8-25-20 MA117C dimitri.pavlov@ttu.edu 2p-3:30 F

Extra Sessions 5+ε p - 6:30p on Wend, Fri

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MA246

Each set of notes completed on your assigned time is worth 10pts

1 homework problem is = 1 pt.

A+	≥ 100	Would only need avg of 5 problems
A	100-80	a week + notes for 105 pts
B	80-60	
F	<60	

Know the definitions and theorems

Make sure you understand the abstract before you solve the problem

Algebra al-jabr: bonesetting

Remarks about sets

	primative notions	
material set theory	sets, $\in$	ZFC
structural set theory	sets, maps of sets, $\times$	Lawvere' ETCS

1) A map of sets $f: A \to B$ has a well defined domain A and a codomain B

The composition $g \circ f$ is only defined if the codomain of f is the domain of g

$$\begin{tikzcd} C & B \arrow[l, "g"'] & A \arrow[l, "f"'] \arrow[ll, bend left, "g \circ f"] \end{tikzcd}$$

This reversed writing comes from Euler in 1700s w/ $f(x)$ and $g(f(x)) := g \circ f(x)$

Text on Set theory: Lawvere, Rosebrugh Sets for Mathematics

Products of Sets

Given A, B sets, we have $A \times B$ which is also set

$A \times B = \{ (a,b) \mid a \in A, b \in B \}$

in ZFC $(a,b) = \{ \{a\}, \{a,b\} \}$

HW 1: prove $(a,b) = (a',b')$ iff $a = a' \wedge b = b'$

Russel's Paradox $\{ x \mid x \notin x \} = A$

if $A \in A \Rightarrow A \notin A$
if $A \notin A \Rightarrow A \in A$ $\Rightarrow$ such a set does not exist

$\Rightarrow$ not all $\{ \}$ are actually sets.

In structural set theory

Given sets A, B, we have $A \times B$ a set

$p_1 : A \times B \mapsto A \qquad p_1(a,b) = a$
$p_2 : A \times B \to B \qquad p_2(a,b) = b$

Given a set C and maps $C \xrightarrow{f} A$, $C \xrightarrow{g} B$
we have,

$(f,g) : C \to A \times B \qquad (f,g)(c) = (f(c), g(c))$

$\underline{\text{Axiom}}$ 1) $p_1(f,g) = f$; $p_2(f,g) = g$

2) $h: C \to A \times B$ $\quad h = (p_1 \circ h, p_2 \circ h)$

$\Bigg\}$ Universal properties of products

$h(c) = (p_1(h(c)), p_2(h(c)))$

$(h_1(c), h_2(c)) = (h_1(c), h_2(c))$

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, "{(f,g)}"] \arrow[dr, phantom, shift left=3ex, "\#"] \arrow[dr, phantom, shift right=3ex, "\#"'] & B \\ A & A \times B \arrow[l, "p_2"] \arrow[u, "p_1"'] \end{tikzcd}$$

\# means the maps are communative

This is called a communative diagram

The empty set $\emptyset$

Given any set A $\quad \emptyset \xrightarrow[\exists!]{} A$

8-27-20

Good texts as Resources

Alufi: Ch 1

Leinster Rethinking Set theory

Group is a set w/

- multiplication: $G \times G \to G \quad (x, y) \mapsto xy$
- inverse: $G \to G \quad x \mapsto x^{-1}$
- identity element $1 \in G \quad \underline{1\!\!1} \to G$

Group Axioms

- Associativity: $\forall x, y, z \in G \quad (xy)z = x(yz)$
- Unitality: $\forall x \in G \quad 1x = x1 = x$
- Inverses: $xx^{-1} = x^{-1}x = 1$

Primodial Example: Permutations of a Set

Fix a set S. Let G be set of permutations of S, ie, bijections $S \to S$

Here multiplication is compositition of maps

$xy := x \circ y$ ← Well defined since x, y are bijections

$1 := 1_S$

$x^{-1} := x^{-1}$

$\forall s \in S$

$$((xy)z)(s) = ((x \circ y) \circ z)(s) = (x \circ y)(z(s)) = x(y(z(s)))$$

$$(x(yz))(s) = (x \circ (y \circ z))(s) = x((y \circ z)(s)) = x(y(z(s)))$$

$\Rightarrow$ Associative

Unitality is easy to verify

Inverses come by definition as permutations are bijections

These are known as the symmetric group S_n

Examples

1) $S = \emptyset$ $\quad \emptyset \xrightarrow[id_\emptyset]{\exists!} \emptyset$ $\quad G = \{ id_\emptyset \}$

This is known as the trivial group

2) $S = \mathbb{1} = \{ * \}$ $\quad \mathbb{1} \xrightarrow[id_\mathbb{1}]{\exists!} \mathbb{1}$ $\quad G = \{ id_\mathbb{1} \}$

Again trivial group

3) $S = \{1, 2\}$ $\quad S \to S$ $\quad G = \{ id_S, (1,2) \}$

This is cyclic group of order 2

A group G is abelian if $\forall x, y \in G$ $\; xy = yx$

Here use addition instead

4) $S = \{1, 2, 3\}$ $\quad S \to S$

$G = \{ (), (1\ 2), (1\ 3), (2, 3), (1\ 2\ 3), (1\ 3\ 2) \}$

This group is non-abelian

$(1\ 2\ 3)(1\ 2) = (1\ 3)$

$(1\ 2)(1\ 2\ 3) = (2\ 3)$

Examples General linear Group $GL_{\mathbb{F}}(n)$ $n \geq 0$

$GL(n) = \{$ invertible matrices of size $n \times n \}$

Here multiplication is just matrix multiplication

$GL(0) = \mathbb{1}$ $GL(1) = \mathbb{R}^\times$ These are the only abelian general linear groups

Examples of Abelian Groups

1) $(\mathbb{Z}, +)$

2) $(\mathbb{Q}, +)$

3) $(\mathbb{Q}^\times, \cdot)$

4) $U(1) = \{ z \in \mathbb{C} \mid |z| = 1 \}$

$(U(1), \cdot)$ This is the unit circle

<u>Example</u> Fix a set I

Consider $\mathbb{Z}^I = \{ \text{maps } I \to \mathbb{Z} \}$

all group operations are pointwise

$x, y \in \mathbb{Z}^I \qquad x, y : I \to \mathbb{Z}$

$x + y \in \mathbb{Z}^I \qquad i \in I \qquad (x+y)(i) := x(i) + y(i)$

$-x \in \mathbb{Z}^I \qquad (-x)(i) := -x(i)$

$0 \in \mathbb{Z}^I \qquad 0(i) := 0$

This is also an abelian group

Remark For any group G and set I

we have $G^I = \{ \text{maps } I \to G \}$

G^I is abelian $\iff$ G is abelian or $I = \emptyset$

Example

G^I_{finite} is a subset of G^I

$x : I \to G$ $\quad x \in G^I_{finite}$

$\iff \left|\{ i \in I \mid x(i) \neq 1_G \}\right| < \infty$

if G is abelian $\implies$ we write

$\bigoplus_{i \in I} G = G^I_{finite}$

9-1-20

Homomorphisms of groups

G, H are groups a homomorphism $f: G \to H$ is a map of sets $f: G \to H$ s.t.

