathbb{C} \times U_\alpha$~~

LEMMA Given $\{U_\alpha\}$, cover of M

preamble:
- Fix $u_\alpha : U_\alpha \longrightarrow \mathcal{L}|_{U_\alpha} \cong \mathbb{C} \times U_\alpha$ non-vanishing sections
- ~~there exists for each α $\sigma_\alpha : U_\alpha \to \mathcal{L}|_{U_\alpha}$ (a section)~~
- define A_α by $A_\alpha \cdot u_\alpha = (\nabla u_\alpha)$
- Recall, $\sigma, \sigma' : M \to \mathcal{L}$ such that $\sigma = f \cdot \sigma'$ for some $f : M \to \mathbb{C}$, $A_\alpha \in \Omega^1(U_\alpha, \mathbb{C})$

Lemma: then we have

$$\boxed{A_\beta = A_\alpha + g_{\alpha\beta}^{-1} dg_{\alpha\beta}}$$

A_β and A_α are compatible via the transition functions

where ~~$g_{\alpha\beta} = u_\beta \cdot u_\alpha^{-1}$~~

$(g_{\alpha\beta} z, x) = u_\beta \cdot u_\alpha^{-1}$

Proof on $U_{\alpha\beta}$

$$A_\alpha \cdot u_\alpha = \nabla(u_\alpha) = \nabla((u_\alpha \cdot u_\beta^{-1}) \cdot u_\beta)$$

$$\overset{\text{Leibniz rule}}{=} \nabla(g_{\alpha\beta} \cdot u_\beta)$$

$$\hookrightarrow = \nabla u_\beta \cdot g_{\alpha\beta} + u_\beta \cdot d g_{\alpha\beta}$$

and we can rewrite:

$$A_\alpha \cdot u_\alpha = A_\beta \cdot u_\beta \overset{\cdot g_{\alpha\beta}}{+} d g_{\alpha\beta} \cdot u_\beta$$

apply $u_{\bar{\alpha}}^{-1}$ on both sides

↑ corresponding trivialization associated to u_α

$$A_\alpha = A_\beta \cdot u_\beta u_{\bar{\alpha}}^{-1} + d g_{\alpha\beta} \cdot u_\beta \cdot u_{\bar{\alpha}}^{-1}$$

$g_{\alpha\beta}$

$$A_\alpha = A_\beta \cdot (g_{\alpha\beta} g_{\alpha\beta}^{-1}) + d g_{\alpha\beta} \cdot u_\beta u_{\bar{\alpha}}^{-1}$$

cancels!

$u_\beta \cdot u_\alpha^{-1}$

$$A_\alpha = A_\beta + d g_{\alpha\beta} g_{\alpha\beta}^{-1}$$

The relation we wanted! ▨

The relation $A_\alpha = A_\beta + d g_{\alpha\beta} g_{\alpha\beta}^{-1}$ is called a "local gauge transformation".

DEF We define the sheaf

$$\Omega^1(-, \mathbb{C}^\times) = (\cdots \to 0 \to \underbrace{C^\infty(-, \mathbb{C}^\times)}_{\deg 1} \to \underbrace{\Omega^1(-, \mathbb{C})}_{\deg 0} \to 0 \to \cdots)$$

PROP $\mathrm{Line}_\nabla(M) \cong H^0(M, \Omega^1(-, \mathbb{C}^\times))$

complex line bundles with connection

sheaf cohomology

takes place of deg 1 Čech cocycles

→

Proof

NOTE

$\text{Line}_\nabla(M)$ group operation is tensor product of underlying bundles

$((\mathcal{L},\nabla),(\mathcal{L}',\nabla')) = (\mathcal{L}\otimes\mathcal{L}', \nabla\otimes\nabla')$

where $\nabla\otimes\nabla'$ is defined using the Leibniz rule

$(\nabla\otimes\nabla')(s\otimes s') = \nabla(s)\otimes s' + s\otimes\nabla'(s')$

Then define $\varphi : \text{Line}_\nabla(M) \longrightarrow H^0(M, \Omega^1(-,\mathbb{C}^\times))$

$(\mathcal{L},\nabla) \longmapsto (g_{\alpha\beta}, A_\alpha)$

$\longrightarrow \prod_\alpha C^\infty(U_\alpha,\mathbb{C}^\times) \xrightarrow{d\log} \boxed{\prod_\alpha \Omega^1(U_\alpha,\mathbb{C})} \ni A_\alpha$

$\downarrow \delta$ $\qquad\qquad\qquad\qquad \downarrow \delta$ restrict to intersections

$\boxed{\prod_{\alpha\beta} C^\infty(U_{\alpha\beta},\mathbb{C}^\times)} \longrightarrow \prod_{\alpha\beta}\Omega^1(U_{\alpha\beta},\mathbb{C})$

all other parts in diagram $=0$

$D(g_{\alpha\beta}, A_\alpha) = ((A_\alpha - A_\beta) - d\log(g_{\alpha\beta}),\ \delta g_{\alpha\beta})$

(1st coord: complex logarithm; 2nd coord)

$0 \overset{\text{iff}}{=}$

$\begin{cases} A_\alpha - A_\beta = g_{\alpha\beta}^{-1}\, dg_{\alpha\beta} & \text{(local gauge transform condition)} \\ \delta g_{\alpha\beta} = 0 & \text{(cocycle condition)} \end{cases}$

so φ lands in $Z^0_{\text{cycles}}(\check{C}(\{U_\alpha\}, \Omega^1(-,\mathbb{C}^\times)))$

(and show φ is well-defined)

$\downarrow$ show φ homomorphism

If $A_\alpha = g_\alpha^{-1} dg_\alpha \Rightarrow$ A_α is isomorphic to the trivial connection for some α,

DEF An $(n-1)$ gerbe is a Čech cocycle $\in \check{C}^0(\{U_\alpha\}, \mathbb{C}^\times[n])$ and w/ connection, is an element of $\check{C}^0(-,\Omega^n(\mathbb{C}^\times))$

Homological Algebra 3/31/20

screen captures taken periodically (see email)

$$X_0 \xleftarrow{p_1} X_1 \xleftarrow{p_2} X_2$$

$$\lim(X_0 \leftarrow X_1 \leftarrow X_2) = \{(x_0, x_1, x_2) \mid p_1(x_1) = x_0,\ p_2(x_2) = x_1\}$$

so $\lim(X_0 \leftarrow X_1 \leftarrow X_2) \cong X_2$

(the initial object in the diagram over which we take the limit)

$$I = \{0 \xleftarrow{-} 1 \xleftarrow{-} 2 \xleftarrow{-} \cdots\}$$

The Reedy model structure on $\mathcal{C}^I \equiv$ the ~~projective~~ injective model structure

fibrant objects are diagrams of the form:

$$X_0 \twoheadleftarrow X_1 \twoheadleftarrow X_2 \twoheadleftarrow \cdots$$

fibrant

$$\text{holim}\, X = \lim_I RX$$

RX: fibrant replacement

$$I = \{0 \xrightarrow{+} 1 \xrightarrow{+} 2 \xrightarrow{+} \cdots\}$$

Reedy model structure on $\mathcal{C}^I \equiv$ the projective model structure

cofibrant objects are diagrams

$$X_0 \rightarrowtail X_1 \rightarrowtail X_2 \longrightarrow \cdots$$

X_0: cofibrant

$$\text{hocolim}\, X = \operatorname{colim}_I RX$$

RX: cofibrant replacement

Remark:

$I = \{0 \to 1 \to 2 \to \cdots\}$, $\mathcal{C} = sSet$, Ch (but NOT Top)

$$\text{hocolim}_I X \ (X_0 \to X_1 \to X_2 \to \cdots) \simeq \operatorname{colim}_I X$$

"filtered colimits are filtered homotopy colimits" $\longrightarrow$

Compute the same homotopy limit of the same tower using another method.

PROJECTIVE MODEL STRUCTURE

F, AF are defined objectwise

Proposition (Bousfield & Kan) A projective cofibrant replacement of the constant weight

$$W : I \longrightarrow \{*\} \xrightarrow{\Delta^\circ} \text{sSet}$$

can be computed as

$$(QW)(i) = N(I/i)$$

$(QW)(i)$: I — $N(I/i)$: nerve construction } wow!

Apply to the special case i

$$I = \{0 \leftarrow 1 \leftarrow 2 \leftarrow 3 \leftarrow \cdots\}$$

$i \in I$ $\quad I/i = \{ i \leftarrow i+1 \leftarrow i+2 \leftarrow \cdots \}$

$$= \operatorname*{colim}_{k \geq i} \{ i \leftarrow i+1 \leftarrow \cdots \leftarrow k \}$$

Nerve

$$N(I/i) = \operatorname*{colim}_{k \geq i} \left(N\{ i \leftarrow i+1 \leftarrow \cdots \leftarrow k \} \right)$$

$$= \operatorname*{colim}_{k \geq i} (\Delta^{k-i})$$

} we can use this to write the homotopy limit of the tower

$$\operatorname*{holim}_{i \in I} X = \lim_{\Delta W} (RX) = \operatorname{eq}\left(\prod_{i \geq 0} (RX_i)^{N(I/i)} \rightrightarrows \prod_{0 \leq j \leq k} (RX_j)^{N(I/k)} \right)$$

$\longrightarrow$

→ homotopy equalizers are specified by

$$I = \{ 0 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 1 \}$$

slice categories:

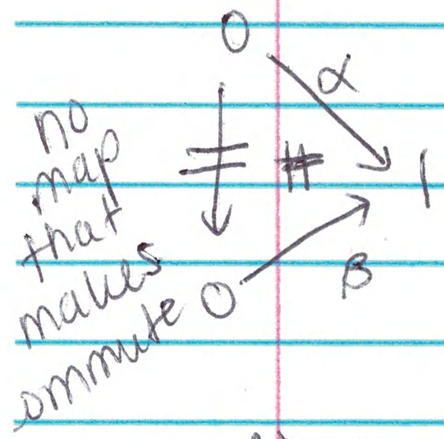
$I/0 = \{*\}$ (identity category; "ido"; the only morphism that lands in 0 is the identity)

$I/1 =$ α, β, id_1 (morphisms that land in 1)

no morphism between α and β

$N(I/0) = \Delta^0$

$N(I/1) =$ 



$X : I \to \mathcal{C}$

assume projectively fibrant (that is, X_0, X_1 are fibrant in $\mathcal{C}$)

so, ~~[illegible]~~ $RX = X$ (X does not need to be derived)

diagram $X_0 \underset{X_\beta}{\overset{X_\alpha}{\rightrightarrows}} X_1$

$$\operatorname*{holim}_{i \in \mathcal{I}} X = \lim_{\mathcal{QW}} RX = \operatorname{eq}\left(X_0^{\Delta^0} \cdot X_1^{\nearrow} \underset{X_1^{N(I/\alpha)}}{\overset{X_\alpha^{\Delta^0}}{\rightrightarrows}} X_1^{\Delta^0} \cdot X_1^{\Delta^0} \right)$$

$$\lim \left(X_0 \xrightarrow{X_\alpha} X_1 \xleftarrow{X_\beta} X_1 \right)$$

nerve construction expresses that equalities are replaced by homotopies

∎

Homological Algebra 4/2/20

Reduction of homotopy colimits to homotopy colimits over Δ^{op} and homotopy coproducts

called "homotopy realizations"
or just "realizations"

Similarly, for limits, (replacing Δ^{op} w/ Δ, coproducts w/ products)

"homotopy totalizations

Reeall,
en $I \in Cat$ (indexing category)
nd
ien $\mathcal{C} \in Cat$ (cocomplete category, has all colimits)

Compute colimit of diagram $\mathcal{D}: I \longrightarrow \mathcal{C}$ as follows:

$$\operatorname*{colim}_I \mathcal{D} = \operatorname{coeq}\left(\coprod_{j \to k} \mathcal{D}(j) \underset{p_2}{\overset{p_1}{\rightrightarrows}} \coprod_{i \in I} \mathcal{D}(i) \right)$$

morphisms objects

p_1 = injection (inclusion) map as $\mathcal{D}(j)$ as a summand of $\coprod_{i \in I} \mathcal{D}(i)$ ($i = j$)

p_2 = map $j \to k$, and then include $\mathcal{D}(k)$ into summand ($i = k$)

formal expression of informal

$\mathcal{C} \in Cat$ (model cat, relative cat, etc), $\mathcal{D}: I \to \mathcal{C}$ diagram

~~hocolim $\mathcal{D} \neq$ hocolim $\coprod^h \mathcal{D}(i_0)$~~

$i_0 \to i_1$

face maps!

$$\underset{I}{\text{hocolim}}\, \mathcal{D} = \underset{\Delta^{op}}{\text{hocolim}} \left(\coprod^h_{i_0} \mathcal{D}(i_0) \rightleftarrows \coprod^h_{i_0 \to i_1} \mathcal{D}(i_0) \rightleftarrows \coprod^h_{i_0 \to i_1 \to i_2} \mathcal{D}(i_0) \cdots \right)$$

i_0 degree 0 ; $i_0 \to i_1$ degree 1 ; $i_0 \to i_1 \to i_2$ degree 2

degeneracy maps!

