# Mathematics 5317 (Introduction to Modern Algebra)

*Fall 2020*

**Homework 10**

*First submission due December 2, 2020.*

**1.** Suppose $\mathsf{C}$ is a category that admits equalizers (i.e., limits for the diagram $I = \{0 \rightrightarrows 1\}$) and small products.

(a) Prove that $\mathsf{C}$ admits all small limits, and the limit of a small diagram $D\colon I \to \mathsf{C}$ can be computed using an equalizer of small products,

$$\mathsf{lim}_{i\in I} D(i) \to \mathsf{eq}\left(\prod_{k\in I} D(k) \rightrightarrows \prod_{f:i\to j} D(i)\right),$$

where the map to the equalizer is induced by the map

$$(p_k)_{k\in I}\colon \mathsf{lim}_{i\in I} D(i) \to \prod_{k\in I} D(k)$$

and the two maps in the equalizer are

$$(D(f)\circ p_i)_{f:i\to j}, (p_j)_{f:i\to j}.$$

(Hint: it may be helpful to expand the above formulas using the known formulas for limits and equalizers in the category $\mathsf{Set}$, to see what is going on concretely.)

(b) Formulate an analogous statement for colimits and prove it by applying part (a) to the category $\mathsf{C}^{\mathsf{op}}$.

**2.** Suppose $I$ and $J$ are categories and $D\colon I \times J \to \mathsf{V}$ is a diagram.

(a) Assuming all limits and colimits below exist, construct a canonical map

$$\mathsf{colim}_{i\in I}\mathsf{lim}_{j\in J} D(i,j) \to \mathsf{lim}_{j\in J}\mathsf{colim}_{i\in I} D(i,j).$$

(b) Give an example of $I$, $J$, $\mathsf{V}$, and $D$ such that the map in part (a) is not an isomorphism.

**3.** Consider a category $I$ with a single object $0$ and a single nonidentity morphism $e\colon 0 \to 0$, where $e\circ e = e$. Suppose $D\colon I \to \mathsf{V}$ is a diagram. Show that the limit of $D$ exists if and only if the colimit of $D$ exists. Show that the limit of $D$ is isomorphic to the colimit of $D$ (more specifically, the corresponding apices are isomorphic). Describe, in concrete terms, the (co)limit for the case $\mathsf{V} = \mathsf{Set}$.

**4.** A morphism $e\colon A \to A$ in a category $\mathsf{C}$ is *idempotent* if $e\circ e = e$. A *splitting* of an idempotent is a pair $(s\colon A \to B, r\colon B \to A)$ such that $rs = \mathrm{id}_A$ and $sr = e$.

(a) Express the data of an idempotent morphism as a diagram $D\colon I \to \mathsf{C}$ (i.e., define $I$ and explain how to construct $D$ from $e$).

(b) Show that $A$ is the colimit of $D$, with $r\colon B \to A$ being the injection map.

(c) Show that $A$ is the limit of $D$, with $s\colon A \to B$ being the projection map.

**5.** A category $I$ is *filtered* if the following conditions are satisfied:

- $I$ is nonempty;
- for any object $i, j \in I$ there is $k \in I$ such that there exist morphisms of the form $i \to k$ and $j \to k$;
- for any pair of parallel arrows $f, g\colon i \to j$ there is an arrow $h\colon j \to k$ such that $hf = hg$.

(a) Show that any directed poset gives rise to a filtered category.

(b) Give an example of a filtered category that does not arise from a construction described in part (a). (Hint: look at other problems.)

(c) Show that the canonical map in Problem (2a) is an isomorphism if $\mathsf{V} = \mathsf{Set}$, $I$ is filtered, and $J$ is finite (i.e., has finitely many morphisms). Hint: the concrete descriptions of limits and colimits in $\mathsf{Set}$ may be useful.

**6.** A *graph* is a quadruple $(V, E, s, t)$, where $V$ is a set of *vertices*, $E$ is a set of *edges*, $s, t\colon E \to V$ are the *source* and *target* maps. Any small category $C$ has an underlying graph $\mathsf{U}(C) = (O, M, s, t)$, with $O$ the set of objects, $M$ the set of morphisms, and $s$ and $t$ being the source and target maps.