$$f(xy) = f(x)f(y)$$

$$f(x^{-1}) = f(x)^{-1}$$

$$f(1_G) = 1_H$$

Properties

1) Let g, f be homomorphisms s.t. $K \xleftarrow{g} H \xleftarrow{f} G$ $\quad g \circ f$ is defined

2) Composition of homomorphisms is associative

3) $id_H \circ f = f \circ id_G = f$

Examples

a) $\mathbb{1} \xrightarrow{\exists !} G$

b) $G \xrightarrow{\exists !} \mathbb{1}$

The 1 elements are mapped to each other and nothing else

c)
$$\begin{tikzcd} G \arrow[r, "1"] \arrow[dr] \arrow[dr, phantom, shift left=3, "\#"] & H \\ & \mathbb{1} \arrow[u] \end{tikzcd}$$

1 is the trivial homomorphism

$$1(g) = 1_H$$

d) $\Sigma_n \longrightarrow \{\pm 1\} \quad n \geq 2$

This is the sign of the permutation
it is unique and nontrivial

Sign homomorphism

- sends all $\underbrace{\text{transpositions}}_{\text{moves exactly two elements}}$ to -1

- Any permutation is a composition of σ transpositions

Then the sign is $\begin{cases} +1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd} \end{cases}$

Observation: A cycle of length k $(a_1\ a_2\ a_3\ \cdots\ a_k)$ has sign $(-1)^{k-1}$

e) $GL_n(\mathbb{R}) \xrightarrow{\det} \mathbb{R}^\times$

$GL_n(\mathbb{C}) \xrightarrow{\det} \mathbb{C}^\times$

Remember $\det(AB) = \det A \det B$

$\det(A^{-1}) = \det(A)^{-1}$

$\det(1_n) = 1$

f) $\Sigma_n \longrightarrow GL_n(\mathbb{R})$

sends $\tau \mapsto \left((i,j) \mapsto \begin{cases} 1, & i = \tau(j) \\ 0, & \text{otherwise} \end{cases} \right)$

$\tau' \tau \mapsto \left\{ \begin{matrix} 1 \text{ in entries} \\ (\tau'(k), k) \end{matrix} \right\} \cdot \left\{ \begin{matrix} 1 \text{ in entries} \\ (\tau(j), j) \end{matrix} \right\}$

$(A'A)_{p,q} = \sum_i A'_{p,i} A_{i,q}$

g)

$$\begin{tikzcd}
& GL_n(\mathbb{R}) \arrow[dr, "\det"] & \\
\Sigma_n \arrow[ur, "P"] \arrow[dr, "\text{sign}"'] \arrow[rr, phantom, "\#"] & & \mathbb{R}^\times \\
& \{\pm 1\} \arrow[ur, "\text{inclusion}"'] &
\end{tikzcd}$$

An inclusion of groups is an injective homomorphism of groups

Remark: In material set theory sometimes require the under lying map of sets is an inclusion of sets

Ex) $\mathbb{Z} \rightarrowtail \mathbb{Q} : (\mathbb{Z} \times \mathbb{Z}^\times)/\sim$

where $\sim : (a,b) \sim (c,d) \Leftrightarrow ad = bc$

This is an injective homomorphism
but techinquely $\mathbb{Z} \not\subseteq \mathbb{Q}$
but its good enough for us

A quotient homomorphism is a surjective homomorphism of groups $G \twoheadrightarrow H$

H is a quotient group of G

<u>Remark</u>: sometimes the map of sets $G \xrightarrow{f} H$ is required to satisfy $\forall h \in H$

$$h = f^*\{h\} = \{g \in G \mid f(g) = h\}$$

<u>Examples</u>

1) $\mathbb{R}^\times \underset{\text{sign}}{\twoheadrightarrow} \{\pm 1\}$

2) $\mathbb{C}^\times \twoheadrightarrow U(1) \qquad z \mapsto \frac{z}{|z|}$

3)
$$\begin{tikzcd}
\mathbb{R}^\times \arrow[r, "\text{abs}"] & \mathbb{R}^\times_{>0} \\
\mathbb{C}^\times \arrow[ur, "\text{abs}"'] &
\end{tikzcd}$$

9-3-20

<u>Examples</u>

Homomorphisms from $\mathbb{Z} \xrightarrow{f} G$

$$f(\overbrace{1+1+\cdots+1}^{n}) = \overbrace{f(1)+f(1)+\cdots+f(1)}^{n}$$

$$f(-1-1-\cdots-1) = -f(1)-f(1)-\cdots-f(1)$$

$\Rightarrow f(n) = n f(1)$ addative notation

$= (f(1))^n$ multiplicative notation

$$\{\text{homomorphisms } \mathbb{Z} \xrightarrow{f} G\} \simeq \{\text{elements of } G\}$$

$f \mapsto f(1)$

A quick digression.

What is an infinite family of sets?

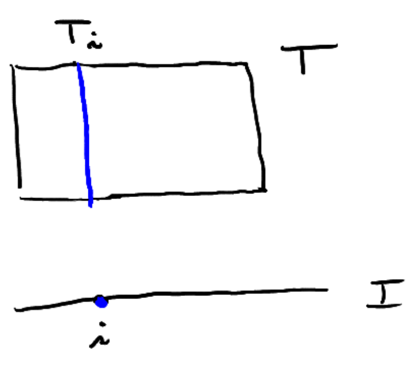
Naive Answer: I set $\forall i \in I$ A_i

Looks like a map $I \to Set$

the set that is a collection of all sets is not a set by Russel's Paradox

Correct Definition: An I-idexed family of sets is a map of sets $T \xrightarrow[p]{} I$

We denote T_i the set $p^*\{i\}$



$\{T_i\}_{i \in I}$

Recall if $f: A \to B$

a) $\forall Q \subset B$ $f^*Q := \{a \in A \mid f(a) \in Q\} = f^{-1}(Q)$ pre image

b) $\forall P \subset A$ $f_* P := \{b \in B \mid \exists a \in A : f(a) = b\} = f(P)$ image

Products of Groups

Let I be a set (possibly infinite)

$\{G_i\}_{i \in I}$ I-index family of groups

The product of $\{G_i\}_{i \in I}$

is a group $\prod_{i \in I} G_i$, together w/

projection homomorphisms

$\{p_j : \prod_{i \in I} G_i \to G_j\}_{j \in I}$

Given K a group $\{K \xrightarrow{f_i} G_i\}_{i \in I}$
homomorphisms

We have a homomorphism

$(f_i)_{i \in I} : K \to \prod_{i \in I} G_i$ s.t.