To take hocolim $\mathcal{D}$ for I and $\mathcal{D}: I \to \mathcal{C}$, take hocolim over Δ^{op} of the diagram

[Take Nerve of indexing category I.]

homotopy colimits over Δ^{op} (homotopy realizations)

we have weighted colimit (for Δ^{op}) $\mathcal{C}$ (simplicial model cat)

functor, $Fun(\Delta, \mathcal{C}) \times Fun(\Delta^{op}, \mathcal{C}) \to \mathcal{C}$

CASE 1

We can put model structure ~~(coarse)~~ (injective-projective) on the weighted colimit functor, makes Quillen functor

- cofibrantly replace weight (objectwise)
- cofibrantly replace diagram

$$Fun(\Delta, \mathcal{C})_{inj} \times Fun(\Delta^{op}, \mathcal{C})_{proj} \to \mathcal{C}$$

$$Fun(\Delta, sSet)_{inj} \times Fun(\Delta^{op}, sSet)_{proj} \to sSet$$

Case 1

trivial weight $W: \Delta \to 1 \xrightarrow{\Delta^0} sSet$ is already injectively cofibrant

$$\operatorname{hocolim}_{\Delta^{op}} \mathcal{D} = \mathbb{L}\operatorname{colim}_{W} \mathcal{D} = \operatorname{colim}_{\mathbb{L}W} \mathbb{L}\mathcal{D}$$

$$= \operatorname{colim}_{W} \mathcal{Q}_{proj}\mathcal{D}$$

already cofibrant (injectively)

equal to ordinary colimit

$$= \operatorname{colim} \mathcal{Q}_{proj}\mathcal{D}$$

CASE 2 (projective - injective)

$$\mathcal{Q}_{proj} W = \left([n] \xrightarrow{proj} N([n]/\Delta)^{op} = N(\Delta^{op}/[n])\right)$$

use Bousfield-Kan Replacement

"fat n-simplex"

$(\mathcal{Q}W)(i) = N(I/i)$

nerve of slice category

cofibrant replacement in the category $\mathcal{C}$ composed with $\mathcal{D}$

$\mathcal{Q}_{\mathcal{C}} \circ \mathcal{D}$

$\mathcal{Q}_{inj}\mathcal{D}$

$$\operatorname{hocolim}_{\Delta^{op}} \mathcal{D} = \mathbb{L}\operatorname{colim}_{W} \mathcal{D} = \operatorname{colim}_{\mathcal{Q}_{proj} W} \mathcal{Q}_{inj}\mathcal{D}$$

CASE 3 (Reedy - Reedy)

Δ Reedy category $(\mathcal{Q}W) = Y$ (Yoneda embedding)

deg = dim $(\mathcal{Q}W)_n = \Delta^n$

$+$ = injective $-$ = surjective

Reedy cofibrant compute latching object

$l_n : L_n X \rightarrowtail X_n$

where latching object $L_n X = \operatorname{colim}_{\substack{m \to n \\ + \\ \neq id}} X_m$ (colimit over all faces of all codimensions)

$L_n \mathcal{Q}W = \operatorname{colim}_{\substack{m \to n \\ + \\ \neq id}} \Delta^m = \partial\Delta^n$

and latching map is inclusion of boundary

$\Rightarrow$ l_n is a cofibration!

$$\operatorname{hocolim}_{\Delta^{op}} \mathcal{D} = \operatorname{colim}_{(\mathcal{Q}_{Reedy} W)} (\mathcal{Q}_{Reedy} \mathcal{D})$$

cofibrant

What does it mean for a diagram to be Reedy cofibrant?

For Δ^{op} Reedy

$+$ = surjective ^{op}

$-$ = injective ^{op}

Latching object

$L_n X = \operatorname{colim}_{n \twoheadrightarrow m} X_m = X_n^{degenerate}$

$\neq id$

latching maps

$l_n : X_n^{deg} \rightarrowtail X_n$

Typically, Reedy cofibrancy easier to satisfy than projective cofibrancy

Examples

Example 1 $\mathcal{C} = sSet$ (Cisinski model category)

$Fun(\Delta^{op}, sSet)$

Reedy

$= Fun(\Delta^{op}, sSet)_{inj}$ (all objects are cofibrant!)

~~Example 2~~ $\mathcal{C} = sSet$

$\operatorname{hocolim}_{\Delta^{op}} X = \operatorname{colim}_{\Delta} X = \Delta \otimes X$

geometric realization of X

Example 2

(see screen captures)

April 2nd, outlook email

Homological Algebra 4/7/2020

The nerve theorem

Theorem (Borsuk, Leray, Weil, ...)

$X \in \mathrm{Top}$ $\{U_i\}_{i\in I}$ (homotopical good cover of X)

↓

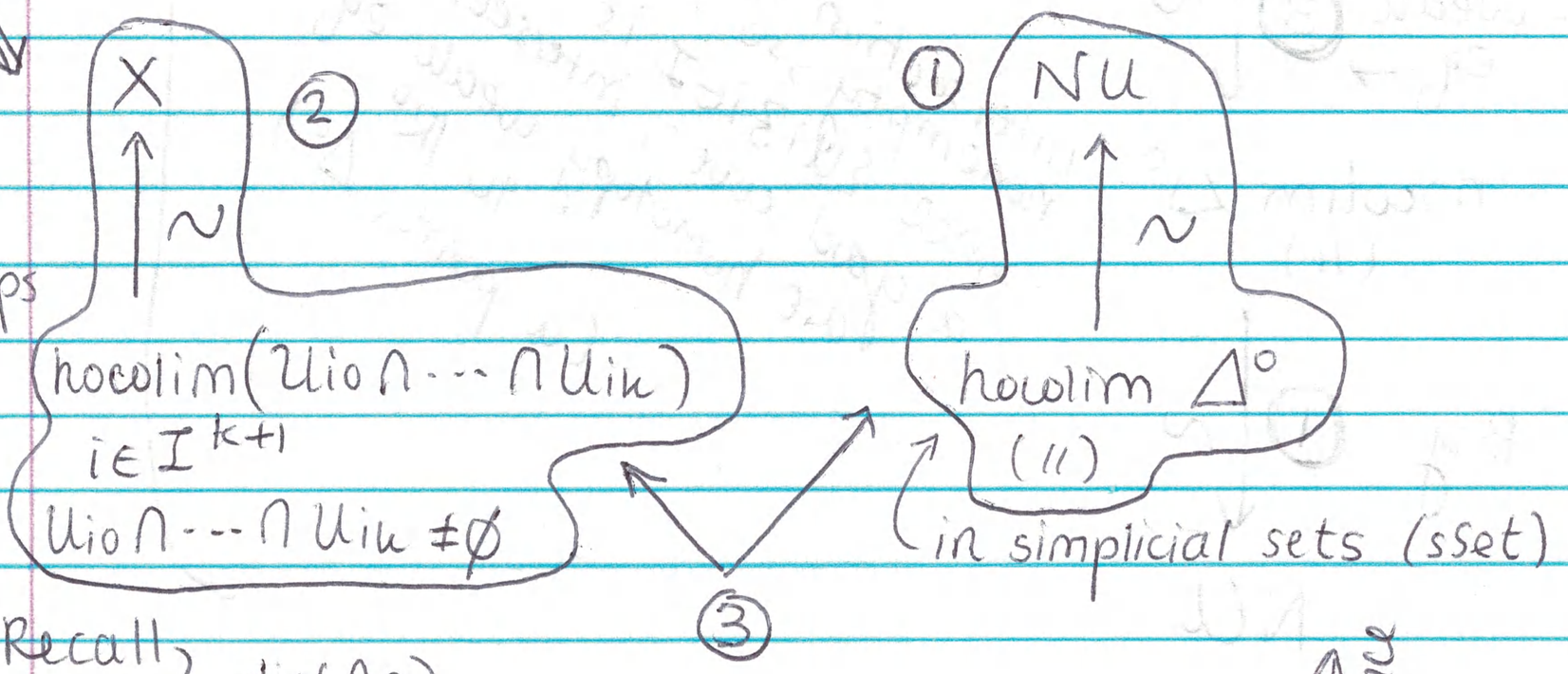
Then $X \simeq |N\mathcal{U}|$ — $U_{i_0} \cap \cdots \cap U_{i_k}$ either $\emptyset$ or weakly homotopy equivalent to $\mathbb{R}^0$

X is weakly (homotopy) equivalent to the nerve of $\mathcal{U}$ — to the geometric realization of $N\mathcal{U}$ (or, $\mathrm{Sing}(X) \simeq N\mathcal{U}$)

Proof of Nerve Theorem

Recall, $(N\mathcal{U})_k = \{ i \in I^{k+1} \mid U_{i_0} \cap \cdots \cap U_{i_k} \neq \emptyset \}$

Show these maps are weak eq.



① We Recall, $\mathrm{dis}(A_n)$

$A \xleftarrow{\sim}$ hocolim $= \mathrm{diag}(n \mapsto \mathrm{dis}(A_n))$

$n \in \Delta^{op}$ ||

hocolim $\coprod_{n\in\Delta^{op}} \Delta^n$ $(m \mapsto (\mathrm{dis}(A_m))_m)$

$n\in\Delta^{op}$ A_n ||

$\sim$ hocolim Δ^0 $(m \mapsto A_m)$

$n\in\Delta^{op}$ ||

$a \in A_n$ A so, the map is a weak eq. equivalence

This is a weak equivalence

Then, for $A = N\mathcal{U}$, we proved →

→ sing X ← ① Apply sing; sing preserves homotopy colimits (move sing inside)

③ Make domains of ① and ② match

Sing hocolim $U_{i_0} \cap \cdots \cap U_{i_k}$

sing X

②

$\text{hocolim}(\text{sing}(U_{i_0} \cap \cdots \cap U_{i_k}))$

(II)

we have a weak eq → ③ ~

$\text{hocolim}\ \Delta^0$

(II)

By ① ~

NC

intersection is nonempty, so since $\{U_i\}_{i \in I}$ is a good cover, intersections are homotopy weak eq to $\mathbb{R}^0$ (a point, Δ^0)

③

② So, now prove the map

$\text{hocolim}(U_{i_0} \cap \cdots \cap U_{i_k}) \longrightarrow X$

(II) is a weak homotopy equivalence

(for then, applying sing gives us ②)

Claim: Here, hocolim = colim

We know how to take $\text{Colim}(U_{i_0} \cap \cdots \cap U_{i_k})$ { Take colimit of underlying sets, and equip it w/ initial or final topology

$\Rightarrow$ $\operatorname{colim}_{\substack{i\in I^{k+1} \\ U_{i_0}\cap\cdots\cap U_{i_k}\neq\emptyset}} (U_{i_0}\cap\cdots\cap U_{i_k}) \xrightarrow{\cong} X$

isomorphic!

$\Vert$

$U_i \sqcup U_j \sqcup U_k \sqcup \cdots$ disjoint union of all elements

$U_i \cap U_j \cdots$ pairs of

glue along intersections

Colimit replaces nonempty intersections $U_{i_0}\cap\cdots\cap U_{i_k}$ $\quad i\in I^{k+1}$, $U_{i_0}\cap\cdots\cap U_{i_k}\neq\emptyset$

with the union of $\{U_i\}_{i\in I}$, which is the whole space X

Alternate proof: (geometric)

For I = linearly ordered

Choose a partition of unity (assume space is paracompact and Hausdorff, so that such a partition exists)

1) $X \xrightarrow{p} |NU|$

(define p for $x\in X$)

choose $x\in X$

$\{f_i : X \to [0,1]\}$ locally finite

then

$I_x = \{i\in I \mid f_i(x)\neq 0\}$ is finite

(by def of partition of unity)

$\operatorname{supp}(f_i)\subset U_i$, $\quad x\in \bigcap_{i\in I_x} U_i \neq \emptyset$

since I is linearly ordered, I_x is linearly ordered,

$I_x \in (NU)_{\#I_x - 1}$ $\longrightarrow$

cardinality of I_x

topology

in the simplex I_x
with coordinates $i \longmapsto f_i(x)$

$\uparrow$
I_x

$\sum_{i \in I_x} f_i(x) = 1$

$\sum_{i \in I} f_i(x) = 1$

partition of unity provides map to nerve!

