Mathematics 5317 (Introduction to Modern Algebra)

Fall 2020

Homework 4

First submission due October 6, 2020.

1. Suppose $f\colon G \to H$ is a homomorphism of groups. The *cokernel* of f is a homomorphism of groups $q\colon H \to Q$ such that $qf = 1$ (here 1 denotes the trivial homomorphism) and the following universal property is satisfied: for any $b\colon H \to B$ such that $bf = 1$ there is a unique $g\colon Q \to B$ such that $gq = b$. Prove that the cokernel exists and is unique up to a unique isomorphism. (Warning: the image of f need not be a normal subgroup of H.)

2. Show that an infinite simple group cannot have a proper simple subgroup of finite index. Here a group G is simple if it has exactly two normal subgroups: G and the trivial subgroup 1. The *index* of $H < G$ is defined as the cardinality of G/H.

3. Suppose $N_i \lhd G_i$ for $i \in I$, where I is an arbitrary set. Prove that

$$\left(\prod_{i\in I} G_i\right) \Big/ \left(\prod_{i\in I} N_i\right) \cong \prod_{i\in I} G_i/N_i.$$

4. Show that all automorphisms of the symmetric group of degree 4 are inner.

5. Suppose $H < \Sigma_n$, where Σ_n is the symmetric group of degree $n \geq 5$. Prove that if Σ_n/H has exactly n elements, then $H \cong \Sigma_{n-1}$. You may use the following result proved in class: for $n \geq 5$, the group Σ_n has no normal subgroups other than Σ_n, A_n, and the trivial subgroup 1.

6. For any $n \geq 0$, the *dihedral group* D_n has as its underlying set $\mathbf{Z}/n\mathbf{Z} \times \{1, -1\}$ with the multiplication $(a, s)(b, t) = (a + sb, st)$.
(a) Prove that this formula indeed defines a group.
(b) Prove that D_{8n} is not isomorphic to $D_{4n} \times \mathbf{Z}/2\mathbf{Z}$.
(c) For an odd $n \geq 3$, prove that D_{2n} is isomorphic to $D_n \times \mathbf{Z}/2\mathbf{Z}$.

7. Recall that $\mathbf{Q}$ is the additive group of rational numbers and $\mathbf{Q}^\times$ is the multiplicative group of (nonzero) rational numbers.
(a) Prove that there are infinitely many homomorphisms of groups $\mathbf{Q}^\times \to \mathbf{Q}$.
(b) Prove that there is exactly one homomorphism $\mathbf{Q} \to \mathbf{Q}^\times$.

8. Suppose $0 \leq i \leq n$. Show that the number of subgroups of $(\mathbf{Z}/2\mathbf{Z})^n$ of order 2^i equals the number of subgroups of $(\mathbf{Z}/2\mathbf{Z})^n$ of order 2^{n-i}.

9. Prove that any group with three or more elements has at least two different automorphisms.

10. Suppose $H < G$. Show that there is a unique $K < G$ such that $H \lhd K$ and if $K' < G$ satisfies $H \lhd K'$, then $K' < K$.