Mathematics 5317 (Introduction to Modern Algebra)

Fall 2020

Homework 9

First submission due November 17, 2020.

1. In this problem, we work in the slice category CRing/A, where A is a commutative ring and CRing is the category of commutative rings. Recall that CRing/A denotes the slice category over A. Suppose $X \in \mathsf{CRing}/A$ is equipped with the following morphisms (in the category CRing/A):

- $\mu\colon X \times X \to X$ (multiplication);
- $X \to X$ (inverse);
- $A \to X$ (neutral).

These morphisms must satisfy the axioms of an abelian group, when expressed diagrammatically, but using products in the category CRing/A instead of Set. For example, the associativity axiom says $\mu \circ (X \times \mu) = \mu \circ (\mu \times X)$, where products and composition are taken in the category CRing/A. Show that any such object X is isomorphic to the following construction: given an A-module M, we set $X = A \oplus M$ with multiplication $(a, m)(a', m') = (aa', am' + a'm)$, the homomorphism $X \to A$ given by $(a, m) \mapsto a$, and $\mu((a, m), (a, m')) = (a, m + m')$.

2. Consider the category of fields, defined as the full subcategory of commutative rings. (Recall that additional properties required of fields are $1 \neq 0$ and $x \neq 0$, $y \neq 0$ implies $xy \neq 0$.) Does this category have initial or terminal objects? Product or coproducts? Equalizers or coequalizers? Same question of the category of fields of a fixed characteristic p, where the *characteristic of a field* is the order of element $1 \in F$, i.e., the smallest p such that $p \cdot 1 = 0$, or 0 if no such p exists.

3. Recall that $\mathbf{Q}[[x]]$ denotes the ring of formal power series over the rational numbers. Show that the only ring endomorphism $\phi\colon \mathbf{Q}[[x]] \to \mathbf{Q}[[x]]$ with $\phi(x) = x$ is the identity.

4. $R = \mathbf{Z} \times \mathbf{Z}$ is a ring with addition and multiplication defined by

$$(a, b) + (c, d) = (a + c, b + d), \qquad (a, b)(c, d) = (ac + ad + bc, bd).$$

Show that there are no nonzero nilpotents in this ring (meaning $x^n = 0$ for $x \in R$ and $n > 0$ implies $x = 0$).

5. Denote by R the ring containing $\mathbf{Q}$, and generated over $\mathbf{Q}$ by two elements x and y with $yx - xy = 1$. Show that R is simple, i.e., has no two-sided ideals other than (0) and R.

6. An R-module X is *injective* if for any injective homomorphism of R-modules $f\colon A \to B$ and any homomorphism of R-modules $g\colon A \to X$ there is a homomorphism of R-modules $h\colon B \to X$ such that $hf = g$. Show that $M \oplus N$ is injective if and only if M and N are injective, where M and N are R-modules.

7. A ring R (assumed to be commutative for the purposes of this problem) is *Noetherian* if every ideal I of R is generated by finitely many elements. Consider $R = \{a_0 + a_1 x + a_2 x^2/2 + \cdots + a_n x^n/n! \mid n \geq 0, a_i \in \mathbf{Z}\}$. Show that R is a subring of $\mathbf{Q}[x]$. Show that R is not Noetherian.

8. Suppose R is a commutative ring. An element $x \in R$ is *nilpotent* if $x^n = 0$ for some $n > 0$.

(a) Show that nilpotent elements of R form an ideal and in the quotient of R by this ideal the element [0] is the only nilpotent element..

(b) Show that any prime ideal of R (i.e., an ideal I such that R/I is an integral domain) contains all nilpotent elements.

(c) Show that the set of nilpotent elements is a maximal ideal of R (i.e., an ideal I such that R/I is a field) if and only if every element of R is either invertible or nilpotent.

9.

(a) Give an example of a commutative ring R with ideals $I \neq J$ such that R/I and R/J are isomorphic as rings.

(b) Let R be a commutative ring and let I and J be ideals in R. Show that if R/I and R/J are isomorphic as R-modules, then one has $I = J$.

10. Suppose R is a ring. Consider the ring $M_n(R)$ of $n \times n$ matrices with coefficients in R for some $n \geq 0$. Show that the only two-sided ideals of $M_n(R)$ are precisely ideals of the form $M_n(I)$, where I is a two-sided ideal of R.