Mathematics 5317 (Introduction to Modern Algebra)

Fall 2020

Homework 8

First submission due November 3, 2020.

1*. In this problem, we work in the category of commutative rings.
(a) Given two commutative rings A and B, equip the abelian group $A\otimes B$ with a structure of a commutative ring.
(b) Show that the resulting commutative ring $A\otimes B$ is the coproduct of A and B. What are the injection maps?
(c) Give an example when the injection maps are not injective.

2*. Suppose A is an abelian group. Equip $T(A)=\bigoplus_{n\geq 0}\bigotimes_{1\leq k\leq n} A=\mathbf{Z}\oplus A\oplus A\otimes A\oplus A\otimes A\otimes A\oplus\cdots$ with a structure of a ring by setting $(a_1\otimes a_2\otimes a_3\otimes\cdots)(a'_1\otimes a'_2\otimes\cdots)=(a_1\otimes a_2\otimes a_3\otimes\cdots\otimes a'_1\otimes a'_2\otimes\cdots)$. Prove the following universal property of $T(A)$: homomorphisms of rings $T(A)\to R$ (where R is an arbitrary ring) are in bijective correspondence with morphisms of abelian groups $A\to \mathsf{U}(R)$, where $\mathsf{U}(R)$ denotes the underlying abelian group of R.

3. In this problem, we work in the category of rings.
(a) Given two rings A and B, equip the abelian group $A*B=\mathbf{Z}\oplus A\oplus B\oplus A\otimes B\oplus B\otimes A\oplus A\otimes B\otimes A\oplus B\otimes A\otimes B\oplus\cdots$ (all possible alternating tensor products of A and B occur in the direct sum) with a structure of a commutative ring.
(b) Show that the resulting ring $A*B$ is the coproduct of A and B. What are the injection maps?

4. Prove or disprove: the abelian group $\mathbf{Q}/\mathbf{Z}$ can be equipped with a structure of a ring (with the given abelian group structure).

5*. Suppose $q\colon A\to B$ is a surjective homomorphism of rings that admits a section, i.e., a homomorphism of rings $s\colon B\to A$ such that $qs=\mathrm{id}_B$. Prove that there is an ideal I in A such that A is isomorphic as a ring to the abelian group $B\oplus I$ equipped with the multiplication $(b,i)(b',i')=(bb',bi'+b'i+ii')$.

6. Show that the group $\mathbf{Q}$ does not have a maximal proper subgroup, i.e., a subgroup $A<\mathbf{Q}$ such that $A\neq\mathbf{Q}$ and if $A<B<\mathbf{Q}$, then $A=B$ or $B=\mathbf{Q}$.

7*. The ring of (complex) *formal power series* in one variable is defined as follows. Its elements are maps of sets $\mathbf{N}\to\mathbf{C}$, where $\mathbf{N}=\{0,1,2,3,\ldots\}$ and $\mathbf{C}$ is the set of complex numbers. We suggestively denote such an element f as $f=\sum_{n\geq 0} f_n x^n$, where x is a 'formal variable'. The abelian group structure is inherited from the product of abelian groups $\mathbf{C}^{\mathbf{N}}$, i.e., group operations are defined indexwise. The multiplicative identity is the element $1\cdot x^0$. The multiplication is defined as follows:

$$\left(\sum_{m\geq 0} f_m x^m\right)\left(\sum_{n\geq 0} g_n x^n\right)=\sum_{p\geq 0}\left(\sum_{\substack{m+n=p\\ m\geq 0,n\geq 0}} f_m g_n\right)x^p.$$

(a) Prove that a formal power series has a multiplicative inverse if and only if $f_0\neq 0$ (here f_0 is the coefficient before x^0, i.e., the free term).
(b) We say that a formal power series $\sum_{n\geq 0} f_n x^n$ is *convergent* if there is a real number $R>0$ such that the sequence $n\mapsto R^n|f_n|$ is bounded. Prove that convergent formal power series form a subring of formal power series.

8. Continuing Problem 7, suggest a nontrivial class of commutative monoids M that contains $\mathbf{N}$ (and other commutative monoids) such that the definition of formal power series continues to make sense for M instead of $\mathbf{N}$. Does part 7(a) remain true?

9. A *pe-group* is a set G with a binary operation $(x,y)\mapsto x/y$ ('division') such that

$$a/a=b/b,\qquad (a/a)/((a/a)/a)=a,\qquad a/(b/c)=(a/((c/c)/c))/b.$$

Homomorphisms of pe-groups are defined in the usual manner.

(a) Define a functor from the category of groups to the category of pe-groups by sends a group G to the pe-group with the same underlying set as G and the division operation $x/y = xy^{-1}$.

(b) Does every pe-group arise from this construction? What does 'p.e.' stand for?

A *torsor* is a set T with a ternary operation $t\colon T \times T \times T \to T$ ('translation') that satisfies the following axioms: $t(b,b,c) = c = t(c,b,b)$, $t(a,b,t(c,d,e)) = t(t(a,b,c),d,e)$.

(c) Show that any group G gives rise to a torsor with the same underlying set and ternary operation $t(a,b,c) = ab^{-1}c$. (This is a good way to think about torsors.)

(d) Define a functor from torsors to pe-groups by sending a torsor (T,t) to the quotient $(T \times T)/\sim$, where $(a,b) \sim (c,d)$ if $b = t(a,c,d)$. Use part (c) to guess what the division operation should be.

10. Continuing Problem 9, show that any set with a transitive action of a group gives rise to a torsor. Can you reverse this construction?