② $|NU| \simeq |sd(NU)| \longrightarrow X$

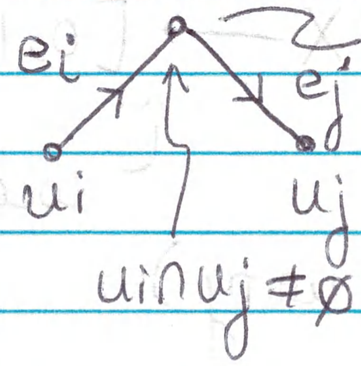
$\uparrow$ barycentric subdivision

This map is adjoint to $sd(NU) \longrightarrow Sing\, X$

$U_i \longmapsto x \in U_i$ (any point x in U_i)

0-simplices of the nerve

and for 1-simplices of the nerve



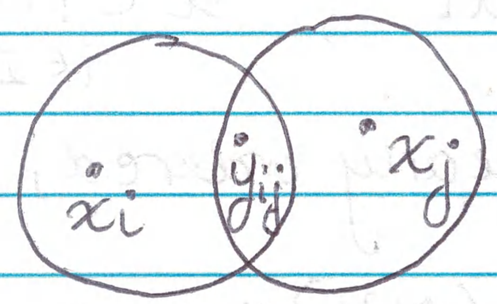
$U_i \cap U_j \neq \emptyset$

$U_i \cap U_j \longmapsto y \in U_i \cap U_j$

such a point exists because of def of nerve

and similarly for 2-simplices

For edges e_i and e_j map to any map



$e_i \longmapsto x_i \longrightarrow y_{ij}$ (contraction in U_i)

$e_j \longmapsto x_j \longrightarrow y_{ij}$ (contraction in U_j)

Mapping Cone

Ch: chain complexes

Def $f: X \to Y$ chain maps

$$X \underset{0}{\overset{f}{\rightrightarrows}} Y \longrightarrow \operatorname{hocoker} f$$

homotopy cokernel

A specific way to compute hocoker f is called the "mapping cone of f"

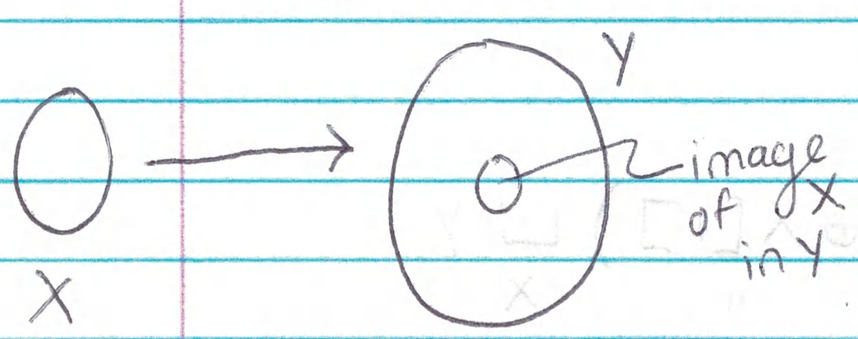
Motivation: sSet* (pointed simplicial sets)
comes w/ map to base point *

generic coequalizer $X \rightrightarrows Y \longrightarrow \operatorname{coeq}(f,g)$

$\parallel$

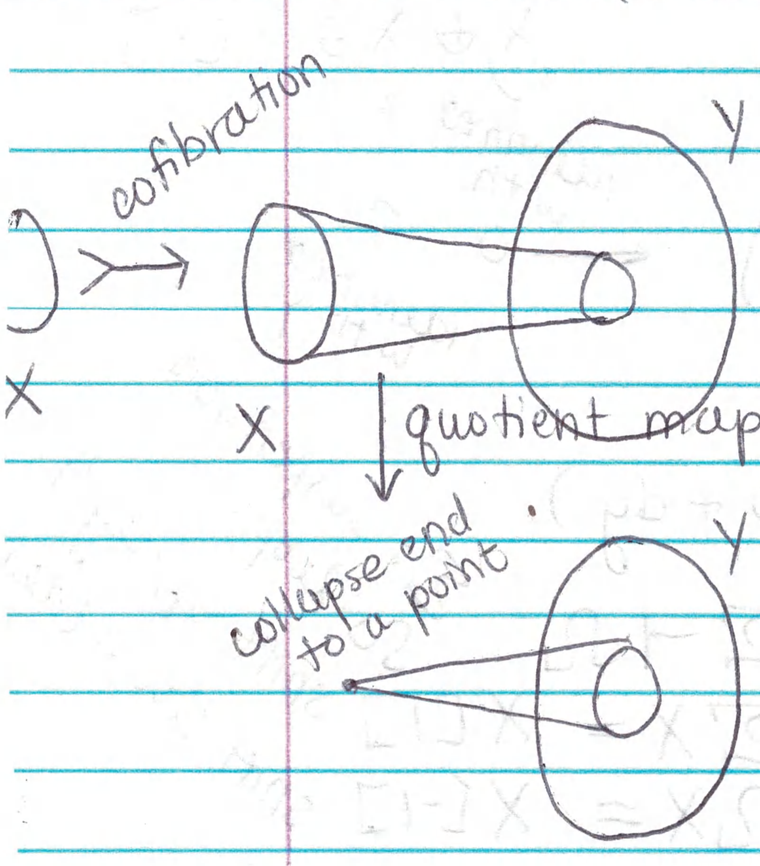
$\operatorname{hocoeq}(f,g)$ if $X \sqcup X \to Y$ is a $[f,g]$ cofibration

For simplicity, fix $g = *$



Replace Y w/ something weakly equivalent:

$(\Delta^1 \times X) \underset{X}{\sqcup} Y$ (cylinder)



Now, take quotient of mapping cylinder construction; you get mapping cone!

Examples see video w/ Dr. Pawlou (mapping cones) →

$e_i \longmapsto x_i \overset{\partial}{\underset{\partial}{\rightrightarrows}} u_{ij}$ (contraction in u_j)

$$* \underset{X}{\sqcup} (\Delta^1 \times X) \underset{X}{\sqcup} Y = \text{cone}(f)$$

(also works for mapping cones of chain complexes)

Mapping Cones for Chain complex

$f: X \longrightarrow Y$ chain map

$$\text{cone}(f) = 0 \underset{X}{\sqcup} (N\mathbb{Z}[\Delta^1] \otimes X) \sqcup Y$$

- 0: zero chain complex
- $N\mathbb{Z}[\Delta^1]$: normalized chains of Δ^1 (Dold-Kan)
- $\otimes$: tensor product (Eilenberg-Zilber)

Recall, $N\mathbb{Z}[\Delta^1]$

$=$

$\mathbb{Z} \oplus \mathbb{Z} \xleftarrow{\begin{pmatrix}-1\\1\end{pmatrix}} \mathbb{Z}$

two vertices (deg 0) — one edge (deg 1)

so, $\text{cone}(f) = 0 \underset{X}{\sqcup} (X \oplus X \oplus X[1]) \underset{X}{\sqcup} Y$

∂x ∂x ∂x $\quad = \quad \mathbb{Z}[1] \otimes X$

Recall, differential map

$X \oplus X \oplus X[1]$

$(a \quad b \quad c)$

$\uparrow x_n \quad \uparrow x_n \quad \uparrow x_{n-1}$

$-id$, $+id$

$(a, b, c) \overset{d}{\longmapsto} (da - c, db + c, dc)$

$X \oplus X \oplus X[1]$

identified with $0 \longleftarrow$

identified with Y

$\text{cone}(f) = X[1] \oplus Y$

differential $d(x, y) = (dc, f(c) + dy)$

$x \in X_{n-1}$

$y \in Y$

suspension and loops

in Ch, $\Sigma X = X[1]$ shift up (deg 1)

$\Omega X = X[-1]$ shift down (deg 1)

Here $\Sigma \dashv \Omega$ is an equivalence

Σ: suspension

Ω: loops

$\Sigma \dashv \Omega$

Def | A "stable" model category is a model category that is (i) pointed $0 \xrightarrow{\sim} 1$ is a weake eq, (ii) $\Sigma \dashv \Omega$ is a Quillen eq.

0 — initial object

1 — terminal object

and there is a canonical such map

(NOTE: stable category also called stable ∞-category)

stable homotopy theorem

canonical

For Examples of stable model categories (see video lecture)

Homological Algebra (4/9/20)

stable model categories (e.g. spectra)

Recall,

$$sSet \xrightarrow{\mathbb{Z}[-]} sAb \xrightarrow[\text{(Dold-Kan)}]{N} Ch_{\geq 0}$$

simplicial chains (w/ coeffs in A)

$Ch_{\geq 0} \xrightarrow{-\otimes A}$ } coefficients $Ch_{\geq 0}$

Homology:

$$H_n(X, A) = H_n(N\mathbb{Z}[X] \otimes A) = \pi_n(\mathbb{Z}[X] \otimes A)$$

$N\mathbb{Z}[X]$: enriched over abelian groups

Cohomology:

$$sSet \xrightarrow{\mathbb{Z}[-]} sAb \xrightarrow[\text{Hom}(-,A)]{} CsAb = Fun(\Delta, Ab)$$

cosimplicial abelian group

$CsAb \xrightarrow{N}$ } simplicial cochains (w/ coeffs in A)

nonneg. graded cochain complex $\{ coCh_{\geq 0}$

$=$

non pos. graded chain complex $\{ Ch_{\leq 0}$

$$H^n(X, A) = H^n(N\,\text{Hom}(\mathbb{Z}[X], A))$$
$$H_{-n}(N\,\text{Hom}(\mathbb{Z}[X], A))$$
$$= \pi_0(\text{Hom}(\mathbb{Z}[X], \Gamma(A[n])))$$

These properties will be expressed in terms of homotopy limits and colimits

Note: Here A: fixed, X will vary; we are interested in the properties of the functors:

$$X \mapsto \mathbb{Z}[X] \otimes A \qquad X \mapsto \text{Hom}(\mathbb{Z}[X], \Gamma(A[n]))$$

Homological Algebra (4/14/20)

simplicial
$\mathcal{C}$: model category
w/ sufficiently strong axioms

Cohomology Theories

then a cohomology theory is a functor $sSet^{op} \xrightarrow{F} \mathcal{C}$ (homotopy continuous)

F is uniquely determined by a point, Δ^0

$F(X) \xrightarrow{\sim} F(\Delta^0)^X$ (assuming $F(\Delta^0)$ is fibrant)

analogy different for homology theories

But, for **Homology Theories** ...

Recall previous example

$$\left[\text{Ex: } X \longmapsto \underbrace{U}_{\text{SSet}} \underbrace{\Gamma}_{\text{Dold-Kan functors}} \underbrace{(N\mathbb{Z}[X] \otimes A)}_{\text{truncation}} \right]$$

$N\mathbb{Z}[X] \otimes A$: homotopy cocontinuous functor

Γ: homotopy continuous functor (preserves homotopy limits)

U: homotopy cocontinuous functor (preserves homotopy colimits)

Ch

already left Quillen!

Recall NOTE: Homotopy pushout squares coincide w/ homotopy pullback squares! (in a stable model category!)

So, preserving homotopy limits same as preserving homotopy colimits

DEFINITION A "homology theory" is a functor $F: sSet \longrightarrow \mathcal{C}$ such that

Here, examples of $\mathcal{C}$ include

$\mathcal{C} = sSet_{(*)}$ pointed simplicial sets

$\mathcal{C} = Ch$ chain complexes

(i) F preserves weak equivalences

(ii) F sends homotopy pushout squares to homotopy pullback squares

(iii) $F(0) \xrightarrow{\sim} 1$

(iv) F preserves filtered colimits

(continued...)

Recall, colimit of diagram indexed by filtered category } **filtered colimit**

* combination of colimits taken over directed posets and walking idempotent category

I is **filtered** if

(i) $I \neq \emptyset$

(ii) $\forall A, B \in I$, $\exists C \in I$ such that morphisms to C exist (always)

$A \to C$, $B \to C$

such that $e \cdot e = e$ — "walking idempotent" — "filtered"

(iii) $A \overset{f}{\underset{g}{\rightrightarrows}} B \xrightarrow{h} C$ such that $hf = hg$

(continued...)

Each of the properties in the previous definition has a name

(i) homotopical functor

(ii) excisive functor

(iii) reduced functor

(iv) $\aleph_0$-accessible functor (usually, restricted to sSets, so this functor doesn't appear)

Homology Theories are called "spectra"

(they are a special application of more general spectra, or, simplicial functors)

$HA = (X \longmapsto \mathcal{U}\Gamma\mathcal{C}(N\mathbb{Z}[X] \otimes A)$

Eilenberg-Maclane space

$\mathcal{E}, F$, $[\mathcal{E}, \Sigma^n F]$ Eilenberg-Maclane spectra

Ex: $\mathcal{E} = F = H(\mathbb{Z}/2)$ "steenrod squares"

Goal: Construct a model structure on the category of functors,

$sset_{(*)} \longrightarrow sset_{(*)}$ ($\mathcal{C}$) any model category w/restriction, whose fibrant objects are objectwise fibrant, reduced, excisive, $\aleph_0$-accessible homotopical functors

Objects: Functors

Functors: Natural Transformations

Fibrant Objects →

We want to construct a model structure on the category of functors $sSet_{(*)} \to sSet_{(*)}$

First issue:

functors $sSet \to sSet$ do not form a (locally small) category

[$\hom(\mathcal{F}, \mathcal{F}')$ is not a set, because $sSet$ is a proper class]

But! We only need fibrant to be $\aleph_0$-accessible!

