Mathematics 5317 (Introduction to Modern Algebra)

Fall 2020

Homework 7

First submission due October 27, 2020.

1. Construct two functors $D\colon \mathsf{Set} \to \mathsf{Set}$ and $I\colon \mathsf{Set}^{\mathsf{op}} \to \mathsf{Set}$ such that for any set X we have $D(X) = I(X) = 2^X$, where 2^X denotes the set of all subsets of X. (In other words, you must define D and I on morphisms and prove that composition and identity maps are respected.)

Consider the following four categories:

- Group, the category of groups and homomorphisms of groups;
- $\mathsf{Group}_{\mathsf{inj}}$, the category of groups and injective homomorphisms of groups;
- $\mathsf{Group}_{\mathsf{surj}}$, the category of groups and surjective homomorphisms of groups;
- $\mathsf{Group}_{\mathsf{iso}}$, the category of groups and isomorphisms of groups;

A functor $\mathsf{Group} \to \mathsf{C}$, where C is a category, can be restricted to $\mathsf{Group}_{\mathsf{inj}}$ or $\mathsf{Group}_{\mathsf{surj}}$. A functor $\mathsf{Group}_{\mathsf{inj}} \to \mathsf{C}$ or $\mathsf{Group}_{\mathsf{surj}} \to \mathsf{C}$ can be restricted to $\mathsf{Group}_{\mathsf{iso}}$. However, extending a functor defined on a smaller subcategory to a bigger subcategory is not always possible. In the following problems, you must find the largest subcategory (i.e., Group, $\mathsf{Group}_{\mathsf{inj}}$, $\mathsf{Group}_{\mathsf{surj}}$, $\mathsf{Group}_{\mathsf{iso}}$) on which a given functor can be defined.

2. Recall that $\mathsf{Z}(G) = \{g \in G \mid \forall a \in G\colon ag = ga\}$ is the center of G and $\mathtt{Inn}(G)$ is the subgroup of $\mathtt{Aut}(G)$ comprising inner automorphisms. Find the largest subcategory on which a given functor can be defined.

(a) The functor $G \mapsto \mathsf{Z}(G)$, $f \mapsto f'$, where f' is the restriction of f.

(b) The functor $G \mapsto \mathtt{Inn}(G)$.

3. Find the largest subcategory on which a given functor can be defined.

(a) The functor $G \mapsto \mathtt{Aut}(G)$.

(b) The functor $G \mapsto \mathtt{Out}(G) = \mathtt{Aut}(G)/\mathtt{Inn}(G)$.

4. Suppose C is a category that admits binary products. Consider the functor $\Pi\colon \mathsf{C} \times \mathsf{C} \to \mathsf{C}$ that sends an object $(X, X') \mapsto X \times X'$ and $(f\colon X \to Y, f'\colon X' \to Y') \mapsto f \times f'\colon X \times X' \to Y \times Y'$. Consider also the functor $\Delta\colon \mathsf{C} \to \mathsf{C} \times \mathsf{C}$ that sends $X \in \mathsf{C}$ to $(X, X) \in \mathsf{C} \times \mathsf{C}$ and $f\colon X \to Y$ to (f, f).

(a) Show that the above assignments indeed define functors.

(b) Establish a bijective correspondence between morphisms $\Delta(W) \to (X, X')$ and $W \to \Pi(X, X')$, where $W \in \mathsf{C}$ and $(X, X') \in \mathsf{C} \times \mathsf{C}$.

(c) Formulate an analogous statement for Δ and $\amalg\colon \mathsf{C} \times \mathsf{C} \to \mathsf{C}$ ($(X, X') \mapsto X \sqcup X'$). Prove your claim by reducing to (a) and (b) using the notion of an opposite category.

5. Recall (Homework 1, Problem 9) that for any sets A, B, C, we have a bijection between sets of maps

$$A \to C^B = \mathtt{hom}(B, C), \qquad A \times B \to C.$$

This problem develops an analogue of this construction for G-sets. Fix an arbitrary group G and consider the category of G-sets. We already proved that G-sets admit small products. Prove that for any G-sets X and Y one can construct a G-set $\mathtt{Hom}(X, Y)$ such that morphisms of G-sets

$$W \to \mathtt{Hom}(X, Y)$$

are in bijection with morphisms of G-sets

$$W \times X \to Y.$$

6. Fix a group G.

(a) Prove that the following construction defines a functor:

$$\otimes\colon \mathsf{Set}_G \times \mathsf{Set}_G \to \mathsf{Set}, \qquad (X, Y) \mapsto X \otimes Y := (\mathsf{U}(X) \times \mathsf{U}(Y))/\!\sim,$$

where the functor U discards the G-action and $\sim$ is an equivalence relation defined as follows: $(x, y) \sim (x', y')$ if there is $g \in G$ such that $x' = gx$ and $y' = gy$. (The use of the symbol $\otimes$ is deliberate, but this is not the same operation as the tensor product of abelian groups.)

(b) Prove or disprove: for any G-sets X and any set Y there is a G-set $\mathsf{Hom}(X, Y)$ such that for any G-set W the set of morphisms of G-sets

$$W \to \mathsf{Hom}(X, Y)$$

is in bijection with the set of maps (of sets)

$$W \otimes X \to Y.$$

(Notice that this part is quite different from Problem 5, despite the formal similarity.)

7. Prove that it is not possible to define for any groups G and H a group $\mathsf{Hom}(G, H)$ such that the set of homomorphisms of groups

$$F \to \mathsf{Hom}(G, H)$$

is in bijection with the set of homomorphisms of groups

$$F \times G \to H.$$

8. Suppose $A \in \mathsf{Ab}$ and $\{B_i\}_{i \in I}$ is a family of abelian groups.

(a) Define a canonical isomorphism (and prove that it is an isomorphism)

$$\bigoplus_{i \in I} (A \otimes B_i) \to A \otimes \bigoplus_{i \in I} B_i.$$

(b) Define a canonical (nontrivial) homomorphism

$$A \otimes \prod_{i \in I} B_i \to \prod_{i \in I} (A \otimes B_i).$$

Give an example of A and $\{B_i\}_{i \in I}$ for which this homomorphism is not an isomorphism.

9. Given an example of an injective homomorphism of abelian groups $f\colon A \to B$ such that $A \ncong 0$ and the canonical homomorphism

$$f \otimes f\colon A \otimes A \to B \otimes B, \qquad a \otimes a' \mapsto f(a) \otimes f(a')$$

is not injective. (You may use as ingredients examples established in class.)

10. Suppose that A is an abelian group such that for any injective homomorphism $f\colon B \to C$ the induced map

$$A \otimes f\colon A \otimes B \to A \otimes C, \qquad a \otimes b \mapsto a \otimes f(b)$$

is injective. Prove that for any $n \in \mathbf{Z}$ and $a \in A$, if $na = 0$, then $n = 0$ or $a = 0$. Extra bonus point: prove the converse statement.