# Mathematics 5317 (Introduction to Modern Algebra)

*Fall 2020*

**Homework 2**

*First submission due September 22, 2020.*

Consider the following groups:

- $2^X$: subsets of a given set $X$, with the symmetric difference $A \oplus B := (A \setminus B) \cup (B \setminus A)$ as the multiplication operation.
- $\Sigma_X$: bijections of sets $X \to X$ for a given set $X$, with the composition $f \circ g$ as the multiplication operation.
- $\mathbf{Q}^\times$: the group of invertible rational numbers with multiplication, likewise for $\mathbf{R}^\times$ and $\mathbf{C}^\times$.

In the following problems, "construct all possible homomorphisms" means you have to give an explicit construction of all homomorphisms and prove that there are no others.

**1.**

(a) For a finite set $X$, construct all possible homomorphisms $2^X \to \{\pm 1\}$ (the right side is the group with two elements).

(b) For an arbitrary set $X$, construct all possible homomorphisms $\Sigma_X \to 2^X$. (Recall that we classified all possible homomorphisms $\Sigma_X \to \{\pm 1\}$ in class.)

(c) Extra bonus point: for a finite set $X$, construct all possible homomorphisms $2^X \to \Sigma_X$.

**2.** Construct all possible homomorphisms $\mathbf{Q}^\times \to \mathbf{Z}$. (The left side refers to invertible rational numbers with multiplication, whereas the right side refers to integer numbers with addition.) You may use the following "Fundamental Theorem of Arithmetic": any positive rational number $q$ admits a unique representation of the form

$$q = \prod_{p \in \mathbf{P}} p^{n_p},$$

where $\mathbf{P} = \{2, 3, 5, 7, \ldots\}$ is the set of all prime numbers, $n_p$ are integer numbers, and the set $\{p \in \mathbf{P} \mid n_p \neq 0\}$ is finite (explain why the infinite product is well-defined in this case).

**3.** In this problem, $f\colon A \to B$ is a homomorphism of groups.

(a) Show that injective homomorphisms of groups $f\colon A \to B$ can be equivalently characterized by the following property: for any homomorphisms of groups $g, h\colon C \to A$, the equality $fg = fh$ implies $g = h$.

(b) Show that surjective homomorphisms of groups $f\colon A \to B$ can be equivalently characterized by the following property: for any homomorphisms of groups $g, h\colon B \to C$, the equality $gf = hf$ implies $g = h$. Assume all groups to be abelian if it helps. Extra bonus point for establishing the nonabelian case.

**4.** In this problem, $f\colon A \to B$ is a homomorphism of groups. Prove or disprove:

(a) Injective homomorphisms of groups $f\colon A \to B$ can be equivalently characterized by the following property: there is a homomorphism of groups $g\colon B \to A$ such that $gf = \mathrm{id}_A$.

(b) Surjective homomorphisms of groups $f\colon A \to B$ can be equivalently characterized by the following property: there is a homomorphism of groups $g\colon B \to A$ such that $fg = \mathrm{id}_B$.

**5*.** Given a group $A$, show that the following data are equivalent by defining mutually inverse constructions (1) $\to$ (2), (2) $\to$ (1), (1) $\to$ (3), (3) $\to$ (1), (2) $\to$ (3), (3) $\to$ (2) (and proving your claims). In class, we established (and proved) some of the directions, you only have to supply (and prove) the remaining ones.

(1) A partition of $A$ into disjoint nonempty subsets such that for any $a \in A$ the maps $A \to A$ given by $g \mapsto ag$ and $g \mapsto ga$ send any subset from this partition to some other (possibly the same) subset from this partition.

(2) An equivalence relation $R$ on $A$ that is compatible with the group structure, meaning the subset $R \subset A \times A$ is a subgroup.

(3) A surjective homomorphism of groups $q\colon A \to Q$ such that for every $b \in Q$ we have $b = q^*\{b\} := \{a \in A \mid q(a) = b\}$.

**6*.** Suppose $\{G_i\}_{i\in I}$ is a family of groups and $(R, \{p_i\colon R \to G_i\}_{i\in I})$ is its product. Suppose $(R', \{p_i'\colon R' \to G_i\}_{i\in I})$ is another product of the same family. Show that there is exactly one homomorphism $g\colon R \to R'$ such that $p_i' \circ g = p_i$ for all $i \in I$. Is $g$ an isomorphism?

**7*.** Suppose $G$ is a group and $A$ is an abelian group. Consider the set of homomorphisms $G \to A$. Equip this set with a group structure. The resulting group is denoted by $\mathsf{Hom}(G, A)$. Is this group abelian?

**8*.** In this problem, $\mathsf{Hom}(G, A)$ denotes the group constructed in Problem 7. Construct *injective* group homomorphisms as indicated. Recall that $\mathbf{U}(1) = \{z \in \mathbf{C} \mid |z| = 1\}$ is the circle group.
(a) $\mathbf{R} \to \mathsf{Hom}(\mathbf{R}, \mathbf{U}(1))$;
(b) $\mathbf{Z} \to \mathsf{Hom}(\mathbf{U}(1), \mathbf{U}(1))$;
(c) $\mathbf{R} \to \mathsf{Hom}(\mathbf{R}, \mathbf{R})$;
(d) $\mathbf{R} \to \mathsf{Hom}(\mathbf{R}, \mathbf{R}^\times)$;
(e) $\mathbf{R} \to \mathsf{Hom}(\mathbf{R}^\times, \mathbf{R})$;
(f) $\mathbf{R} \to \mathsf{Hom}(\mathbf{R}^\times, \mathbf{R}^\times)$.

**9.** Prove or disprove: there is $n \geq 3$ such that the symmetric group $\Sigma_n$ is isomorphic to the product $\{\pm 1\} \times A_n$, where $A_n$ denotes the subgroup of $\Sigma_n$ comprising permutations of sign $+1$.

**10.** Prove or disprove: there is $n \geq 3$ such that the general linear group $\mathsf{GL}_n(\mathbf{R})$ is isomorphic to the product $\mathbf{R}^\times \times \mathsf{SL}_n(\mathbf{R})$, where $\mathsf{SL}_n(\mathbf{R})$ denotes the subgroup of $\mathsf{GL}_n(\mathbf{R})$ comprising matrices of determinant 1.