(NOT homotopy $\aleph_0$-accessible... here, the definitions happen to coincide!)

Solution:

$\aleph_0$-accessible functors are determined by their values on compact simplicial sets.

(finitely many, nondegenerate)

Observation: Any simplicial set $X \in sSet$ can be presented as a colimit of compact subspaces:

$$X \cong \operatorname{colim}_{\substack{Y \subset X \\ Y \text{ compact}}} Y$$

} filtered category and filtered colimits ≡ homotopy filtered colimits (automatically derived)

Hence, we will take only compact functors

$$sSet_{(*)} \xrightarrow{\text{compact}} sSet_{(*)}$$

} Now, only countably many objects (objects in this category form a set)

and essentially replace the "$\aleph_0$-accessible" condition with the definition:

$$F(X) = \operatorname{colim}_{\substack{Y \subset X \\ Y \text{ compact}}} F(Y)$$

Previous Goal:

$\mathrm{Fun}(sSet_{(*)}, \mathcal{C})$

model structure

New Goal:

$\mathrm{Fun}(sSet^{\text{compact}}_{(*)}, \mathcal{C})$

model structure

Recall, Left Bousfield localization: $\mathcal{E} \longleftarrow \mathcal{L}$ full subcategory

Replace model structure with another model structure (same cofibrations, but more weak equivalences)

Ex: presheaves ← sheaves

Localization: forcing more morphisms to be isomorphisms / weak equivalences

(category) (∞-category)

$\mathcal{E} \longmapsto \mathcal{E}[S^{-1}]$ localization

objects are the same; morphisms localized

(Localization extracts a "nice" subcategory)

so, in $Fun(sSet_{(*)}^{compact}, \mathcal{C})$

perform a **LEFT BOUSFIELD LOCALIZATION**

① homotopical

$\forall A \xrightarrow{f \sim} B$ in $sSet_{(*)}$

$$\begin{array}{ccc} F(A) & \xrightarrow{\sim} & F(B) \\ \| & & \| \\ Hom_{\mathcal{C}}(Y(A), F) & & \| \\ \| & & \| \\ Hom_{\mathcal{C}}(Y(B), F) & & \| \\ \| & & \| \\ Hom_{\mathcal{C}}(Y(A), F) & \xrightarrow{\sim} & Hom(Y(B), F) \end{array}$$

Yoneda Functor $Y(A): sSet_{*}^{compact} \longrightarrow sSet$

$X \longmapsto RHom(A, X)$

and the induced map of Yoneda embeddings in the opposite direction:

$Y(A) \xleftarrow[Y(f)]{} Y(B)$

② reduced

reduced $F(*) \xrightarrow{\sim} 1$

// initial object, terminal object

$RHom(Y(*), \exists) \longrightarrow RHom(\emptyset, F)$

induced by the map

Yoneda map: $\emptyset \longrightarrow Y(*)$

③ excisive

$(sSet_{(*)}^{compact})$ category

pushout diagram in source category

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & \ulcorner & \downarrow \\ C & \longrightarrow & D \end{array}$$

homotopy pullback diagram in target category $(\mathcal{C})$

$$\begin{array}{ccc} F(A) & \longrightarrow & F(B) \\ \downarrow & \ulcorner h & \downarrow \\ F(C) & \longrightarrow & F(D) \end{array}$$

By part ①, we can assume

$$\begin{array}{ccc} A & \rightarrowtail & B \\ A & \rightarrowtail & C \end{array}$$ are cofibrations

$$\begin{array}{ccc} F(A) & \xrightarrow{\sim} & F(B) \overset{h}{\times}_{F(D)} F(C) \\ \| & & \| \\ \mathrm{RHom}(Y(A), F) & \hookrightarrow & \mathrm{RHom}(Y(B) \overset{h}{\sqcup}_{Y(D)} Y(C), F) \end{array}$$

homotopy pushout

$$S_1 = \{ Y(A) \leftarrow Y(B) \mid f: A \to B \}$$

$$S_2 = \{ \emptyset \longrightarrow Y(*) \}$$

where $X \longmapsto \emptyset$ and $X \longmapsto *$ in $Y(*)$

$$S_3 = \left\{ Y(B) \overset{h}{\sqcup}_{Y(D)} Y(C) \to Y(A) \;\middle|\; \begin{array}{ccc} A & \rightarrowtail & B \\ \downarrow & & \downarrow \\ C & \to & D \end{array} \right\}$$

and

$$S = S_1 \cup S_2 \cup S_3$$

MAIN POINT

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* Homology Theories are Fibrant Objects *

How to construct spectra/homology theories!

Take any functor

$$s\mathrm{Set}^{\mathrm{compact}}_{(*)} \longrightarrow \mathcal{C}$$

and fibrantly replace

Example

$\mathcal{C} = s\mathrm{set}_{(*)}$

F is inclusion $\left(F : s\mathrm{Set}^{\mathrm{compact}}_{(*)} \hookrightarrow s\mathrm{set}_{(*)} \right)$

$\mathbb{S}$: sphere spectrum (spectral analog of rings)

$\mathbb{N}, \mathbb{Z}, (\mathbb{S}), \mathbb{Q}, \mathbb{R}, \mathbb{C}$

natural integer sphere rational real complex

... Introducing ... (drum roll) SMASH PRODUCTS! $\hat{}$

ALGEBAA	Analogies	HIGHER ALGEBRA "Brave New Algebra"
UNSTABLE		
sets		Kan complexes
groups		simplicial groups
abelian groups		$Ch_{\geq 0} \cong sAb$
		$\downarrow H$ functor passes from unstable to stable
STABLE		
abelian groups		spectra
$\mathbb{Z}$		$\mathbb{S}$ sphere spectrum
$\otimes$		$\wedge$ smash product
rings		ring spectrum
modules		module spectrum
$Mod_{\mathbb{Z}} \cong Ab$	$\longrightarrow$	$Mod_{\mathbb{S}} \cong$ spectra
modules over $\mathbb{Z}$ are abelian groups		modules over $\mathbb{S}$ are spectra

chain complex can be shifted w/ no loss of information

tensor product

$\vee$

There is a canonical map

$\mathbb{S} \longrightarrow H\mathbb{Z}$

$\not\simeq$

NOT a weak equivalence

Homological Algebra (4/16/20)

Homological Algebra 4/21/20

Higher Algebra

Cotangent complex

Definition $\mathcal{C}$: category (typical example : CRing commutative ring Spaces ^{op})

$A \in Ob(\mathcal{C})$

(A : space)

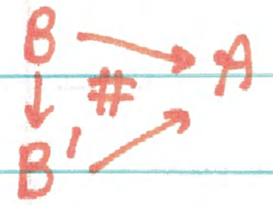
($Mod_A^{\mathcal{C}}$)

A module over A is an abelian group object in the slice category $\mathcal{C}/A$

(slice category

$Ob(\mathcal{C}/A)$ = morphisms $B \to A$

$Mor(\mathcal{C}/A)$ = commutative triangles



PROPOSITION

$\mathcal{C}$ = CRing

$A \in \mathcal{C}$ (A : commutative ring)

$Mod_A^{CRing} \cong Mod_A$ (in the classical algebraic sense)

Ψ

$\longmapsto M$

Given an A-module, M

construct an object of the slice category

so we have,

$B = A \oplus M \longmapsto M$

isomorphism!

construct an element of $\mathcal{C}/A$ by specifying some object of $\mathcal{C}$ (specify B) and a morphism ($B \to A$)

Hence, let $B = A \oplus M \in \mathcal{C}$

(a commutative ring)

Where multiplication is defined

$(a, m) \cdot (a', m') = (a \cdot a', a \cdot m' + a' m)$

"square zero extension"

The morphism $B \to A$ is given by $\Psi \quad \Psi$

$(a, m) \mapsto a$

projection onto A component.

"square zero extension" SZ

abelian group objeet structure on $(B \to A) \in \mathcal{C}/A$

$=$ "G (a particular element of the slice category)"

$1 \to G$

identity

$$\begin{array}{ccc} B & \xrightarrow{G} & A \\ {\scriptstyle f}\uparrow & \nearrow_{\#} & \\ A & \xrightarrow[id]{} & \end{array}$$

$a \in A$

$f(a) = (a, 0) \in B = A \oplus M$

$G \to G$

inverse

$$\begin{array}{ccc} B & \xrightarrow{G} & A \\ {\scriptstyle f}\uparrow & \nearrow_{\#} & \\ B & \xrightarrow[G]{} & \end{array}$$

$(a, m) \in B$

$f((a,m)) = (a, -m)$

$G \times G \to G$

multiplication

$$\begin{array}{ccc} B & \xrightarrow{G} & A \\ {\scriptstyle f}\uparrow & \nearrow & \\ B \times_A B & & \text{projection map} \end{array}$$

fiber product

$(a, m) \not\in (a, m')$

over fiber product A so first component must be the same

$f((a, m, m')) = (a, m + m')$

$\mathrm{Mod}_A^{\mathrm{CRing}} \cong \mathrm{Mod}\, A$

$\mathrm{Mod}_A^{\mathrm{CRing}} \xleftarrow{SZ} \mathrm{Mod}\, A$

"square zero extension"

and the other functor

$\mathrm{Mod}_A^{\mathrm{CRing}} \xrightarrow{ZS} \mathrm{Mod}\, A$

is given by mapping B to the kernel of the map G ...

restriction "ZS" → Given some $G \in \mathrm{Mod}_A^{\mathrm{CRing}}$

$B \xrightarrow{G} \mathcal{A}$, we can construct some $M \in \mathrm{Mod}_{\mathcal{A}}$ as follows:

from the identity property in the abelian group structure on G

$$\begin{array}{ccc} B & \xrightarrow{G} & \mathcal{A} \\ f \uparrow & \# & \\ \mathcal{A} & \xrightarrow{\quad} & \end{array} \qquad \mathrm{id}$$

we can see that $B \cong A \oplus M$ for some $M = \ker G$ where the isomorphism is given by

$$B \longrightarrow A \oplus M$$
$$b \longmapsto (G(a),\ b - f(G(a)))$$
$$B \longleftarrow A \oplus M$$
$$f \oplus \iota$$

Exercises

categorical modules for various categories

$\mathcal{C} = \mathrm{Ring}$,
$\mathrm{Mod}_A^{\mathcal{C}} \cong {}_A\mathrm{Mod}_A$ (bimodules over $\mathcal{A}$)

$\mathcal{C} = \mathrm{LieAlgebra}$,
$\mathrm{Mod}_A^{\mathcal{C}} \cong \mathrm{Mod}_{\mathcal{A}}$ (Lie Algebra modules)

$\mathcal{C} = G\text{-Set}$, $G \in \mathrm{Group}$
$\mathrm{Mod}_A^{\mathcal{C}} \cong$ [illegible]

$\mathcal{C} = C^\infty\mathrm{Ring}$ (automatically commutative)
$\mathrm{Mod}_A^{\mathcal{C}} = \mathrm{Mod}_{\mathcal{A}}$ (regular $\mathcal{A}$-module, in the usual algebraic sense)

$\mathcal{C}$: category w/ finite limits

DERIVATIONS

$G \in \mathrm{Mod}_A^{\mathcal{C}}$, $\mathcal{A} \in \mathcal{C}$

DEFINITION A (G-valued) derivation on $\mathcal{A}$ is a section of $B \xrightarrow{G} \mathcal{A}$,
that is, a morphism $s : \mathcal{A} \longrightarrow B$
such that $G \circ s = \mathrm{id}_{\mathcal{A}}$

Example $\mathcal{C} = CRing$

$B = A \oplus M$

derivation (section) $G(a,m) = a$ $\quad B \xrightarrow{G} A$, $(a,m) \longmapsto a$

$s : A \longrightarrow B = A \oplus M$, homomorphism of commutative rings $s(1) = (1,0)$

we require $\quad s(a \cdot a') = s(a)s(a')$

~~s(a) = (a, t(a)), t : A → M~~

$s(a) = (a, t(a))$, $t : A \longrightarrow M$

since $s(1) = (1,0)$

$(1, t(1)) = (1,0)$

so $t(1) = 0$

since $s(a \cdot a') = s(a)\, s(a')$

$(a \cdot a', t(a \cdot a')) =$

$(a, t(a)) \cdot (a', t(a'))$

formula for square zero multiplication !!