$\forall j \in I$ $p_j \circ (f_i)_{i \in I} = f_j$

$$\begin{tikzcd}
& \prod_{i \in I} G_i \arrow[d, "p_j"] \\
K \arrow[ur, "(f_i)_{i \in I}"] \arrow[r, "f_j"'] & G_j
\end{tikzcd}$$

and for any $g : K \to \prod_{i \in I} G_i$

we have $g = (p_i \circ g)_{i \in I}$

The universal property of product homomorphisms $K \to \prod_{i \in I} G_i$ are in a cannonical bijective correspondence w/ familys of homomorphisms $\{f_i : K \to G_i\}_{i \in I}$

Thm Any family of groups has a product

Proof The underlying set $U(\prod_{i\in I} G_i)$

is taken to be $\prod_{i\in I} U(G_i)$

Group operations are defined index wise

e.g. $(g_i)_{i\in I} \cdot (h_i)_{i\in I} = (g_i \cdot h_i)_{i\in I}$

where $(g_i)_{i\in I}$ $\quad g_i : \mathbb{1} \to U(G_i)$

$$\Rightarrow (g_i)_{i\in I} : \mathbb{1} \to \prod_{i\in I} U(G_i)$$

Group Axioms are verified index wise

The projection homomorphisms $p_j : \prod_{i\in I} G_i \to G_j$

are given by maps of sets

$$U(p_j) : U(\prod_{i\in I} G_i) \longrightarrow U(G_j)$$

$$\shortparallel$$

$$\prod_{i\in I} U(G_i)$$

note operations are preserved

Given K a group and $\{K \xrightarrow[f_i]{} G_i\}_{i \in I}$

The homomorphism

$$(f_i)_{i \in I} : K \to \prod_{i \in I} G_i$$

is given by the map of sets

$$\mathcal{U}((f_i)_{i \in I}) : \mathcal{U}(K) \to \mathcal{U}(\prod_{i \in I} G_i) = \prod_{i \in I} \mathcal{U}(G_i)$$

$$\shortparallel$$

$$(\mathcal{U}(f_i))_{i \in I}$$

This also preserves group operations

For any $g : K \to \prod_{i \in I} G_i$ we have

$g = (p_i \circ g)_{i \in I}$ because

$$\begin{aligned} \mathcal{U}(g) &= \mathcal{U}((p_i \circ g)_{i \in I}) \\ &= (\mathcal{U}(p_i \circ g))_{i \in I} \\ &= (\mathcal{U}(p_i) \circ \mathcal{U}(g))_{i \in I} \end{aligned}$$

Axiom of products of sets $\square$

Examples

$$\mathbb{C}^\times \xrightarrow{\sim} \mathbb{R}^\times_{>0} \times U(1)$$

$$z \longmapsto \left(|z| , \frac{z}{|z|} \right)$$

$$\mathbb{R}^\times_{>0} \times U(1) \xrightarrow{\sim} \mathbb{C}^\times$$

$$(r, u) \longmapsto ru$$

9-8-20

Def

$f_1 : A_1 \to B_1$

$f_2 : A_2 \to B_2$

$f_1 \times f_2 : A_1 \times A_2 \longrightarrow B_1 \times B_2$

$(f_1 \times f_2)(a_1, a_2) = (f_1(a_1), f_2(a_2))$

univ. prop. to define $f_1 \times f_2$ need

to define
$$\begin{tikzcd} A_1 \times A_2 \arrow[r] \arrow[dr, "p_1"'] & B_1 \\ & A_1 \arrow[u, "f_1"'] \arrow[u, phantom, "\#", shift left=3] \end{tikzcd}$$

and
$$\begin{tikzcd} A_1 \times A_2 \arrow[r] \arrow[dr, "p_2"'] & B_2 \\ & A_2 \arrow[u, "f_2"'] \arrow[u, phantom, "\#", shift left=3] \end{tikzcd}$$

Def

$f : A \to B$ $\quad C$ set

$f \times C := f \times id_C$

Kernels and Quotients

Kernel of a homomorphism of groups

$G \xrightarrow[f]{} H$ is the subgroup

$$\ker f := \{ g \in G \mid f(g) = 1 \}$$

Lemma $\ker f$ is a subgroup

Proof Closure: $x, y \in \ker f$

$$f(xy) = f(x) f(y) \quad f \text{ homo}$$
$$= 1 \cdot 1$$
$$= 1$$

$\implies xy \in \ker f$

Inverses: $x \in \ker f$

$$f(x^{-1}) = f(x)^{-1} \quad \text{homo}$$
$$= 1^{-1}$$
$$= 1^{-1}$$

Identity $f(1) = 1$

$\Rightarrow 1 \in \ker f$

$\Rightarrow \ker f$ is a subgroup $\square$

Examples

1) $\Sigma_n \xrightarrow[\text{sign}]{} \{\pm 1\}$

$\ker(\text{sign}) = A_n$ The alternating group

$|A_n| = \frac{1}{2}|\Sigma_n| = \frac{n!}{2} \quad \forall n \geq 2$

2) $GL_n(\mathbb{R}) \xrightarrow{\det} \mathbb{R}^\times$

$\ker(\det) = SL_n(\mathbb{R})$ The Special Linear Group

Eulers Formula

$\exp(x+yi) = \exp(x)(\cos y + i \sin y)$

3) $\mathbb{C} \xrightarrow[\exp]{} \mathbb{C}^{\times}$ $\quad \exp(z) = \sum_{k \geq 0} \frac{1}{k!} z^k$

$$\ker(\exp) = 2\pi i \mathbb{Z}$$
$$= \{ z \in \mathbb{C} \mid \exists\, m \in \mathbb{Z} \text{ s.t. } z = 2\pi i \cdot m \}$$

What subgroups can be kernels of some homomorphism $G \xrightarrow[f]{} H$?

Lemma $\ker f$ satisfies the following

$$\forall g \in G \ \forall k \in \ker f \quad \underbrace{g^{-1} k g}_{\text{conjugation}} \in \ker f$$

Proof

We know $f(k) = 1$

WTS $f(g^{-1}kg) = 1$

$$
\begin{aligned}
f(g^{-1}kg) &= f(g)^{-1} f(k) f(g) \\
&= f(g)^{-1} \cdot 1 \cdot f(g) \\
&= 1
\end{aligned}
$$

$\Rightarrow g^{-1}kg \in \ker f$ $\square$

Let $K \overset{\text{subgroup}}{<} G$. K is a normal subgroup

if $\forall g \in G \ \forall k \in K \ \ g^{-1}kg \in K$

We write $K \triangleleft G$

Rk kernels are always normal subgroups

Remark If G is abelian then every subgroup is normal.

Example of non-Normal groups

$K < \Sigma_n \quad n \geq 3$

$K = \{ 1, (1\ 2) \}$

$K \ntriangleleft G$ as $g = (1\ 3)$

$$g^{-1} (1\ 2)\, g = (1\ 3)(1\ 2)(1\ 3)$$
$$= (2\ 3) \notin K$$

What normal subgroups arise as kernels?

ALL OF THEM!

By the quotient group construction

$\Rightarrow$ Normal subgroups could just be defined as kernels of homomorphisms.

The universal property of quotients of sets by equivalence relations.

