Mathematics 5317 (Introduction to Modern Algebra)

Fall 2020

Homework 5

First submission due October 13, 2020.

Which sets of data given below define categories? Warning: some sets of data are incomplete, e.g., do not specify composition or identities. You must reconstruct all missing data. This is a creative process and sometimes it can have more than one answer, in which case any correct answer will do. Some items below have negative answers.

1. Sets and relations. More precisely, given two sets X and Y, morphisms $X \to Y$ are *relations* from X to Y, i.e., subsets of $X \times Y$. Two relations $R \subset X \times Y$ and $S \subset Y \times Z$ are composed as follows: $R \circ S = \{(x, z) \mid \exists y \in Y\colon (x, y) \in R \wedge (y, z) \in Z\}$.

2. Objects are sets. Morphisms $A \to B$ are maps $f\colon A \to 2^B$ such that $\{f(a)\}_{a \in A}$ is a partition of B into disjoint nonempty subsets.

3. Sets and partially defined functions. A *partially defined function* $X \to Y$ is a function $A \to Y$, where $A \subset X$. If $x \in X$, we say that f is *defined* on x (or: $f(x)$ is defined) if $x \in A$. Partially defined functions are composed as follows: if $f\colon X \to Y$ and $g\colon Y \to Z$ are partially defined function, then the composition gf is defined on $x \in X$ if f is defined on x and g is defined on $f(x)$, in which case $(gf)(x) := g(f(x))$.

4. Poset: objects are partially ordered sets (i.e., a set X with a relation R that is reflexive ($x \le x$), transitive ($x \le y$ and $y \le z$ implies $x \le z$), and antisymmetric ($x \le y$ and $y \le x$ implies $x = y$). Morphisms are functions that preserve the order: if $x \le y$, then $f(x) \le f(y)$.

In the following problems, recall that the universal property of binary products requires the following.

(1) The projection maps $A \times B \to A$ and $A \times B \to B$ must be morphisms in the given category;
(2) The maps $Z \to A$ and $Z \to B$ given as an input to the universal property must be morphisms in the given category;
(3) The resulting map $Z \to A \times B$ is a morphism in the given category.

In particular, if we work (say) in the category of sets and surjective maps, then (1), (2), and (3) must be surjective. Also, there is no (a priori) reason why products in this category must coincide with products in the category of sets, i.e., we need not have (a priori) $A \times_1 B = A \times_2 B$, where $\times_1$ denotes the product in the category of sets and maps of sets, whereas $\times_2$ denotes the product in the category of sets and (say) surjective maps of sets. (Sometimes this *does* happen, but you have to prove it separately.)

5. Does the category of sets and injective maps of sets admit products?

6. Does the category of sets and surjective maps of sets admit products?

The next few problems require the following definitions.

- An *epimorphism* in a category C is a morphism $f\colon X \to Y$ such that for any $g, h\colon Y \to Z$ the equality $gf = hf$ implies $g = h$.
- A *section* of a morphism $f\colon X \to Y$ is a morphism $s\colon Y \to X$ such that $fs = \mathrm{id}_Y$.
- An *monomorphism* in a category C is a morphism $f\colon X \to Y$ such that for any $g, h\colon W \to X$ the equality $fg = fh$ implies $g = h$.
- A *retraction* of a morphism $f\colon X \to Y$ is a morphism $r\colon Y \to X$ such that $rf = \mathrm{id}_X$.
- An *isomorphism* in a category C is a morphism $f\colon X \to Y$ such that there is $g = f^{-1}\colon Y \to X$ for which $gf = \mathrm{id}_X$ and $fg = \mathrm{id}_Y$.

7. See above for definitions.

(a) Show that a morphism that admits a section is an epimorphism. Give an example of a morphism in some category that is an epimorphism, but does not admit a section.
(b) Show that a morphism that admits a retraction is a monomorphism. Give an example of a morphism in some category that is a monomorphism, but does not admit a retraction.

(c) Prove or disprove: in any category for which the product below exists, the projection maps

$$p_j\colon \prod_{i\in I} A_i \to A_j$$

are epimorphisms.

8. See above for definitions.

(a) Give an explicit description of monomorphisms and epimorphisms in the category Poset.

(b) Give an example of a morphism in some category that is an epimorphism and a monomorphism, but is not an isomorphism.

(c) Show that a split epimorphism (i.e., an epimorphism with a section) that is also a monomorphism must necessarily be an isomorphism.

Given a category C and an object $X \in \mathsf{C}$, define the following two categories.

The *slice category* (alias *overcategory*) $\mathsf{C}/X = \mathsf{C} \downarrow X = \mathsf{C}_{/X}$ is defined as follows.

- Objects are morphisms in C of the form $A \to X$.
- Morphisms from $f\colon A \to X$ to $f'\colon A' \to X$ are morphisms in C of the form $g\colon A \to A'$ such that $f'g = f$:

$$\begin{array}{ccc} A & \searrow^{f} & \\ {\scriptstyle g}\downarrow & & X. \\ A' & \nearrow_{f'} & \end{array}$$

A good way to think about objects in a slice category is that we have a family of objects of C fibered over X. For instance, if $\mathsf{C} = \mathsf{Set}$, then an object of Set/X is a family of sets indexed by X.

The *coslice category* (alias *undercategory*) $X/\mathsf{C} = X \downarrow \mathsf{C} = \mathsf{C}_{X/}$ is defined as follows.

- Objects are morphisms in C of the form $X \to A$.
- Morphisms from $f\colon X \to A$ to $f'\colon X \to A'$ are morphisms in C of the form $g\colon A \to A'$ such that $gf = f'$:

$$\begin{array}{ccc} & \nearrow^{f} & A \\ X & & \downarrow{\scriptstyle g} \\ & \searrow_{f'} & A'. \end{array}$$

9. Suppose C is a category that admits all small products. Show that for any object $X \in \mathsf{C}$, the coslice category X/C admits all small products.

10. Show that for any set X, the slice category Set/X admits all small products.