$(a \cdot a', a \cdot t(a') + a' \cdot t(a))$

so, $t(a \cdot a') = a\,t(a') + a' \cdot t(a)$

Leibniz Rule!

This required multiplicative property defines what it means for a section to be a derivation

Both modules and derivations can be defined in a completely abstract setting !!

DEFINITION $\mathcal{C}$: category w/ finite limits

The tangent category, $T\mathcal{C}$

It has **objects**: $A \in \mathcal{C}$, $G \in \mathrm{Mod}_A^{\mathcal{C}}$ pairs (A, G)

and **morphisms**: $(A,G) \longrightarrow (A',G')$

pairs $(f: A \to A',\ G \xrightarrow{p} f^*G')$ pairs (f,p)

union of all the module categories

where $f^*B' \longrightarrow B'$

pullback of G' along morphism f: $f^*G' \to G'$, $A \xrightarrow{f} A'$

morphism in the category of A-modules Mod_A

(the category has finite limits, so pullbacks exist)

alternatively, we have morphisms:

$$\begin{array}{ccc} A & \xleftarrow{G} & B \\ {\scriptstyle f}\downarrow & \# & \downarrow{\scriptstyle h} \\ A' & \xleftarrow[G']{} & B' \end{array}$$

where h preserves abelian group structure

We have a forgetful functor

$$\begin{array}{ccc} T\mathcal{C} & \xrightarrow{U} & \mathcal{C} \\ (A,G) & \longmapsto & A \\ (f,p) & \longmapsto & f \end{array}$$

Example $(A \in \mathrm{CRing},\ M \in \mathrm{Mod}_A)$ $\qquad f^*M'$

morphism $(A,M) \longmapsto (A',M')$ $\qquad\qquad =$

which is a pair $(f: A \to A',\ p: M \to M')$ $\qquad\qquad f^*M'$

f: ring morphism; p: morphism of modules over different rings

Square Zero Extensions

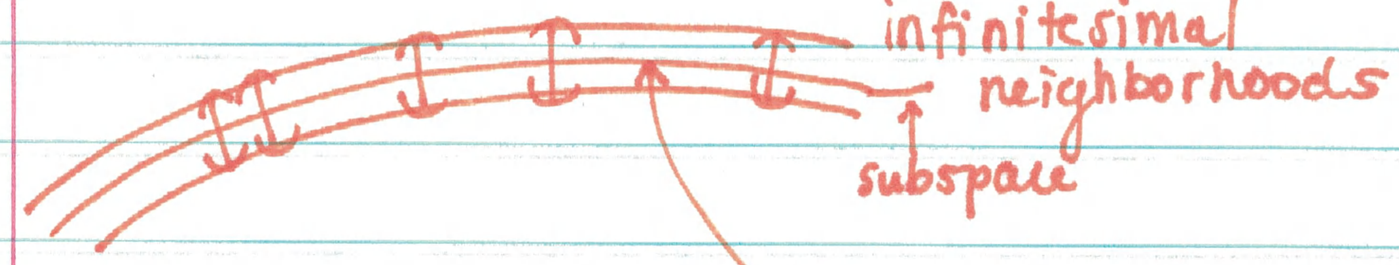
Geometric intuition:

$A \xleftarrow{} B = A \oplus M$

spectra functor contravariant: $\mathrm{spec}\, A \hookrightarrow \mathrm{spec}\, B$ (inclusion)

} many elements whose square is 0

functions on infinitesimal neighborhoods



B: algebra of functions on entire region

A: restricts to algebra of functions on subspace

M: kernel of surjection
(all functions on larger region that vanish on subspace)

Recall $\mathbb{R}_{\varepsilon/\varepsilon^2}$ w/ elements $a+b\varepsilon$

The cotangent bundle / Kähler differentials

DEFINITION $\mathcal{C} \underset{U}{\overset{\Omega}{\rightleftarrows}} T\mathcal{C}$

$\Omega \dashv U$, Ω: Kähler differential functor
(left adjoint to forgetful functor)

$X \in \mathrm{Man}$ $\quad \Omega'_X \in \mathrm{VBun}_X$ (vector bundles over X)

$X \to X'$, $\Omega'_{X'} \xrightarrow{f^*} \Omega'_X$

Take $\mathcal{C} = C^\infty\mathrm{Ring} \supset \mathrm{Man}^{op}$

Then $\Omega|_{\mathrm{Man}^{op}}$: $X \longmapsto \Omega'_X$ (vector bundle, 1-forms)

$$\begin{array}{ccc} X & & \Omega'_X \\ \uparrow & \longmapsto & \downarrow \\ X' & & \Omega'_{X'} \end{array}$$

(Ω can recover the cotangent bundle)
(derivations recover tangent bundle)

Example $\mathcal{C} = C^\infty\text{Ring} \supset \text{Man}^{op}$

$\Omega(C^\infty M) = \Omega^1 M \in \text{Mod}_{C^\infty M}$

PROPOSITION

$\mathcal{C} = \text{CRing}$, $A \in \mathcal{C}$

describe $\Omega A \in \text{Mod}\, A$

classical definition

Kähler differential $\quad \forall D, \exists! h$

$$\begin{array}{ccc} A & \xrightarrow{d} & \Omega A \\ & \searrow^{D} & \downarrow h \\ & & M \end{array} \qquad \begin{array}{l} d: \text{derivation} \\ \# \\ \exists! \ A\text{-linear map } h \\ M \in \text{Mod}\, A \end{array}$$

Then, here, for $G' \in T_{\mathcal{C}}$, $B' \xrightarrow{G'} A'$, $B' \cong A' \oplus M'$

$$\begin{array}{ccc} \{\Omega A \to G'\} & \cong & \{A \to UG\} \\ \| & & \| \\ A \oplus \Omega' A \xrightarrow{g} A' \oplus M' & & A' \\ \downarrow \quad \# \quad \downarrow & & \\ A \xrightarrow{f} A' & & \end{array} \qquad f,\ {}_A\Omega' A \to {}_A M'$$

$g: A \oplus \Omega' A \longrightarrow A' \oplus M'$ homomorphism of rings

$g(a,m) = (f(a), h(a,m))$

$g(1,0) = (f(1), h(1,0))$ $\qquad f(1) = 1$ ✓

$\| \qquad\qquad\qquad\qquad\quad h(1,0) = 0$ ✓

$(1,0)$

w/ derivation properties!

Views derivations as homomorphisms between square zero extensions!

$$\Omega'_A = \langle da \mid a \in A \rangle \Big/ \begin{matrix} d(a+b) = da + db \\ d(ab) = b\cdot da + a\cdot db \end{matrix}$$

Ω'_A ∩ Mod_A

$$A \longrightarrow \Omega' A$$
$$a \longmapsto da$$

usual Kähler differential

DEFINITION $s\mathcal{C} \underset{sU}{\overset{s\Omega}{\rightleftarrows}} sT\mathcal{C}$ is a Quillen adjunction

Left-derived simplicial Kähler functor $\mathbb{L}(s\Omega)$ = cotangent complex

simplicial tangent category (simplicial abelian group objects ≡ co-chain complexes)

$\mathcal{C} = CRing$, $\mathbb{L}\Omega : sCRing \longrightarrow coCh$

Homological Algebra 4/23/20

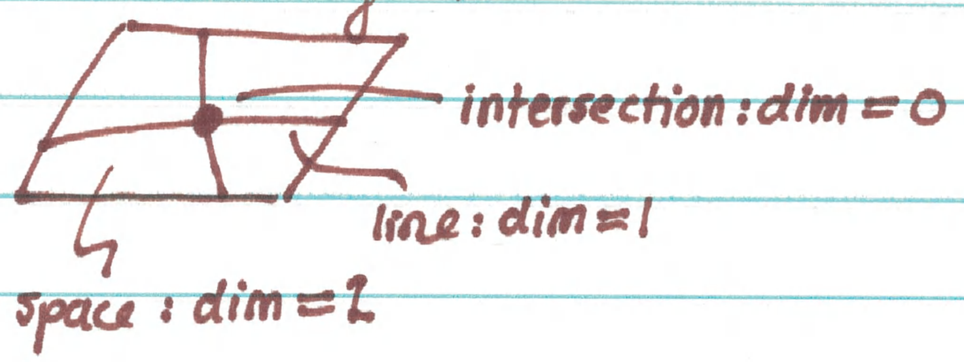
derived manifolds

DEFINITION $DMan := sC^\infty Ring^{op}_{germ}$

~~derived manifolds~~

Recall, $Man^{op} \subset C^\infty Ring$

{ derived manifolds allow us to intersect things that are not transversal



{ derived tensor product reaveals non transversal intersections

$$DMan = sC^\infty Ring^{op}_{germ} = Fun(\Delta^{op}, C^\infty Ring_{germ})^{op}$$

"germ-determined"
mild, technical restriction

Observation #1

Intersections can be expressed via pullbacks

pullback

$$\begin{array}{ccc} A \times_X B & \longrightarrow & B \\ \downarrow & \lrcorner & \downarrow \text{inclusion} \\ A & \longrightarrow & X \\ & \text{inclusion} & \end{array} \Rightarrow A \times_X B = A \cap B$$

The derived intersection can be computed via homotopy pullbacks

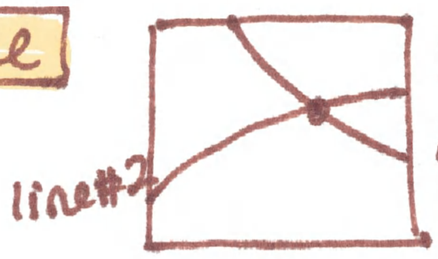
Observation #2

Homotopy limits in $DMan \equiv$ homotopy colimits in $sC^\infty Ring_{germ}$

Observation #3

$$\begin{array}{ccc} \mathbb{L} & & \mathbb{L} \\ C^\infty A \otimes C^\infty B & \longleftarrow & C^\infty B \\ C^\infty X \dashv h & & \uparrow \\ \uparrow & & \\ C^\infty A & \longleftarrow & C^\infty X \end{array} \qquad \underbrace{- \otimes -}_{\text{derived tensor products}}$$

Example

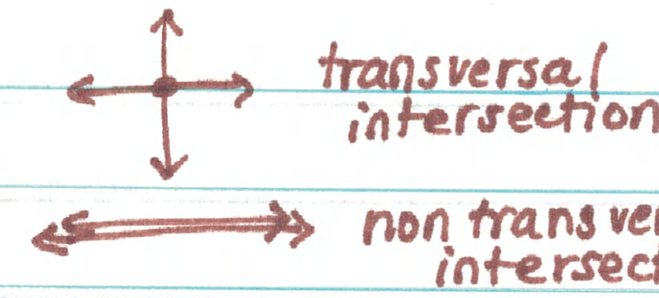


intersection

$$\begin{array}{ccc} I & \longrightarrow & \mathbb{R} \\ \downarrow & & \downarrow \\ \mathbb{R} & \longrightarrow & \mathbb{R}^2 \end{array}$$

inclusion of line #1

inclusion of line #2



Recall, $C^\infty\mathbb{R} = \{\text{smooth functions } \mathbb{R} \longrightarrow \mathbb{R}\}$

algebraic diagram

compute homotopy pushout

$$\begin{array}{ccc} C^\infty I & \longleftarrow & C^\infty\mathbb{R} \\ \uparrow & & \uparrow \\ C^\infty\mathbb{R} & \longleftarrow & C^\infty\mathbb{R}^2 \end{array}$$

free C^∞ Ring on two variables

$C^\infty\{x,y\}$

$$C^\infty\mathbb{R} \otimes_{C^\infty\mathbb{R}^2} C^\infty\mathbb{R}$$

both derived / non derived
transversal / non transversal

$$C^\infty\{x\} \otimes_{C^\infty\{x,y\}} C^\infty\{y\}$$

(nLab)

C^∞ Ring ("smooth algebra") is an algebra A over $\mathbb{R}$ for which not only the product operation $\cdot : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ lifts to the algebra product $A\times A \longrightarrow A$, but ALSO for which every smooth map $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ lifts to a map $A(f): A^n \longrightarrow A^m$ in a compatible way

$$\begin{aligned} Alg_{\mathbb{R}} &= (\underset{\text{set}}{A},\ +: A\times A \longrightarrow A,\ \cdot : A\times A \longrightarrow A) \\ &= (\underset{\text{set}}{A},\ \forall f \in \mathbb{R}[x_1,\dots,x_n],\ A^n \xrightarrow{Af} A,\ \text{associativity property}) \end{aligned}$$

→

→ To compute the homotopy pushout:

DERIVED TRANSVERSAL CASE

$$\begin{array}{ccc} \mathbb{R}[x,y] & \longrightarrow & \mathbb{R}[x] \\ \downarrow & & \downarrow \\ \mathbb{R}[y] & \longrightarrow & D \end{array}$$

Replace the top map by a cofibration

Recall,

Cofibrations in Real Algebras (same in C^∞Rings)

Here, cofibrant objects are retracts of free DGA's

so, cofibrations are retracts of relative free DGA's

$$A \longrightarrow A \otimes_{\mathbb{R}} \text{Free}(B)$$

inclusion

$$\uparrow_{\mathbb{R}}$$
$$Ch_{\geq 0}$$

The cofibrant replacement of f is the chain map Qf (a morphism of DGA's; it preserves multiplication)

Recall, model structure

weak equivalence = quasi-isomorphism

fibration = degreewise surjection

$$Ch^{\mathbb{R}}_{\geq 0} \underset{U}{\overset{\text{Free}}{\rightleftarrows}} DGA_{\mathbb{R}}$$

model structure

fibrations, weak equivalences are created by U.