X set $R \subset X \times X$ equivalence relation

$$X \xrightarrow{q} X/R$$

univer prop: $\forall Y$ set

$$\{X/R \xrightarrow{\alpha} Y\} \to \{X \xrightarrow{\beta} Y \mid \begin{array}{l}\forall x_1, x_2 \in X \\ x_1 R x_2 \Rightarrow \beta(x_1) = \beta(x_2)\end{array}\}$$

$$\alpha \mapsto \alpha \circ q$$

$$\begin{tikzcd} X \arrow[r, "\beta"] \arrow[d, "q"'] \arrow[dr, phantom, "\#"] & Y \\ X/R \arrow[ur, "\alpha"'] & \end{tikzcd}$$

Typical Application

α is well defined ↓

1) Start w/ $\beta : X \to Y$

2) Prove: $\forall x_1, x_2 \in X$ if $x_1 R x_2 \Rightarrow \beta(x_1) = \beta(x_2)$

3) Conclude $\exists! \alpha : X/R \to Y$ s.t. $\alpha \circ q = \beta$

Prop. $K < G$

$R \subset U(G) \times U(G)$ s.t.

$g R g' \iff g g'^{-1} \in K$

R is an equivelence relation and

$U(G) / R$ is denoted by

$G/K \leftarrow$ a set not a group

of right cosets

A right coset of a group G w/ subgrup K is a subset of $U(G)$ of the form

$Kg := \{ h \in G \mid \exists k \in K \text{ s.t. } h = kg \}$ for some $g \in G$

R is reflexive as $xRx \iff xx^{-1} \in K$

$\iff 1 \in K$

R is symmetric as if xRy

$\implies xy^{-1} \in K$

$\implies yx^{-1} = (xy^{-1})^{-1} \in K$ as $K < G$

$\implies yRx$

R is transative as if $xRy \wedge yRz$

$\implies xy^{-1} \in K$ and $yz^{-1} \in K$

$xz^{-1} = xy^{-1}yz^{-1} \in K$ as $K < G$

$\implies xRz$

$\implies R$ is an equivelence Relation

What is the equivalence class of $g \in G$

$$[g] = \{h \in G \mid hRg\}$$

$$= \{h \in G \mid hg^{-1} \in K\}$$

$$= \{h \in G \mid \exists k \in K \text{ s.t. } hg^{-1} = k\}$$

$$= \{h \in G \mid \exists k \in K \text{ s.t. } h = kg\}$$

$$= Kg$$

$\Rightarrow$ The equivalence classes are just the right cosets $\square$

9-10-20

Correction from previous lecture

G/K = the set of left cosets $\{gK\}$ $x \sim y$ iff $x^{-1}y \in K$

$G\backslash K$ = the set of right cosets $\{Kg\}$ $x \sim y$ iff $xy^{-1} \in K$

lemma $K < G$ is normal $\Leftrightarrow$

$\forall g \in G \quad gK = Kg$

$(\Rightarrow)$ $\forall g \in G \quad gK = Kg \Rightarrow gKg^{-1} = K$

$\Rightarrow K$ is normal

$(\Leftarrow)$ if K is normal $\Rightarrow gKg^{-1} \subset K$

$\Rightarrow gKg^{-1} = K \Rightarrow gK = Kg$

$\square$

Prop if $K \trianglelefteq G$, then G/K has a group structure induced from G.

<u>Proof</u> Proof of Existence

Id: $1_{G/K} := 1_G \cdot K \in G/K$

Inverses: given $a \in G/K$ $\quad a^{-1} = \{g \in G \mid g^{-1} \in a\} \in G/K$

$$\Updownarrow$$

$$(gK)^{-1} = K^{-1}g^{-1} = Kg^{-1} = g^{-1}K \in G/K$$

multiplication:

$a, b \in G/K \quad a \cdot b := \{g \cdot h \mid g \in a, h \in b\}$

$$(gK)\cdot(hK) = gKhK = ghKK = ghK \in G/K$$

Proof of Group Axioms

Associativity

WTS $(g_1 k \cdot g_2 k) \cdot g_3 k = g_1 k \cdot (g_2 k \cdot g_3 k)$

$$\| \qquad\qquad \|$$

$$g_1 k g_2 k g_3 k$$

This follows b/c G is associative

Identity $\quad k \in G/k$

$$k \cdot (gk) = gkk = gk$$

$$(gk) \cdot k = gk$$

Inverses:

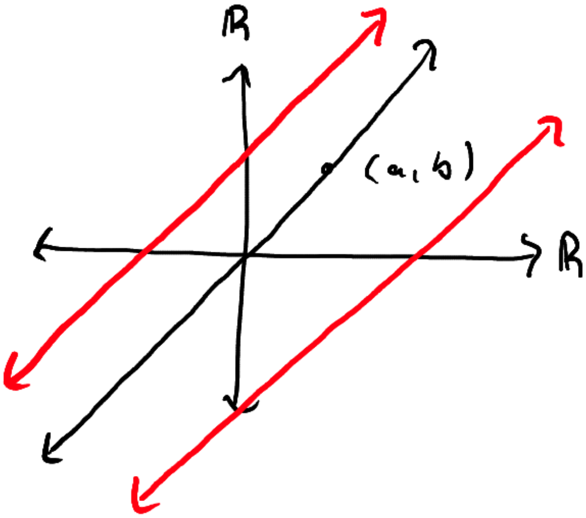
$$gkg^{-1}k = gg^{-1}kk = k = g^{-1}gkk$$
$$= g^{-1}kgk$$

$\Rightarrow G/k$ is a group $\square$

Examples

$$\mathbb{R} \longrightarrow \mathbb{R}^2 \longrightarrow Q = \mathbb{R}^2/\mathbb{R} \simeq \mathbb{R}$$

$$t \longmapsto t \cdot (a,b) \qquad (a,b) \neq (0,0)$$



These are cosets

$(u,v) \in \mathbb{R}^2$

The intersection of $(u,v) + (a,b)\cdot\mathbb{R}$ with the line $\{x=0\}$

$$= (\underbrace{u + ta}_{=0},\ \underbrace{v + tb}_{= v - \frac{u}{a}b})$$

$t = -u/a$

$$\mathbb{R}^2 \to \mathbb{R}$$

$$(u,v) \mapsto v - \frac{u}{a} b$$

if $K \trianglelefteq G$, then the quotient of G by K

is $q : G \to Q$ s.t. q is surjective

and $\ker q = K$

$$K \rightarrowtail G \twoheadrightarrow Q$$

Thm $G \to G/K$ is the quotient

Proof

The homomorphism $G \to G/K$

sends $g \mapsto gK$

it is a homomorphism as $g_1 g_2 \mapsto g_1 g_2 K = g_1 K g_2 K$

it is surjective as if $a \in G/K$

$\Rightarrow a = gK \quad \forall g \in a$

kernel is K

$$\ker = \{g \in G \mid gK = K\} = K \qquad \square$$

Thm The quotient is unique up to a unique isomorphism.

