Mathematics 5317 (Introduction to Modern Algebra)

Fall 2020

Homework 1

First submission due September 15, 2020.

An asterisk marks particularly important problems.

All problems must be accompanied by complete proofs, even if the statement does not mention proofs explicitly. If you claim it, you must prove it.

1. Fix some sets a and b.
(a) Define $(a,b)=\{\{a\},\{a,b\}\}$. Prove that $(a,b)=(a',b')$ if and only if $a=a'$ and $b=b'$.
(b) Same question for $(a,b)=\{\{\emptyset,\{a\}\},\{\{b\}\}\}$.
(c) Prove or disprove: (b) remains valid if we define $(a,b)=\{\{\emptyset,a\},\{b\}\}$.

2. In this problem, $f\colon A\to B$ is a map of sets.
(a) Show that injective (i.e., one-to-one) maps of sets $f\colon A\to B$ can be equivalently characterized by the following property: for any $g,h\colon C\to A$, the equality $fg=fh$ implies $g=h$.
(b) Show that surjective (i.e., onto) maps of sets $f\colon A\to B$ can be equivalently characterized by the following property: for any $g,h\colon B\to C$, the equality $gf=hf$ implies $g=h$.

3. In this problem, $f\colon A\to B$ is a map of sets.
(a) Show that bijective (i.e., one-to-one and onto) maps of sets $f\colon A\to B$ can be equivalently characterized by the following property: there is $g\colon B\to A$ such that $gf=\mathrm{id}_A$ and $h\colon B\to A$ such that $fh=\mathrm{id}_B$. Prove that $g=h$.
(b) Prove or disprove: injective (i.e., one-to-one) maps of sets $f\colon A\to B$ can be equivalently characterized by the following property: there is $g\colon B\to A$ such that $gf=\mathrm{id}_A$.

4*. Given a set A, show that the following data are equivalent by defining mutually inverse constructions (1) → (2), (2) → (1), (1) → (3), (3) → (1), (2) → (3), (3) → (2) (and proving your claims).
(1) A partition of A into disjoint nonempty subsets, i.e., a set B such that for any $b\in B$ we have $b\neq\emptyset$, for any $a,b\in B$ we have $a\cap b=\emptyset$, and $\bigcup_{b\in B}b=A$.
(2) An equivalence relation R on A, i.e., a subset $R\subset A\times A$ (notation: xRy means $(x,y)\in R$) such that xRx, xRy implies yRx, xRy and yRz imply xRz.
(3) A surjective map $q\colon A\to Q$ such that for every $b\in Q$ we have $b=q^*\{b\}:=\{a\in A\mid q(a)=b\}$.

5*. Given sets A and B, define $A\sqcup B:=\{0\}\times A\cup\{1\}\times B$.
(a) Construct canonical maps $\iota_0\colon A\to A\sqcup B$ and $\iota_1\colon B\to A\sqcup B$.
(b) Given a set C and maps of sets $f\colon A\to C$ and $g\colon B\to C$, construct a canonical map $[f,g]\colon A\sqcup B\to C$ such that $[f,g]\circ\iota_0=f$, $[f,g]\circ\iota_1=g$.
(c) Prove that for any $h\colon A\sqcup B\to C$ we have $h=[h\circ\iota_0,h\circ\iota_1]$.

6. For any of the following sets of data, prove or disprove that they form a group. Is the group abelian? (Sometimes the answer depends on X, be sure to analyze all X.)
(a) Subsets of a given set X, with the union as the multiplication operation.
(b) Subsets of a given set X, with the symmetric difference $A\oplus B:=(A\setminus B)\cup(B\setminus A)$ as the multiplication operation.
(c) Maps of sets $X\to X$ for a given set X, with the composition $f\circ g$ as the multiplication operation.
(d) Same as (c), but maps are required to be bijections.
(e) $\mathbf{Z}$ (integers) with the group operation $(x,y)\mapsto x+y+xy$.
(f) $(-\pi/2,\pi/2)$ with the group operation $(x,y)\mapsto\arctan(\tan(x)+\tan(y))$.

7. Fix an arbitrary set S. Recall the cycle notation for permutations: $(a_1\,a_2\,\ldots\,a_n)$ denotes the permutation that maps $a_1\mapsto a_2$, $a_2\mapsto a_3$, …, $a_{n-1}\mapsto a_n$, $a_n\mapsto a_1$. Given a permutation

$$\sigma=(a_{1,1}\,a_{1,2}\,\ldots\,a_{1,k_1})(a_{2,1}\,a_{2,2}\,\ldots\,a_{2,k_2})\ldots(a_{m,1}\,a_{m,2}\,\ldots\,a_{m,k_m}),$$

write down a cycle presentation for the permutation $\tau\sigma\tau^{-1}$, where τ is an aribtrary permutation of S. (The answer should be presented in terms of $a_{i,j}$ and values of τ.)

8. Construct the required structures and prove that they define groups.

(a) Define a group structure on the set $\mathbf{N} = \{0, 1, 2, 3, 4, \ldots\}$ of natural numbers.

(b) Define a nonabelian group structure on the set $\mathbf{Z}$ of integer numbers.

9*. Given sets A, B, denote by $B^A = \{f : A \to B\}$ the set of all maps of sets $A \to B$.

(a) Construct a map of sets $e : A \times B^A \to B$. (Your construction must allow for positive solutions to (b) and (c).)

(b) Given a map $f : A \times X \to B$, explain how to construct from it a map $\langle f \rangle : X \to B^A$. (Here $\langle f \rangle$ is an ad hoc notation created for the specific purposes of this problem. Your construction must allow for a positive solution to (c).)

(c) Given a map $g : X \to B^A$, prove that $g = \langle e \circ (A \times g) \rangle$.

10. Suppose G is a group (not necessarily abelian) such that $g^3 = g$ for all $g \in G$. Prove that G is abelian.