(morphism in $DGA_{\mathbb{R}}$ is fibration/weak eq iff its image under U is a fibration/weak equivalence in $Ch_{\geq 0}$)

so, $\mathbb{R}[x,y] \xrightarrow{f} \mathbb{R}[x]$ is not a cofibration (it is not even an inclusion)

↓ Replace w/ something quasi-isomorphic that is a cofibration

FREE RESOLUTION!

$$\begin{array}{ccc} \mathbb{R}[x,y] & \xrightarrow{Qf} & \mathbb{R}[x,y] \\ \uparrow & & \uparrow \\ 0 & & \mathbb{R}[x,y] \end{array}$$

map of chain complexes!

multiplication by y

so that $H_0 \cong \mathbb{R}[x]$ (since image of differential map is all multiples of y)

and $H_n \cong 0$ $n \geq 1$

$$\mathbb{R}[y] \overset{\mathbb{L}}{\otimes}_{\mathbb{R}[x,y]} \mathbb{R}[x]$$

$$\mathbb{R}[y] \otimes_{\mathbb{R}[x,y]} \begin{pmatrix} \mathbb{R}[x,y] \\ \uparrow \cdot y \\ \mathbb{R}[x,y] \end{pmatrix} = \begin{pmatrix} \mathbb{R}[y] \\ \uparrow \cdot y \\ \mathbb{R}[y] \end{pmatrix} \in DGA_{\mathbb{R}}$$

we replaced

$$\begin{array}{ccc} \mathbb{R}[x,y] & \xrightarrow{f} & \mathbb{R}[x] \\ \uparrow & & \uparrow \\ \mathbb{R}[x,y] & \xrightarrow[\mathbb{Q}f]{} & \mathbb{R}[x,y] \\ \uparrow & & \uparrow \\ 0 & & \mathbb{R}[x,y] \\ \uparrow & & \uparrow \end{array}$$

with

These have the same homology so are indeed quasi-isomorphic

and the result, $\begin{array}{c} \mathbb{R}[y] \\ \uparrow \cdot y \\ \mathbb{R}[y] \end{array}$ is quasi-isomorphic as a chain to $\mathbb{R}$

$$\begin{array}{ccc} \uparrow & & \uparrow \\ \mathbb{R}[y] & \xleftarrow[\text{quiso}]{\simeq} & \mathbb{R} \\ \uparrow \cdot y & & \uparrow \\ \mathbb{R}[y] & & 0 \\ \uparrow & & \uparrow \end{array}$$

same homology!

DERIVED TRANSVERSAL CASE

so, we get the same answer in the regular transversal case

DERIVED NON-TRANSVERSAL CASE

$$\begin{array}{ccc} \mathbb{R}[x,y] & \longrightarrow & \mathbb{R}[x] \\ \downarrow & \ulcorner h & \downarrow \\ \mathbb{R}[x] & \longrightarrow & D \end{array}$$

w/ same cofibrant replacement for top map

But now, the tensor product is:

derived non-transversal

$$\mathbb{R}[x] \overset{L}{\otimes}_{\mathbb{R}[x,y]} \left(\begin{array}{c} \mathbb{R}[x,y] \\ \uparrow \cdot y \\ \mathbb{R}[x,y] \end{array} \right) = \left(\begin{array}{c} \mathbb{R}[x] \\ \uparrow \cdot 0 \\ \mathbb{R}[x] \end{array} \right)$$

$$\mathbb{R}[x] \overset{L}{\otimes}_{\mathbb{R}[x,y]} \mathbb{R}[x]$$

Compared to non-derived non-transversal case:

$$H_0 \left(\begin{array}{c} \mathbb{R}[x] \\ \uparrow \cdot 0 \\ \mathbb{R}[x] \end{array} \right) = \mathbb{R}[x]$$

The homology in degree 0 ~~[illegible]~~ is non-derived non-transversal case!

But now, we have a nontrivial first homology

$$H_1 \left(\begin{array}{c} \mathbb{R}[x] \\ \uparrow \cdot 0 \\ \mathbb{R}[x] \end{array} \right) = \mathbb{R}[x]$$

} captures additional info about nontransversal lines intersect two identical lines in a plane, we have one normal direction

So $\left(\begin{array}{c} \mathbb{R}[x] \\ \uparrow \cdot 0 \\ \mathbb{R}[x] \end{array} \right) \cong \mathbb{R}[x, y']$ where x: degree $=0$, y': degree $=1$, $(y')^2 = 0$

Geometrically,

$\mathbb{R}[x,y']$ ⟷ $\mathbb{R}[x]$

the algebra of functions of the derived intersection:

$\mathbb{R}[x,y']$

y' captures the normal direction (infinitesimal neighborhood)

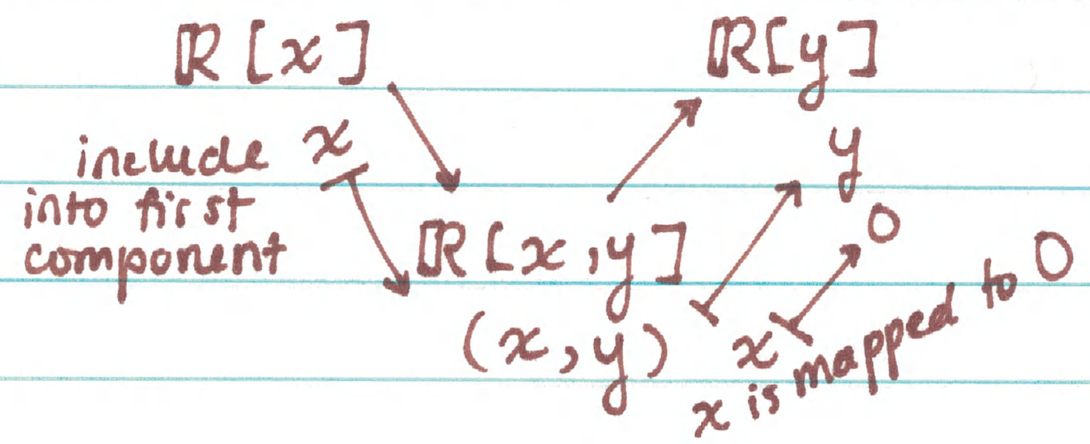
Recall, computations (ordinary tensor product)

$$\mathbb{R}[x] \otimes_{\mathbb{R}[x,y]} \mathbb{R}[y] \cong \mathbb{R}$$

The intersection of two lines on a plane is a single point

TRANSVERSAL COMPUTATION

$$x \cdot a \otimes b = a \otimes 0 \cdot b = a \otimes 0 = 0$$



similarly, $a \otimes y \cdot b = a \cdot 0 \otimes b = 0 \otimes b = 0$

so, $(a + x \cdot c) \otimes (b + y \cdot d)$

$= a \otimes b = ab \otimes 1$

NONTRANSVERSAL COMPUTATION

$$\mathbb{R}[x] \otimes_{\mathbb{R}[x,y]} \mathbb{R}[x] \cong \mathbb{R}[x]$$

(tensor product of two copies of $\mathbb{R}[x]$ over $\mathbb{R}[x,y]$; the y doesn't do anything here!)

so, we have (the underived case)

$$C^\infty\{x\} \otimes_{C^\infty\{x,y\}} C^\infty\{y\} \cong \mathbb{R}$$

$$C^\infty\{x\} \otimes_{C^\infty\{x,y\}} C^\infty\{x\} \cong C^\infty\{x\}$$

I'm not going to feed you some second rate formalism

Now, we can do the same thing for derived tensor products.

Homological Algebra 4/28/20

Recall, from last time:

$$\begin{array}{ccc} P & \longrightarrow & \mathbb{R} \\ \downarrow & \overset{h}{\nearrow} & \downarrow 0 \\ \mathbb{R} & \xrightarrow[h]{} & \mathbb{R}^2 \end{array} \qquad T^*P = (\mathbb{R} \xleftarrow{0} \mathbb{R})$$

$$\upsilon\dim P = \dim T^*P = \chi(T^*P) = 0$$

Although the image of P looks like an infinitesimal line, it has degree 0; infinitesimal direction is in the odd direction here.

So, (interesting!)

Normally, intersection of ~~[illegible]~~ line with itself has dimension 1 ($\mathbb{R}^1$)

Here, intersection of line w/ itself has degree 0 ($\mathbb{R}^0$)

Example

$$\begin{array}{ccc} P & \longrightarrow & \mathbb{R}^0 \\ \downarrow & \overset{h}{\nearrow} & \downarrow \\ \mathbb{R}^a & \xrightarrow{f} & \mathbb{R}^b \end{array}$$

Compute $C^\infty P$ as a C^∞DGRing

(closed elements in deg 0 now equipped w/ ring structure)

PROPOSITION

C^∞DGRing is Quillen equivalent to simplicial C^∞ rings.

REFERENCE

(Homological Algebra for Superalgebras of Differentiable Functions)

– Carchedi, Roytenberg

DEFINITION A C^∞ differential graded ring is a commutative differential graded algebra A over $\mathbb{R}$ together with a C^∞-ring structure on A_0^{closed}

→ Rewrite as homotopy pushout diagram of C^∞DGRings

(We are solving for $P = \{f = 0\}$

f_i is a smooth function on $\mathbb{R}^a$)

$$\begin{array}{ccc} \mathbb{R} & \longleftarrow & \mathbb{R}\{y_1, \dots, y_b\} \\ y_i \mapsto 0 \uparrow & & \uparrow \\ \mathbb{R}\{x_1, \dots, x_a\} & \longleftarrow & \mathbb{R}\{y_1, \dots, y_b\} \\ \text{"a"-variables} & & u_i \mapsto f_i \end{array}$$

derived

→ compute the homotopy pushout:

diagram in original category

$$\begin{array}{ccc} C^\infty p & \longleftarrow & \mathbb{R} \\ \uparrow h & \star & \uparrow y_i \mapsto 0 \\ \mathbb{R}\{x_1,\dots,x_a\} & \longleftarrow & \mathbb{R}\{y_1,\dots,y_b\} \\ f_i & \longleftarrow & y_i \end{array}$$

Replace!

To compute the homptopy pushout of diagram (★), we must replace the right map by a cofibration!

When $b=1$, we proceed as before (example from 4/23 notes)

But, when $b>1$:

A cofibrant replacement for

$$\mathbb{R}\{y_1,\dots,y_b\} \longrightarrow \mathbb{R}$$

This is, in fact, a tensor product of maps

$$\mathbb{R}\{y_1,\dots,y_b\} = \bigsqcup_{1\le i\le b} \mathbb{R}\{y_i\} \longrightarrow \mathbb{R}$$

$\otimes_{\mathbb{R}}$

(Recall, 4/23) free DG Ring on ch complex concentrated in degree 0

can be computed as

$$\bigsqcup_{1\le i\le b} \left(\mathbb{R}\{y_i\} \longrightarrow \mathbb{R}\{y_i\}\langle y_i' \rangle \right)$$

(y_i in degree 0, y_i' in degree 1)

copy of $\mathbb{R}$ indexed by y_i — copy of $\mathbb{R}$ indexed by y_i'

such that $dy_i' = y_i$

(by computing each cofibrant replacement individually, and putting answers together)

weakly equivalent to $\mathbb{R}$

$$\mathbb{R}\{y_0,\dots,y_b\}\langle y_1',\dots,y_b' \rangle$$

(all "0" — all "1")

$$C^\infty(\mathbb{R}^b) \otimes_{\mathbb{R}} \wedge^* \mathbb{R}^b$$

w/ differential $dy_i' = y_i$

cofibrant Replacement!

$$\mathbb{R}\{y_1,\dots,y_b\}$$

$$\left(\mathbb{R}\{y_i\} \xleftarrow{(-\cdot y_i)} \mathbb{R}\{y_i\} \right) \quad (0 \leftarrow 1)$$

as a chain complex

"Koszul complex"

so, now we have a diagram

$$\begin{array}{ccc} (\mathbb{R}\{y_0,\dots y_b\}\langle y_1',\dots,y_b'\rangle) & & \\ C^\infty(\mathbb{R}^b)\otimes\wedge^*\mathbb{R}^b & \longleftarrow & \mathbb{R}\{x_j\}\otimes\mathbb{R}\{y_i\}\otimes\wedge^*\mathbb{R}^b \\ \uparrow & & \uparrow \\ \mathbb{R} & & \mathbb{R}\{y_i\} \\ & & \uparrow \\ & & \mathbb{R}\{x_i\}\langle y_i'\rangle \\ & & (dy_i' = f_i) \\ & & \uparrow \\ \mathbb{R}\{y_1,\dots,y_b\} & \longleftarrow & \mathbb{R}\{x_1,\dots,x_a\} \end{array}$$

and computing the ordinary pushout of this diagram will give us the homotopy pushout of the original diagram!