Thm The Universal Prop of Quotients of Groups

$\forall$ groups H we have $K \trianglelefteq G$ $\quad K \rightarrowtail G \twoheadrightarrow Q$

a bijection

$$\left\{ \begin{array}{c} \text{homomorphisms} \\ Q \xrightarrow[\varphi]{} H \end{array} \right\} \xrightarrow{\varphi \mapsto \varphi \circ q} \left\{ \begin{array}{c} \text{homomorphisms} \\ G \xrightarrow[\psi]{} H \\ \text{s.t. } \psi|_K = 1 \end{array} \right\}$$

$\Updownarrow$

$\psi(k) = 1_H$

Typical Applications

$$\begin{tikzcd}
 & G \arrow[dr, two heads, "q"] \arrow[dd, "\varphi"] & \\
K \arrow[ur, tail] \arrow[dr, "1"'] \arrow[r, phantom, "\#"] & & Q \arrow[dl, dashed, "\exists! \psi"] \\
 & H &
\end{tikzcd}$$

1) Start w/ some homomorphism $G \to H$

2) Verify φ is trivial on K

ie $\varphi_* K = \{1_H\}$

3) Conclude $\exists! \psi : Q \to H$ s.t.

$\psi \circ q = \varphi$

Proof of Universal Prop

- For any φ $\quad \varphi \circ q|_K = 1$
  because $q|_K = 1$

- $\varphi \mapsto \varphi \circ q$ is injective
  as q is surjective
  (Suppose $\varphi \circ q = \varphi' \circ q$ $\quad$ as q surjective
  $$\Rightarrow \varphi \circ q \circ q^{-1} = \varphi' \circ q \circ q^{-1}$$
  $$\Rightarrow \varphi = \varphi'$$
  $\Rightarrow$ injective)

- $\varphi \mapsto \varphi \circ q$ is surjective
  Let $G \xrightarrow{\psi} H$ s.t. $\psi|_K = 1_H$
  Define $\underset{\underset{Q}{\cap}}{\varphi(p)} = \psi(g)$ where $q(g) = p$

if g' is another choice s.t. $q(g') = p$

$\Rightarrow \psi(g) = \psi(g')$ as

$$\psi(gg'^{-1}) = 1$$

$$\Leftarrow q(gg'^{-1}) = q(g)q(g')^{-1} = 1$$

$$\Rightarrow gg'^{-1} \in K$$

and $\psi|_K = 1_H$

- φ is a homomorphism

Verification is easy $\square$

9-15-20

The universal property of quotients of groups $\{G/K \to H\} \longleftrightarrow \{G \xrightarrow{\varphi} H\}$

$\varphi|_K = 1_H$

Thm if $G \overset{\varphi}{\twoheadrightarrow} H$, then

$$G/\ker\varphi \simeq H$$

$$\begin{tikzcd}
 & G/\ker\varphi \arrow[dr, "\psi"] & \\
\overbrace{\ker\varphi}^{K} \arrow[r, tail] & G \arrow[u] \arrow[r, "\varphi"'] & H
\end{tikzcd}$$

- ψ is surjective as the image of ψ equals the image of φ
- ψ is injective as $\Leftrightarrow \ker\psi = \{1\}$

$\ker\psi = \{gk \mid g \in G,\ \varphi|_{gk} = \{1\} \subset H\}$

$\square$

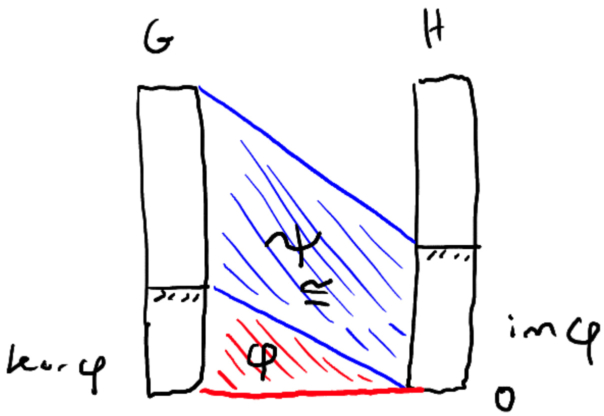
The Homomorphism Theorem

Corr $G \xrightarrow{\varphi} H$ any homomorphism

then

$$\begin{tikzcd}
 & G \arrow[r, two heads] \arrow[dr] & \operatorname{im}\varphi \arrow[r, tail] & H \\
\ker\varphi \arrow[ur, tail] & & G/\ker\varphi \arrow[u, "\cong", "\psi"'] &
\end{tikzcd}$$

and



Corr if you have a sequence of homomorphisms

$$K \rightarrowtail G \twoheadrightarrow H$$ and $K \rightarrowtail G$ is the kernel of $G \twoheadrightarrow H$, then

$$G/K \cong H$$

Examples

$$G/\mathbb{1} \simeq G \quad \text{and} \quad G/G \simeq \mathbb{1}$$

$$\mathbb{Z} \overset{(-)n}{\rightarrowtail} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}/n = \mathbb{Z}_n$$

Dmitri doesn't like this one lol

$$\mathbb{Z} \rightarrowtail \mathbb{Q} \twoheadrightarrow \mathbb{Q}/\mathbb{Z}$$

Rational Circle

$$\mathbb{Z} \overset{2\pi i}{\rightarrowtail} \{z \in \mathbb{C} \mid \operatorname{Re} z = 0\} \overset{\exp}{\twoheadrightarrow} U(1)$$

$$\{z \in \mathbb{C} \mid \operatorname{Re} z = 0\} \cong \mathbb{R}$$

$$\Rightarrow \mathbb{R}/\mathbb{Z} \simeq U(1)$$

Unit Circle

The abelianization of a group G

is $G \to A$ (A abelian)

that satisfies the following universal prop:

For any abelian group A' and $\varphi' : G \to A'$

$\exists! \ \alpha : A \to A'$ s.t. $\alpha \varphi = \varphi'$

$$\begin{tikzcd} & A \arrow[dd, dashed, "\alpha"] \\ G \arrow[ur, "\varphi"] \arrow[dr, "\varphi'"'] \arrow[r, phantom, "\#"] & \\ & A' \end{tikzcd}$$

Let G be a group $S \subset \mathcal{U}(G)$

The subgroup generated by S is the

smallest subgroup of G whose underlying set

contains S.

ie $S \subset \mathcal{U}(K)$ and if $K' < G$ s.t. $S \subset \mathcal{U}(K)$

then $K \subset K'$

K exists as $K = \cap K'$

$$K = \left\{ \prod_{i=1}^{m} s_i \;\middle|\; m \geq 0,\ s_i \in S \cup S^{-1} \right\}$$

Prop There is a unique way to define maps of sets $G^I \to G$ for all finite totally ordered I that satisfy the generalized associative property

$$\begin{tikzcd}
G^{\prod_{i=1}^{p} I_i} \arrow[rr] \arrow[d, "\cong"] \arrow[drr, phantom, "\#"] & & G \\
\prod_{i=1}^{p} G^{I_i} \arrow[r] & \prod_{i=1}^{p} G = G^p \arrow[ur] &
\end{tikzcd}$$

Let G be a group then $[G, G]$ the commutator of G

generated by the set of commutators

$\{[g_1, g_2] \mid g_1, g_2 \in G\}$

$[g_1, g_2] := g_1 g_2 g_1^{-1} g_2^{-1}$

Theorem $G \to G/_{[G,G]}$

is the abelianization of G

<u>Proof</u> if $G \xrightarrow{\varphi} A$, then $[G, G] < \ker\varphi$

because $\varphi([g_1, g_2]) = [\varphi(g_1), \varphi(g_2)] = 1$

if you have $\varphi' : G \to A'$, then

$[G, G] < \ker\varphi'$ $\Rightarrow$

$$\begin{tikzcd} & G/[G,G] \arrow[d, "\exists!"] \\ G \arrow[ur] \arrow[r, "\varphi'"'] & A' \end{tikzcd}$$

Show quotient is abelian

9-17-20

Coproducts

Coproducts of Groups = free products

Coproducts of Abelian Groups = Direct Sums

Let $\{A_i\}_{i \in I}$ be a family of abelian groups

The direct sum of $\{A_i\}$ is

an abelian group $\bigoplus_{i \in I} A_i$ together w/

injection homomorphisms

$\iota_i : A_i \to \bigoplus A_i$ that satisfy the following

universal property.

