Mathematics 5317 (Introduction to Modern Algebra)

Fall 2020

Homework 3

First submission due September 29, 2020.

1. In this problem, G denotes an arbitrary group.

(a) Show that the set of all automorphisms (i.e., invertible homomorphisms) $\varphi\colon G \to G$ of a group G itself forms a group $\mathsf{Aut}(G)$, with composition as multiplication.

(b) An automorphism (i.e., invertible homomorphism) $\varphi\colon G \to G$ of a group G is called *inner* if there is $g \in G$ such that for all $h \in G$ we have $\varphi(h) = ghg^{-1}$. Prove that for $G = \mathsf{GL}(2, \mathbf{R})$ the automorphism $\varphi\colon G \to G$ that sends $A \mapsto (A^{-1})^t$ (the transpose of the inverse matrix) is not an inner automorphism.

2. In this problem, G denotes an arbitrary group.

(a) Consider the center

$$\mathsf{Z}(G) = \{z \in G \mid \forall g \in G\colon gz = zg\}.$$

Show that $\mathsf{Z}(G)$ is a normal subgroup of G.

(b) Prove that the quotient of the inclusion $\mathsf{Z}(G) \to G$ is isomorphic to the subgroup $\mathsf{Inn}(G)$ of $\mathsf{Aut}(G)$ comprising all inner automorphisms (see Problem 1).

3. In this problem, G denotes an arbitrary group.

(a) Prove that inner automorphisms (Problem 1) form a normal subgroup of the group $\mathsf{Aut}(G)$ of automorphisms of G.

(b) Construct a nontrivial homomorphism

$$\mathsf{Aut}(G)/\mathsf{Inn}(G) \to \mathsf{Aut}(\mathsf{Z}(G)).$$

4. Given a group G, consider

$$N = \{\sigma \in \mathsf{Aut}(G) \mid \forall g \in G\colon \sigma(g)g^{-1} \in \mathsf{Z}(G)\}.$$

(See Problem 2 for a definition of $\mathsf{Z}(G)$.) Prove that N is a normal subgroup of $\mathsf{Aut}(G)$.

5. Prove that if A and B are normal subgroups of a group G such that G/A and G/B are both abelian, then the group $G/(A \cap B)$ exists and is abelian.

6. Suppose H and K are subgroups of a group G. Recall the notation:

$$HK = \{hk \mid h \in H, k \in K\} = \{g \in G \mid \exists h \in H, k \in K\colon g = hk\}.$$

(a) Show that $HK = KH$ if and only if HK is a subgroup of G.

(b) Show that the cardinality of HK equals $|H| \cdot |K|/|H \cap K|$.

7. Suppose G is a group such that $G/\mathsf{Z}(G)$ is cyclic. Show that G is abelian.

8. Suppose G is a group such that the map $G \to G$ that sends $g \mapsto g^2$ for any $g \in G$ is a homomorphism of groups. Show that G is abelian.

9. In this problem, G is a finite abelian group. A *group character* of G is a homomorphism $\chi\colon G \to \mathbf{C}^\times$. Denote by $\hat{G}$ the set of group characters of G. Show that $\hat{G}$ is a group with the operation of pointwise multiplication, i.e., $\chi_1\chi_2 = (g \mapsto \chi_1(g)\chi_2(g))$. Show that if G is cyclic, then G is isomorphic to $\hat{G}$.

10. In this problem, you may use the results of Problem 7 from Homework 1.

(a) Show that if $H < G$, then for any $g \in G$ we have $gHg^{-1} < G$.

(b) Classify all subgroups of Σ_3, the symmetric group of degree 3.

(c) Classify all normal subgroups of Σ_4, the symmetric group of degree 4.