Batalin-Vilkovisky complex
"BV-complex"

$\mathbb{R}\{x_j\}\langle y_i'\rangle$ $(dy_i' = f_i)$ where

compute homology:

0-homology $H_0 = Z_0/B_0 = \mathbb{R}\{x_j\}/(f_1,\dots,f_b)$ w/no odd variables (since odd variables have degree!)

ideal generated image of functions

corresponds to map

$$\mathbb{R}\{x_j\} \longrightarrow \mathbb{R}\{x_j\}/(f_1,\dots,f_b)$$
$$\mathbb{R}^a \longleftarrow \{f=0\}$$
(not derived!)

so $H_0 = C^\infty(\{f=0\})$

we recover the usual nonderived zero locus (intersection)

1-homology $H_1 = Z_1/B_1$ $\quad Z_1 = \ker\left(\bigoplus_{1\le i\le b}\mathbb{R}\{x_j\} \xrightarrow{(f_1,\dots f_b)} \mathbb{R}\{x_j\}\right)$

$= \{(g_i) \mid g_i : \mathbb{R}^a \to \mathbb{R}, \sum f \cdot g = 0\}$

PROPOSITION If f is a submersion at $0 \in \mathbb{R}^b$
(if $f(x)=0$ for ~~some~~ some $x \in \mathbb{R}^a$)

$\mathbb{R}^a \xrightarrow{f} \mathbb{R}^b$ then $(Tf)(x) : T_x\mathbb{R}^a \twoheadrightarrow T_{f(x)}\mathbb{R}^b$

↑ induced map of tangent spaces

$\parallel \quad \parallel$

$\mathbb{R}^a \xrightarrow{(Df)(x)} \mathbb{R}^b$ derivative

KEYWORDS FOR PROOF

Koszul complex

regular sequence

then, $C^\infty p \simeq H_0 C^\infty p$

(derived interseetion same as ordinary intersection)

Recall, **COTANGENT COMPLEX**

$\mathcal{C}$: category ($\mathcal{C} = C^\infty Ring$, $\mathcal{C} = C^\infty DGRing$)

$B \in \mathcal{C}$

"Modules" $Mod_B^{\mathcal{C}} := AbGroup(\mathcal{C}/B)$

$T\mathcal{C} = Mod^{\mathcal{C}} := (B \in \mathcal{C},\ G \in Mod_B^{\mathcal{C}})$ vector bundle over B

$\mathcal{C} = spaces^{op}$
$T\mathcal{C} = VectBun^{op}$

map between tangent spaces

$f \downarrow$

$(B' \in \mathcal{C},\ G' \in Mod_{B'}^{\mathcal{C}})$ vector bundle over B'

morphism $B \xrightarrow{f} B'$ smooth map

$G \xrightarrow{g} f^*G'$ morphism of vector bundles

Forgetful functor

$\mathcal{C} \underset{u}{\leftarrow} T\mathcal{C}$ "forgetful functor"

$\mathcal{C} \underset{u}{\overset{\Omega}{\underset{\leftarrow}{\rightarrow}}} T\mathcal{C}$ left adjoint: Kähler differentials

Ex: CRing

$B \longmapsto (B, \Omega'_B)$

↑ "cotangent bundle"

Now, we compute:

$\mathbb{L}\Omega(C^\infty P)$

$\mathbb{L}\Omega$ preserves homotopy colimits

$$\begin{array}{ccc} P & \longrightarrow & \mathbb{R}^0 \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle 0} \\ \mathbb{R}^a & \xrightarrow[f]{} & \mathbb{R}^b \end{array} \qquad \begin{array}{ccc} C^\infty P & \longleftarrow & \mathbb{R} \\ \uparrow{\scriptstyle h} & & \uparrow \\ C^\infty\mathbb{R}^a & \xleftarrow[C^\infty f]{} & C^\infty\mathbb{R}^b \end{array}$$

Recall, $C^\infty P = \mathbb{R}\{x_0, \dots, x_a\}\langle y_1', \dots, y_b' \rangle$

(such that $dy_i' = f_i$)

and we have that: (no dx_j, since they vanish)

$\mathbb{L}\Omega_\bullet(C^\infty P) = \Omega_\bullet(C^\infty P)$

$= (C^\infty P, \Omega^1 C^\infty P)$

d = chain differential

∂ = Kähler differential

where

$\Omega^1_{C^\infty P} = \langle \underbrace{\partial x_1, \dots, \partial x_a, \partial y_1', \dots, \partial y_b'}_{\text{generators}} \rangle$

(in terms of generators and relations)

$d\partial x_j = 0,\ d\partial y_i' = \partial f_i$ (relations)

here $d\partial y_i' = \partial dy_i'$

$= \partial(f_i)$

computed using usual rules

$= \sum_j \left(\frac{\partial f_i}{\partial x_j}\right) \partial x_j$

($\frac{\partial f_i}{\partial x_j}$: ordinary partial derivative)

$\upsilon\dim P = \chi(\Omega^1 P) = a - b$

(χ: Euler characteristic (alternating sum of dimensions of homology groups))

punchline:

The virtual dimension is always equal to the codimension

DERIVED STACKS

Homological Algebra 4/30/20

manifolds in $C^\infty_{bad}\text{-rings}^{op}$ MOTIVATION

bad theory holim, hocolim
no internal Hom

↳ good theory of homotopy limits and internal hom

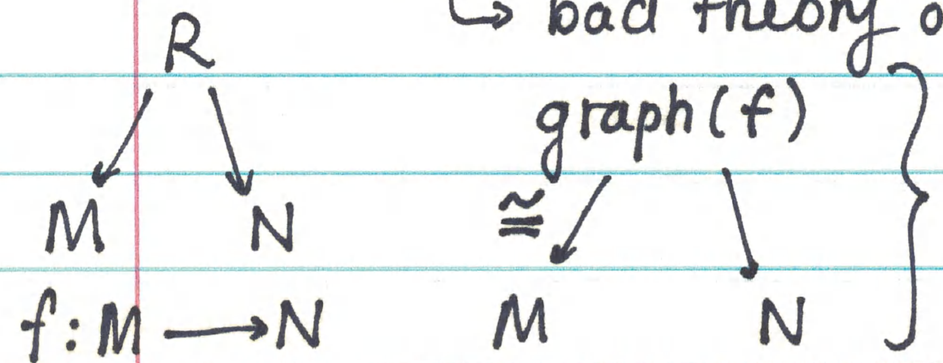
↳ bad theory of hocolim, but always exists.

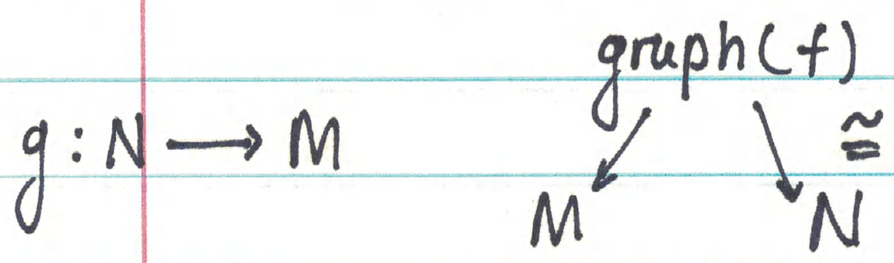
↓ embedding into stacks (simplicial presheaves)

↓ embedding into
derived manifolds in $sC^\infty\text{-rings}^{op}$

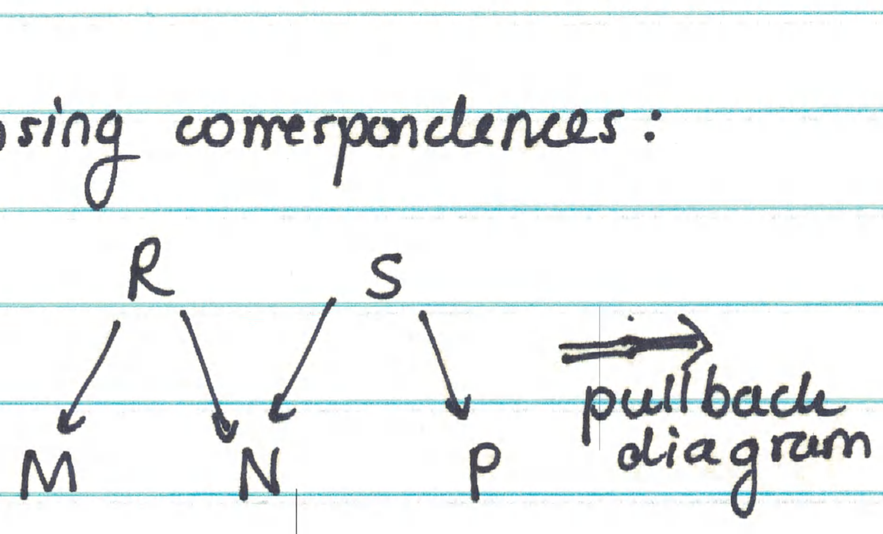
↳ excellent theory of homotopy limits!

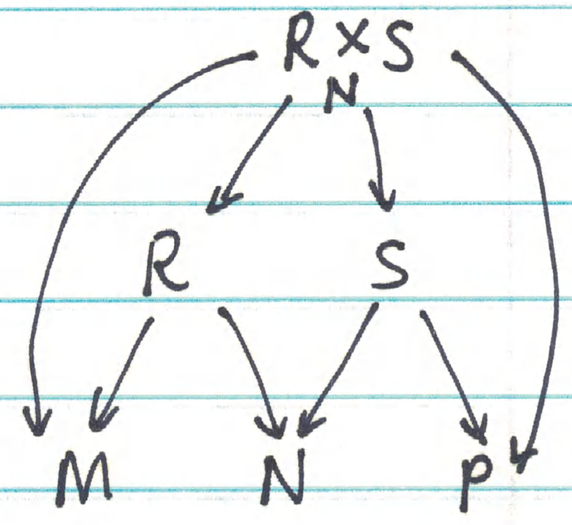
↳ bad theory of homotopy colimits, but always exists.





composing correspondences:





What happens if this pullback is not transversal?

$G \curvearrowright M$ and we want to compute

$$\dim(M/G) = \dim M - \dim G$$

RESOURCES: Toën/Vezzosi, Homotopical Algebraic Geometry I
Lurie, Spectral Algebraic Geometry (2319 pages)

Toën, Higher and Derived Stacks: A Global Overview (survey paper)

Moezdijk/Toën, Simplicial Methods for operads and algebraic geometry

We will define a model category that satisfies

(1) $\dim(M/G) = \dim M - \dim G$

(2) $\dim R \underset{N}{\times} S = \dim R + \dim S - \dim N$

so it will have an excellent theory of limits/colimits

DEFINITION (1) $C^\infty\text{Ring} = \text{Fun}_{\text{product-preserving}}(\text{Cart}, \text{Set})$

(2) There is an adjunction:

$$\text{Set} \underset{\mathcal{U}=ev_{\mathbb{R}^1}}{\overset{\text{Free}}{\rightleftarrows}} C^\infty\text{Ring}$$

evaluation

Free $\dashv \mathcal{U}$ adjoint functors

$A \in \text{Set}$ finite $\Rightarrow \text{Free}(A) = C^\infty(\mathbb{R}^A, \mathbb{R})$

(3) $R \in C^\infty\text{Ring}$ is "finitely-generated" if there is a (surjective) quotient map onto $\mathbb{R}$

$$C^\infty(\mathbb{R}^n, \mathbb{R}) \xrightarrow{\exists} \mathbb{R}$$

(similar to algebra definition of "finitely-generated")

(continued...)