$\forall$ abelian group B

$$\left\{ \bigoplus_{i \in I} A_i \xrightarrow{\varphi} B \right\} \xrightarrow{\simeq} \prod_{i \in I} \{ A_i \to B \}$$

$$\left((-) \circ \iota_i \right)_{i \in I}$$

$$\begin{tikzcd}
 & \bigoplus_{i \in I} A_i \arrow[d, "\varphi"] \\
A_i \arrow[ur, "\iota_i"] \arrow[r, "\varphi \circ \iota_i"'] \arrow[ur, phantom, "\#"] & B
\end{tikzcd}$$

Typical Application

given $f_i : A_i \to B$

construct $\varphi : \bigoplus_{i \in I} A_i \to B$ s.t. $\varphi \iota_i = f_i$

Thm Direct sums always exist

<u>Proof</u>

Take $K < \prod_{i \in I} A_i$

$K = \{ x \in \prod A_i \mid |\operatorname{supp} x| < \infty \}$

$\operatorname{supp} x := \{ i \in I \mid x_i \neq 0 \}$

Let $\iota_i : A_i \to K \qquad a \mapsto x, \quad x_j = \begin{cases} a & j = i \\ 0 & j \neq i \end{cases}$

This is a homomorphism

Let B be any abelian group

$$\{ K \xrightarrow{\varphi} B \} \longrightarrow \prod_{i \in I} \{ A_i \xrightarrow{f_i} B \}$$

<u>Surjective</u>: Given $(f_i)_{i \in I}$ construct φ

$$\underset{\substack{\wedge \\ K}}{\varphi(x)} := \underbrace{\sum_{i \in I} f_i(x_i)}_{\text{all, but a finite amount are } 0} \in B$$

$$(\varphi \circ \iota_i)(a) = \varphi(x) = f_i(a)$$

<u>injective</u> Let $\varphi_1, \varphi_2 : k \to B$

$\forall i \in I$ Let $\varphi_1 \circ \iota_i = \varphi_2 \circ \iota_i$

finitely many non zero $\hat{x}_i$

Let $x \in k$ $\quad x = \sum_{i \in I} \hat{x}_i \quad \hat{x}_i = \left(j \mapsto \begin{cases} x_i & j = i \\ 0 & j \neq i \end{cases} \right)$

$$\varphi_1(x) = \sum_{i \in I} \varphi_1(\hat{x}_i) = \sum_{i \in I} \varphi_1(\iota_i(x))$$

|| assumption

$$\varphi_2(x) = \sum_{i \in I} \varphi_2(\hat{x}_i) = \sum_{i \in I} \varphi_2(\iota_i(x))$$

$\square$

Example

$$\mathbb{Z}/2\mathbb{Z} \oplus \bigoplus_{p \in \mathbb{P}} \mathbb{Z} \xrightarrow{\cancel{\phi}} \mathbb{Q}^\times$$

$\Rightarrow$ $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Q}^\times$ $\xleftarrow{\text{induced by}}$ $\mathbb{Z} \to \mathbb{Q}^\times$

$1 \mapsto -1$

$n \mapsto (-1)^n$

$2m \mapsto 1$

$2m+1 \mapsto -1$

$$\bigoplus_{p \in \mathbb{P}} \mathbb{Z} \to \mathbb{Q}^\times$$

$\Rightarrow$ $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Q}^\times$

$[0] \to 1$

$[1] \to -1$

$$\bigoplus_{p \in \mathbb{P}} \mathbb{Z} \to \mathbb{Q}^\times$$

$\Rightarrow$ $\forall p \in \mathbb{P}$ $\mathbb{Z} \to \mathbb{Q}^\times$

$1 \mapsto p$

$n \mapsto p^n$

The Fundemental Theorem of Arithmetic

the homomorphism $\#$ is an isomorphism

That is any $q \in \mathbb{Q}^\times$ is a unique presentation $q = (-1)^s \prod_{p \in \mathbb{P}} p^{n_p}$

$s \in \mathbb{Z}/2\mathbb{Z}$ $n_p \in \mathbb{Z}$ $\{p \in \mathbb{P} \mid n_p \neq 0\}$ is finite

9-22-20

A category is $\mathcal{C}$

- a class of objects Ob
- a class of morphisms Mor
- maps $\mathrm{Mor} \underset{t}{\overset{s}{\rightrightarrows}} \mathrm{Ob}$

$$\mathrm{Ob} \xrightarrow{\mathbf{id}} \mathrm{Mor}$$

$$\underset{s,\,\mathrm{Ob}_j,\,t}{\mathrm{Mor} \times \mathrm{Mor}} \xrightarrow{\bullet} \mathrm{Mor}$$

s: source

t: target

" $\{(g,f) \in \mathrm{Mor} \times \mathrm{Mor} \mid s(g) = t(f)\}$

such that

$s \cdot \mathbf{id} = \mathrm{id}_{\mathrm{Ob}}$ $\quad$ $t \cdot \mathbf{id} = \mathrm{id}_{\mathrm{Ob}}$ $\quad$ $(g \circ (h \circ f)) = ((g \circ h) \circ f)$

$s \circ \bullet = s \circ p_2$ $\quad$ $t \circ \bullet = t \circ p_1$ $\quad$ $\mathrm{id}_Y \circ f = f \cdot \mathbf{id}_X = f$

$f: X \to Y$

We write $X \in \mathcal{C}$ instead of $X \in \mathrm{Ob}\,\mathcal{C}$

and $f: X \to Y$ $\quad$ $X \xrightarrow{f} Y$ instead of $f \in \mathrm{Mor}\,\mathcal{C}$, $s(f) = X$, $t(f) = Y$

$\Rightarrow$ The Axioms can be written as

$$X \xrightarrow{id} X$$

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "g \circ f"'] & & Z \end{tikzcd}$$

$$\begin{tikzcd} W \arrow[d, "f"'] \arrow[dr, "g \circ f"] \arrow[rr, bend left, "h \cdot g \cdot f"] & & Z \\ X \arrow[r, "g"'] \arrow[urr, bend right, "h \cdot g"'] & Y \arrow[ur, "h"] & \end{tikzcd}$$

The primordial example

Set: the category of sets

Ob: all sets

Mor: maps of sets

s: domain   t: codomain

id = identity map $\bullet$ = composition

Other Examples

Group is a Category

Ob: groups

Mor: homomorphisms of groups

(Properties are transfered from sets)

Ab is a category of abelian groups

Ob: abelian groups

Mor: homomorphisms

This is a full subcategory

b/c it has some groups and all

the morphisms of the taken groups

<u>Example</u>

- Set_{inj} Mor: injective maps of sets

- Set_{surj} Mor: surjective "

- Set_{bij} Mor: bijective "

Ob: sets.