(a) Show that any graph $G = (V, E, s, t)$ admits a *free category* $\mathsf{F}(G)$ with the following universal property: for any small category $C$, the set of functors $\mathsf{F}(G) \to C$ is canonically isomorphic to the set of morphisms of graphs $G \to \mathsf{U}(C)$.

(b) Given an explicit description (i.e., identify all morphisms) of the free category $\Psi$ on the following graph (draw a picture): $V = \{0, 1\}$, $E = \{S\colon 1 \to 0, T\colon 1 \to 0, I\colon 0 \to 1\}$, where the notation $A\colon b \to c$ means that $s(A) = b$, $t(A) = c$.

(c) Formulate and prove a generalization of part (a): define the notion of a category freely generated by a graph with relations (e.g., using a universal property), and prove its existence. (For an example of relations, see part (d).)

(d) Given an explicit description (i.e., identify all morphisms) of the free category $\Psi$ on the graph from part (a), subject to the relations $SI = \mathrm{id}_0$, $TI = \mathrm{id}_0$.

**7.** Consider the category $\Psi$ from Problem 5. The colimit of a diagram $D\colon \Psi \to C$ is known as a *reflexive coequalizer*. (Hint: the universal property from Problem 5 may be useful in identifying the concrete data associated with the diagram $D$.)

(a) Suppose $S$ is a set and $R \subset X \times X$ is a relation on $S$. Construct a reflexive coequalizer diagram $\Psi \to \mathsf{Set}$ that sends $0 \mapsto S$, $1 \mapsto R$, $S \mapsto p_0$, $T \mapsto p_1$, $I \mapsto (s \mapsto (s, s))$. Prove that $X/R$ is the colimit of this diagram.

(b) Suppose $f\colon G \to H$ is a surjective homomorphism of groups. Take $K = G \times_H G = \{(g, g') \mid f(g) = f(g')\}$. Here $G \times_H G$ is the limit of the diagram of groups $G \to H \leftarrow G$ (both morphisms are $f$) and as such, automatically has a group structure. Show that the assignment $0 \mapsto G$, $1 \mapsto K$, $S \mapsto p_0$, $T \mapsto p_1$ ($p_i$ are projection maps) and $I \mapsto (g \mapsto (g, g))$ define a reflexive coequalizer diagram. Prove that $H$ is the colimit of this diagram.

(c) Formulate and prove an analogue of (b) for ideals of commutative rings.

**8.** A category $I$ is *sifted* if the following conditions are satisfied:

- $I$ is nonempty;
- for any objects $i, j \in I$ the category of *cospans* from $i$ to $j$ in $I$ is connected. Here a cospan from $i$ to $j$ is a diagram $i \to k \leftarrow j$ in $I$; a morphism of cospans from $i \to k \leftarrow j$ to $i \to k' \leftarrow j$ is a morphism $k \to k'$ that makes the two triangles with vertices $i, k, k'$ respectively $j, k, k'$ commute; finally, a category is *connected* if any two objects can be connected by a chain of morphisms going in either direction (e.g., a zig-zag).

(a) Show that any filtered category is sifted.

(b) Prove that the walking coequalizer, i.e., $0 \rightrightarrows 1$ is not a sifted category.

(c) Prove that the walking reflexive coeqalizer $\Psi$ from Problem 5 is a sifted category. (A good "formula" to keep in mind: sifted colimits = filtered colimits + reflexive coequalizers.)

(d) Show that the canonical map in Problem (2a) is an isomorphism if $\mathsf{V} = \mathsf{Set}$, $I$ is filtered, and $J$ is finite discrete (i.e., has finitely objects and only identity morphisms, so $J$-limits are finite products). Hint: the concrete descriptions of limits and colimits in $\mathsf{Set}$ may be useful.

**9.** Consider the category $\Delta$, whose objects are finite nonempty totally ordered sets and morphisms are nondecreasing maps of sets ($x \leq y$ implies $f(x) \leq f(y)$). Prove that $\Delta^{\mathsf{op}}$ is sifted (Problem 8).

**10.** Suppose $\mathsf{C}$ is a variety of algebras (take the category of groups if you want). Show that the forgetful functor $\mathsf{C} \to \mathsf{Set}$ (assume there is a single underlying set for simplicity) preserves sifted colimits (you can take just reflexive coequalizers for simplicity).