Lemma $M \in \mathrm{Man}$

$\mathcal{C}^\infty M$ is a finitely generated C^∞-ring

proof | $M \hookrightarrow \mathbb{R}^b$ induces a map in opposite direction: Whitney Embedding Thm

$C^\infty(\mathbb{R}^b) \twoheadrightarrow C^\infty M$ ▨

④ $R \in C^\infty\mathrm{Ring}$ is "germ-determined"

if $\exists R \hookrightarrow \prod_{i \in I} S_i$, S_i: "local" C^∞-ring

$0 \neq 1$ and whenever $(x+y) \in S_i^\times$ (invertible) than either x or y is invertible: $x \in S_i^\times$ or $y \in S_i^\times$

or, equivalently, $R = C^\infty(\mathbb{R}^A)/I$ (I is closed under locally finite infinite sums)

where quotient is the same as in ordinary commutative rings!

locally finite infinite sums: $\sum_{k \in K} f_k$, $f \in I$: ideal, $f_k : \mathbb{R}^A \to \mathbb{R}$

$\exists$ open cover $\{V_j\}$ of $\mathbb{R}^A$, such that, $\forall j \in J$

$\{k \in K \mid f_k|_{V_j} \neq 0\}$ is finite

---

k
fork
kangaroo
krazy glue
Turkish
lucky
thinking

change k to k' so as not to confuse with h or K (capital)

DEFINITION $\mathbb{G} := (C^\infty\mathrm{Ring}_{\text{germ-determined}}^{\text{finitely-generated}})^{op}$

$\mathbb{G} \subset \mathbb{L} = (C^\infty\mathrm{Ring}_{\text{finitely-generated}})^{op}$

"C^∞-locus" (loci)

We will let $\mathbb{G}$ be our "site" (previously, site was Cart, and we defined sheaves and presheaves)

We will introduce a Grothendieck Topology

DEFINITION: $\mathbf{G}$ is equipped with the Grothendieck Topology generated by the coverage

$$\{C^\infty(U_k)/I|_{U_k} \xleftarrow[\text{homomorphism}]{\text{restriction}} C^\infty(\mathbb{R}^n)/I\}$$

where $k \in K$

and $\{U_k \hookrightarrow \mathbb{R}^n\}_{k\in K}$ is an open cover of $\mathbb{R}^n$.

open subsets

DEFINITION $c\mathbf{G}$ ("cosimplicial G") is defined:

$$c\mathbf{G} := (sC^\infty\text{Ring}^{\substack{\text{finitly-generated}\\ \text{germ-determind}}})^{op}$$

$$= \text{Fun}(\Delta, \mathbf{G})$$

(weak equivalences are transferred from sSet)

$c\mathbf{G}$ is a (small) relative site, i.e., a relative category $(\mathcal{C}, W)$ with a coverage.

(W: weak equivalences)

($c\mathbf{G}$ can be thought of as derived germ-determined C^∞-loci)

DEFINITION $(\mathcal{C}, W)$: relative site

$sPSh(\mathcal{C}) := \text{Fun}(\mathcal{C}^{op}, sSet)$ proj

perform left Bousfield localization at:

Both of these classes become weak equivalences

① (first class of morphisms) $\forall C(U) \longrightarrow Y(X)$

$\forall$ covering family $\{U_k \to X\}$

② (second class of morphisms) $Y(X) \longrightarrow Y(X')$

$\forall X \xrightarrow{\sim} X'$

fibrant objects

"derived C^∞-stacks" $\longrightarrow$

Fibrant objects $\mathcal{F} : \mathcal{C}^{op} \to sSet$

(a) $\mathcal{F}(X) \in Kan$, $\forall X \in \mathcal{C}^{op}$

(b) $\mathcal{F}(X) \to \left(\operatorname{holim} \mathcal{F}(U_{i_0} \times_X \cdots \times_X U_{i_k}) \right)$

↑ diagram w/ intersections, double intersections, maps, etc

(c) $\mathcal{F}(X') \xrightarrow{\sim} \mathcal{F}(X)$ if $X \xrightarrow{\sim} X'$

($\mathcal{F}$ preserves weak equivalences)

using the generic site, G, we obtain derived C^∞-stacks

[NOTE: The above definition is a generalization of the sPSh's defined earlier; replacing weak equivalences (above) by isomorphisms recovers the previous notion of sPSh.]

DEFINITION

Take $(\mathcal{C}, W) = (cG, \text{weak equivalences})$

we Get derived C^∞-stacks!

DEFINITION ① $G \hookrightarrow cG \hookrightarrow Fun(cG^{op}, sSet)$ projective, local

(hom fully faithful)

embedding #1 — constant — Yoneda

Recall, "derived Yoneda" preserves homotopy limits

$X \longmapsto (W \longmapsto \mathbb{R}Hom(W, X))$

$X \in cG$, $W \in cG$

but, does not preserve homotopy colimits

DEFINITION $Fun(Cart^{op}, sSet)$ projective local

↑↓ Left Quillen functor

$Fun(cG^{op}, sSet)$ local projective $\longrightarrow$

Left adjoint map:

$$\begin{array}{ccc} \mathrm{Cart} & \longrightarrow & \mathbb{G} \\ \mathbb{R}^n & \nearrow & \downarrow \\ & \searrow & \mathbb{G} \\ & & C^\infty(\mathbb{R}^n)^{op} \end{array}$$

Right adjoint map:

$$\begin{array}{ccc} \mathrm{Fun}(\mathrm{Cart}^{op}, \mathrm{sSet}) & \underset{\text{restriction}}{\overset{\mathcal{E}}{\rightleftarrows}} & \mathrm{Fun}(\mathbb{G}^{op}, \mathrm{sSet}) \\ Y \uparrow & \# & \uparrow Y \\ \mathrm{Cart} & \longrightarrow & \mathbb{G} \end{array}$$

The functor $\mathcal{E}$, when derived ($\mathbb{L}\mathcal{E}$),
preserves homotopy colimits!
(in general, does NOT preserve homotopy limits)

CONCLUSION

In derived C^∞-stacks:

$$\dim(M/G) = \dim M - \dim G$$

$$\dim(R \underset{N}{\times}^h S) = \dim R + \dim S - \dim N$$

(excellent theory of holim/hocdim,
in that it behaves as expected
geometrically)

Homological Algebra 5/5/20

BV-BRST formalism studies algebra of functions on derived critical locus of action function

BV- derived geometry BRST - stacks, etc.

∫ infinite-dimensional integrals over a space

INTEGRAL

M: finite-dimensional, oriented, smooth manifold

(assumed only for ease of calculation, but not required)

DeRham complex:

$$\Omega^0 M \longrightarrow \Omega^1 M \longrightarrow \cdots \longrightarrow \Omega^{\dim M-1} M \longrightarrow \Omega^{\dim M}_M \longrightarrow 0$$

[NOTE: Must be compactly supported]

$$H^{\dim M}_{dR}(M)$$

(which is the cokernel of the map $\Omega^{\dim M-1} M \longrightarrow \Omega^{\dim M} M$)

Let $n = \dim M$

Then,

$\Omega^n M$

$\int$

$H^n_{dR}(M)$

$\cong$

$H^n(M, \mathbb{R})$

$\cong$

$\mathbb{R}$

The composition is called "integration" ($\int$)

Derivative map

$C^\infty \mathbb{R} \longrightarrow C^\infty \mathbb{R}$

w/ kernel constant real functions; cokernel canonically isomorphic to $\mathbb{R}$

How do we define integrals over infinite-dimensional spaces M? →

$\mu \in \Omega^n M$

$\left[\mu \in \mathrm{Dens}\, M \quad \text{(density)}\right.$

$\left. \Omega^n M \otimes_{C^\infty M} \mathrm{Or}(M) \quad \text{(orientation bundle)} \right]$ ← tensor with

Recall, we can define a canonical map:

$$\Omega^n M \otimes \Lambda^k TM \xrightarrow{\cong} \Omega^{n-k} M$$

This isomorphism produces a map $\forall k$:

$$\Omega^k M \to \Lambda^k TM$$

identify forms $\Omega^{n-k} M$ w/ multi-vector fields of complementary degree.

smooth sections of exterior powers of tangent bundle

$$\Lambda^n TM \xrightarrow{\Delta\mu} \Lambda^{n-1} TM \to \cdots \to \Lambda^1 TM \to \Lambda^0 TM$$

$$\Lambda^1 TM = \mathfrak{X} M \quad \text{vector fields}$$

$$\Lambda^0 TM = C^\infty M \quad \text{smooth functions}$$

$\Delta\mu$ (differential) "divergence" (called such because, like classical divergence, it produces a smooth function from a vector field)

... and this chain complex is defined also for infinite-dimensional spaces, M

REFERENCE: Ph.D thesis of Owen Gwilliam

"Factorization algebras and free field theories"

Q: How does $\Delta\mu$ change as μ changes?

Replace μ by $\exp\left(\frac{i}{\hbar} S\right) \cdot \mu$

$S: M \to \mathbb{R}$ "action" ← smooth function defined on same space as μ, so defined on M

and the exponential expression can be presented as a composition: →

$$2\pi\hbar\mathbb{R} \xrightarrow[\frac{i}{\hbar}\cdot(-)]{} \mathbb{C} \xrightarrow[\exp(-)]{} \mathbb{C}^\times \quad \text{exact sequence}$$

$$\begin{array}{c} \cup \\ 2\pi\hbar\mathbb{Z} \end{array}$$

"quantized" integer lattice

and

$2\pi\hbar = h$ "planck's constant"

QUESTIONS:
(Dirac charge argument)
(explain parallel transport)

Recall,

$$\begin{array}{ccc} \Omega^k & \xrightarrow{d} & \Omega^{k+1} \\ \cong\uparrow & & \uparrow\cong \\ \wedge^{n-k} & \xrightarrow{\Delta\mu} & \wedge^{n-k-1} \end{array}$$

$$\begin{array}{ccc} \wedge^k & \xrightarrow{L_{ds}} & \wedge^{k-1} \\ \| & & \\ \Gamma(\wedge^{k-1}TM)\wedge\Gamma(TM) & & \end{array}$$

Replace μ by μ_S

where $\mu_S = \exp(\frac{i}{\hbar}S)\cdot\mu$

PROPOSITION

$$\Delta\mu_S = \Delta\mu + \frac{i}{\hbar}L_{dS}$$

Multiply $\Delta\mu_S$ by $\frac{\hbar}{i}$

$$(\tfrac{\hbar}{i})\Delta\mu_S = (\tfrac{\hbar}{i})\Delta\mu + L_{ds}$$

and take the classical limit as $\hbar \to 0$

The new differential, L_{ds}, does not depend on μ

PROPOSITION $(\wedge^*\Gamma(TM), L_{ds})$ chain complex is the derived critical locus of the action functional S

$S: M \to \mathbb{R}$
action functional

(alternatively, derived zero locus of dS) ←

The "classical BV-complex"

The Quantum BV-complex is a deformation of the classical BV-complex

QUESTION:
(What is deformation quantization?)

"Review" "Session"

Definition A quantum BV-complex (alias Beilinson-Drinfeld algebra) is a $\mathbb{Z}$-graded algebra over $\mathbb{R}[\![\hbar]\!]$ equipped with $(\mathcal{A})$

$\Delta: \mathcal{A} \longrightarrow \mathcal{A}$, a "differential" of degree 1, $\Delta^2 = 0$, satisfying

$$\Delta(ab) - (\Delta a)b - (-1)^{|a|} a(\Delta b) = \hbar\{a, b\}$$

where, $\{-,-\}: \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}$ is a Poisson bracket of degree 1

↓ ENHANCE!
w/ deformation information

The classical limit of a Beilinson-Drinfeld algebra, $\mathcal{A}$, is $\mathcal{A} \otimes_{\mathbb{R}[\![\hbar]\!]} \mathbb{R}$

QUESTION: What is a deformation quantization?

"deformation quantization" from $\mathbb{R}$ upgrade algebraic structure to field $\mathbb{R}[\![\hbar]\!]$

Ex: $M = \mathbb{B}_\nabla G$ (stack of principal G-bundles w/ connection)

$S: \mathbb{B}_\nabla G \longrightarrow \mathbb{R}$ Chern-Simons action functional

computes curvature and substitutes into invariant polynomial in Lie Algebra, takes trace wrt Lie Algebra and integrates

curvature (differential 2-form valued in Lie Algebra)

Ex: $\dim M = 3$

$$S\left(\begin{array}{c} P \\ \downarrow \\ G \end{array}\right) = \frac{1}{4\pi}\int \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$$

$A \in \Omega^1(M, g)$
$\dim M = 3$

RESOURCE: Owen Gwilliam thesis or Kevin Costello paper/book

$\mathbb{B}_\nabla G$: $\mathrm{Map}^{op} \longrightarrow \mathrm{Grpd} \xrightarrow{N} \mathrm{sSet}$ (nerve N)