These are non-full categories

Categories are more than just sets w/ structure.

★ <u>Example</u> What is a category of 1 object

$Ob = \{*\}$ $\quad$ $Mor = M$

$$\{*\} \xrightarrow{id} M$$

Here $\bullet : M \times M \to M$ as all morphisms are compatible

Associative: $(h \circ g) \circ f = h \circ (g \circ f)$

unity: $id \circ f = f \circ id = f$

Def **Monoid** is a set M w/ $e \in M$ and $M \times M \to M$ s.t.

1) $(xy)z = x(yz)$

2) $ex = x = xe$

All groups are monoids, $\mathbb{Z}_{\geq 0}$ $+$, $\mathbb{Z}^+$ $\cdot$,

$X \in \mathrm{Set} \quad M = \{X \to X\} = X^X$

$\mu(f, g) := f \circ g \quad e = id_X$

Another example.

In any category $\mathcal{C}$ $\forall X \in \mathcal{C}$

we can form a full subcategory by just using X

This is called $End(X)$ the endomorphism category of X

$\{f \in Mor \mid f : X \to X\}$

$\mu(f, g) = f \circ g \quad e = id_X$

Revisiting $\emptyset$ shows that for any monoid M we can construct a category BM also known as the delooping of M

$m \in M$ $\quad \overset{*}{\circlearrowleft}_m$

What is a category for which there is at most one morphism between any pair of objects ie

if $X \overset{f}{\underset{g}{\rightrightarrows}} Y$, then $f = g$

In this case $Mor \xrightarrow{(s,t)} Ob \times Ob$ is an injection

ie Mor can be identified as a subset of $Ob \times Ob$

That is Mor is a relation R on Ob

$id : Ob \to Mor$
$x \mapsto X \xrightarrow{id_x} X$ ie R is reflexive

We also have that

$$\mathrm{Mor} \times_{s, \mathrm{Ob}, t} \mathrm{Mor} \longrightarrow \mathrm{Mor}$$

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "f \circ g"'] & & Z \end{tikzcd}$$

$\Rightarrow$ XRY and YRZ

also XRZ

$\Rightarrow$ R is transative

Associativity and Unitality hold

bc if $X \overset{f}{\underset{g}{\rightrightarrows}} Y \Rightarrow f = g$

That is such categories can be identified w/ a reflexive and transative relation on the set of objects ie a pre order

<u>Example</u> Posets are sets equipped w/ a reflexive, antisymmetric, transative relation

Thus a Poset is a category!

<u>Unrelated Example</u> Poset

Ob: posets $(X, \leq)$

Mor: order preserving maps

$$(X, \leq) \to (X', \leq')$$

$f: X \to X'$ s.t. if $x_1 \leq x_2 \Rightarrow f(x_1) \leq' f(x_2)$

Don't <u>confuse these two examples</u>

9-24-20

Let $\mathscr{C}$ be a category. An initial object in $\mathscr{C}$ is $X \in \mathscr{C}$ s.t. $\forall Y \in \mathscr{C}$: $\exists ! \, X \to Y$. Notation $X = 0$

The terminal object in $\mathscr{C}$ is $X \in \mathscr{C}$ s.t. $\forall Y \in \mathscr{C}$: $\exists ! \, Y \to X$. Notation $X = \mathbb{1}$

Universal property of initial / terminal objects

Examples	Category	Initial	terminal
	Set	$\emptyset$	$\{*\}$
	Grp	$\mathbb{1}$	$\mathbb{1}$
	Ab	$\mathbb{1}$	$\mathbb{1}$
	BM	exists only if $M = \{e\}$	exists only if $M = \{e\}$
$x \to y$ iff $x R y$ $\to$	R_{preorder}	$0 \le y \; \forall y$ called the bottom element	$y \le 1 \; \forall y$ called the top element

* existence of top/bot is not guarenteed *

Lemma An intial object if it exists is unique up to a unique isomorphism if 0 and $0'$ are initial objects there is a unique morphism $0 \to 0'$ this morphism is an isomorphism. Likewise for terminal objects.

Proof By defintion of an initial object 0 $\exists !$ morphism $0 \xrightarrow{f} 0'$

$0'$ $\exists !$ morphism $0' \xrightarrow{g} 0$

0 $\exists !$ morphism $0 \to 0$ (id_0)

$0'$ $\exists !$ morphism $0' \to 0'$ $(id_{0'})$

We have $fg : 0 \to 0 \Rightarrow fg = id$ also $gf : 0' \to 0' \Rightarrow gf = id_{0'}$

$\Rightarrow f, g$ are isomorphisms $\square$

Let $\mathscr{C}$ be a category and $\{X_i\}_{i\in I}$ is a family of objects in $\mathscr{C}$

The category of cones over $\{X_i\}_{i\in I}$ is defined as follows

Objects are cones : $(A, \{p_i : A \to X_i\})_{i \in I}$ — A: apex, p_i: proj maps

Morphisms $(A, \{p_i : A \to X_i\}_{i\in I}) \longrightarrow (A', \{p_i' : A' \to X_i\}_{i\in I})$

are morphisms $f : A \to A'$ in $\mathscr{C}$ s.t.

$p_i' f = p_i \quad \forall i \in I$

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "p_i"'] & \# & A' \arrow[dl, "p_i'"] \\ & X_i & \end{tikzcd}$$

The identity morphisms and composition are inherited from $\mathscr{C}$.

Again associativity and unitality are inherited from $\mathscr{C}$.

Now must verify composition has closure.

Let $(A, \{p_i : A \to X_i\}_{i\in I}) \xrightarrow{f} (A', \{p_i' : A' \to X_i\}_{i\in I}) \xrightarrow{g} (A'', \{p_i'' : A'' \to X_i\}_{i\in I})$

must show triangle commutes

$$\begin{tikzcd} A \arrow[rr, "f\circ g"] \arrow[dr, "p_i"'] & & A'' \arrow[dl, "p_i''"] \\ & X_i & \end{tikzcd} \Rightarrow \begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "p_i"'] & A' \arrow[r, "g"] \arrow[d, "p_i'"] \arrow[dl, phantom, "\#"] & A'' \arrow[dl, "p_i''"] \arrow[dl, phantom, "\#", near start] \\ & X_i & \end{tikzcd} \Rightarrow \begin{tikzcd} A \arrow[rr, "f\circ g"] \arrow[dr, "p_i"'] & \# & A'' \arrow[dl, "p_i''"] \\ & X_i & \end{tikzcd}$$

$\Rightarrow$ composition is closed.

$\mathscr{C}$ is a category $\{X_i\}_{i\in I}$. The product of $\{X_i\}_{i\in I}$ is the (if it exists) terminal object in the category of cones over $\{X_i\}_{i\in I}$.

