# Mathematics 5317 (Introduction to Modern Algebra)

*Fall 2020*

**Homework 6**

*First submission due October 20, 2020.*

Recall that the *free group* $F(S)$ on a set $S$ is the coproduct $\coprod_{s\in S}\mathbf{Z}$. Accordingly, it has the following universal property: group homomorphisms $F(S)\to G$ are in a canonical bijective correspondence with maps of sets $S\to \mathsf{U}(G)$.

A *system of generators and relations* for a group is a pair $(S,R)$, where $S$ is a set and $R$ is a subset of $F(S)\times F(S)$, where $F(S)$ denotes the free group on the set $S$. The *group generated by this system* is a group $\langle S|R\rangle$ together with a map of sets $f\colon S\to \mathsf{U}(\langle S|R\rangle)$ such that the homomorphism of groups $g\colon F(S)\to\langle S|R\rangle$ induced by $f$ (see the previous paragraph) satisfies $g(r_1)=g(r_2)$ for any $(r_1,r_2)\in R$ and for any other group $G$ with a map $f'\colon S\to\mathsf{U}(G)$ satisfying the same property there is a unique homomorphism $h\colon\langle S|R\rangle\to G$ such that $\mathsf{U}(h)f=f'$.

**1.** Prove that $\langle S|R\rangle$ exists and is unique up to a unique isomorphism.

The typical application of the above universal property is to construct a homomorphism $\langle S|R\rangle\to G$ by constructing a map of sets $S\to\mathsf{U}(G)$ and verifying that its compatible with the given relations.

**2.** Prove that the dihedral group $D_n$ as defined in Homework 4, Problem 6, is isomorphic to the group $\langle x,y|r^n=s^2=(sr)^2=1\rangle$.

**3.** Suppose $G$ is a group and $m\colon G\times G\to G$ is a homomorphism of groups.

(a) Show that if $m(u,g)=g=m(g,u)$ for all $g\in G$ and some fixed $u\in G$, then $u=1$ and $\mathsf{U}(G)$ with the multiplication operation $m$ is a group.

(b) Assuming (a), show that $m(g,h)=gh$, i.e., the resulting group coincides with $G$, and $G$ is abelian.

A *groupoid* is a category in which all morphisms are isomorphisms. Below, we assume groupoids to be small, i.e., their objects will always form sets, not proper classes.

**4.** Suppose a group $G$ acts on a set $X$.

(a) Show that there is a groupoid whose set of objects is $X$ and the set of morphisms is $X\times\mathsf{U}(G)$, with the source and target of $(x,g)$ being $x$ and $g\cdot x$ respectively.

(b) A groupoid is *connected* if it has at least one object and any two objects are isomorphic. Prove that if the action of $G$ on $X$ is transitive, then the resulting groupoid is connected.

**5.** Suppose $X$ is a $G$-set for some group $G$.

(a) For a subgroup $H<G$, compute the set of morphisms of $G$-sets $\mathsf{hom}(G/H,X):=\{G/H\to X\}$, where $G/H$ is equipped with the standard left action of $G$.

(b) For subgroups $H_1<G$, $H_2<G$, and an element $[g]\in G/H_2$ that defines a morphism of $G$-sets $h\colon G/H_1\to G/H_2$ such that $[g]=h(H_1)$, compute the induced map of sets

$$\mathsf{hom}(G/H_2,X)\to\mathsf{hom}(G/H_1,X)$$

that sends $f\colon G/H_2\to X$ to $fh\colon G/H_1\to X$.

Recall that a sequence of homomorphisms of abelian groups

$$A\to B\to C$$

is *exact* if $A\to B$ is the kernel of $B\to C$ and $B\to C$ is surjective. Equivalently, $A\to B$ is injective and $B\to C$ is the cokernel of $A\to B$. Another equivalent characterization is that $A\to B$ is injective, $B\to C$ is surjective, and the image of $A\to B$ coincides with the kernel of $B\to C$.

**6.** Recall the group $\mathsf{Hom}(G, A)$ from Homework 2, Problem 7.

(a) Show that if $C \to D$ is the cokernel (quotient) of $B \to C$ (i.e., $D = C/B$), then $\mathsf{Hom}(D, A) \to \mathsf{Hom}(C, A)$ is the kernel of $\mathsf{Hom}(C, A) \to \mathsf{Hom}(B, A)$.

(b) Show that if $B \to C$ is the kernel of $C \to D$, then $\mathsf{Hom}(C, A) \to \mathsf{Hom}(B, A)$ need not be the cokernel of $\mathsf{Hom}(D, A) \to \mathsf{Hom}(C, A)$.

**7.** A group $G$ is *finitely generated* if it has a finite subset $S$ such that the only subgroup of $G$ that contains $S$ is $G$ itself.

(a) Show that any finitely generated group has a maximal proper subgroup.

(b) Show that the additive group $\mathbf{Q}$ of rational numbers is not finitely generated.

**8.** Suppose a group $G$ has a trivial center and every automorphism of $G$ is inner. If $G \triangleleft H$, show that $H$ is isomorphic to the product of $G$ and another group.

**9.** Suppose $n \geq 2$ and $\sigma \in \Sigma_n$. Show that if $\sigma$ commutes with a permutation in $\Sigma_n$ of sign $-1$, then the conjugacy classes of $\sigma$ in $\Sigma_n$ and $A_n$ are the same. (The *conjugacy class* of $\sigma$ is $\{\tau\sigma\tau^{-1}\}$.)

**10.** Denote by $G$ the free group on a set $\{a, b\}$. Denote by $N$ the *normal* subgroup of $G$ generated by $aba$ and $a^{16}b^5$ (i.e., the intersection of all normal subgroups of $G$ containing these two elements). Show that $G/N$ is abelian.