Notation $\left(\prod_{i\in I} X_i \,,\; p_i : \prod_{j\in I} X_j \to X_i\right)$

* Note if the product exists then it is unique up to a unique isomorphism since products are terminal objects *

Unfolding the definition we get the universal property of products

$\left(\prod_{i\in I} X_i, \{p_i : \prod_{j\in I} X_j \to X_i\}\right)_{i\in I}$ is the product of $\{X_i\}_{i\in I}$

if for any $(A, \{p_i' : A \to X_i\}_{i\in I})$ there is a unique morphism

$A \to \prod_{i\in I} X_i$ s.t.
$$\begin{tikzcd} A \arrow[r] \arrow[dr, "p_i'"'] & \prod_{i\in I} X_i \arrow[d, "p_i"] \\ & X_i \end{tikzcd}$$
$\#$

Reformation
$$\left\{A \to \prod_{i\in I} X_i\right\} \xrightarrow[(p_i\circ(-))_{i\in I}]{\simeq} \prod_{i\in I}\{A \to X_i\}$$
$$\left\{A \to \prod_{i\in I} X_i\right\} = \hom\left(A, \prod_{i\in I} X_i\right) = \mathcal{C}\left(A, \prod_{i\in I} X_i\right)$$
$$\prod_{i\in I}\{A \to X_i\} = \prod_{i\in I}\hom(A, X_i)$$

A typical application: Given $\{A \to X_i\}_{i\in I}$ we construct $A \to \prod_{i\in I} X_i$

Examples

Set : small products (I is a set) exists

Grp : small products exist

Ab : small products exist (inherited from Grp)

BM : Does $\{*\}_{i\in I}$ have a product? $(*, p_i : * \xrightarrow[e_M]{} *)$

universal property $\forall (*, p_i' : * \to *)$ $\exists!\ * \xrightarrow{f} *$ s.t.
$$\begin{tikzcd} * \arrow[r, "f"] \arrow[dr, "p_i"'] & * \arrow[d, "p_i'"] \\ & * \end{tikzcd} \qquad p_i' = p_i f$$
$\#$

if $I \neq \emptyset$ and $M \neq \{e\}$, then the product does not exist.

if $I = \emptyset$ and $M \neq \{e\}$, then again products do not exist (f is not unique)

Thus products only exist if $M \simeq \{e\}$

$\underset{\leq}{R}$ a preorder on a set W:

$\underset{\overset{\frown}{W}}{\{x_i\}_{i\in I}}$ What is the product of $\{x_i\}_{i\in I}$

$(\overset{\in W}{a}, a \leq x_i)$ ie a cone is a lower bound for $\{x_i\}_{i\in I}$

The product $\left(\prod_{i\in I} x_i, \prod_{i\in I} x_i \leq x_i\right)$ $\forall a\ \exists!\ a \leq \prod_{i\in I} x_i$

$\Rightarrow \prod_{i\in I} x_i = \inf_{i\in I} x_i$

Last week saw that Ab has all small products

if P is a poset that admits inf, sup, then P as a category admits all products.

Today we are going to show that Grp has all small products

Recall $\{G_i\}_{i\in I}$ $\quad K < \prod_{i\in I} G_i$ where $K = \{x \mid \operatorname{supp} x \text{ is finite}\}$

$$\{K \xrightarrow{\varphi} H\} \xrightarrow[(\,\cdot\,)\circ\iota_i]{} \prod_{i\in I} \hom(G_i, H) \ni (\psi_i)_{i\in I}$$ this is a bijection for abelian groups

In general this map is not surjective

Let $\varphi: K \to H$ $\quad \varphi(x) = \sum_{i\in I} \psi(x_i) \leftarrow$ does not make sence for nonabelian H

The universal property of restricted sums

The map $\{K \xrightarrow{\varphi} H\} \to \prod_{i\in I} \hom(G_i, H)$ is bijective when the right side is restricted to $(\psi_i)_{i\in I}$ for which

$\forall i\neq j \quad \psi_i(x_i)\psi_j(x_j) = \psi_j(x_j)\psi_i(x_i) \qquad x_i \in G_i \quad x_j \in G_j$

$\Rightarrow \varphi(x) = \prod_{i\in I} \psi_i(x_i)$ makes sence

Observ: in the restricted sum the images of $\iota_i: G_i \to K$ commute

The free product of groups (coproduct of Grp) $\{G_i\}_{i\in I}$

is constructed as a quotient of a monoid, M.

The underlying set of M is the set $\coprod_{n\geq 0}\left(\coprod_{i\in I}\mathcal{U}(G_i)\right)^n$

this finite sequences whose elements are taken from the set $\coprod_{i\in I}\mathcal{U}(G_i)$

multiplication = concatination of finite sequence

neutral element = the finite sequence of length $n=0$ $\emptyset$

associativity $\wedge$ unitality ✓

The equivelence relation on $\mathcal{U}(M)$ is defined as follows:

$g\cdot g'$ are finite sequences $n=2$ s.t. $g,g'\in G_i$ for some $i\in I$

$\Rightarrow g\cdot g' \sim gg'$ $(n=1)$

$e\sim\emptyset$ e $n=1$ $e\in G_i$ for some i $\emptyset$ $n=0$

(We have canonical injection maps $\mathcal{U}(G_i)\xrightarrow[i_i]{}\mathcal{U}(M)$ $g\mapsto(g)$ $n=1$

We want these maps to be homomorphisms of monoids)

$i_i(g\cdot g') = (g\cdot g')$

$i_i(g)\,i_i(g') = (g,g')$. But we still have to close this set of pairs under multiplication w/ arbitrary elements of M

$a\sim a'$, $m\in M$ $\quad m\cdot a\sim m\cdot a'$

$am\sim a'm$

Take the equivelence relation generated by the restriction of pairs concatination:

Claim: $\mathcal{U}(M)/\sim$ is a monoid $\mathcal{U}(M)/_\sim \times \mathcal{U}(M)/_\sim \xrightarrow{m} \mathcal{U}(M)/_\sim$

$m_1\sim m_1'$ $\quad m_2\sim m_2'$ $\Rightarrow$ $m_1m_2 \sim m_1'm_2'$

$\sim m_1m_2' \sim$

Finally $\mathcal{U}(M)/\sim$ has inverses

A finite sequence (word) in $\coprod_{i\in I} U(G_i)$ is reduced iF $x_j \neq 1 \in G_i$ and $\forall j: x_j \in G_i \quad x_{j+1} \in G_{i'} \quad i \neq i'$

Remark: Any equivalence class of words has a unique reduced word.

The free group on a set G is $\coprod_{g\in G} \mathbb{Z} = F_G$

Examples

a) $G = \emptyset \quad F_G = \mathbb{1}$

b) $G = \{*\} \quad F_G = \mathbb{Z}$

c) $G = \{x, y\} \quad F_G = \mathbb{Z} * \mathbb{Z} \qquad \omega \in \mathbb{Z} * \mathbb{Z} \qquad \omega = x^{m_1} y^{n_1} x^{m_2} y^{n_2} \cdots x^{m_k} y^{n_k}$

$m_i \neq 0 \quad i > 1, \quad n_i \neq \quad i < k$

X metric or topological space $X \subset \mathbb{R}^n$.

X has a base point $* \in X$. The fundemental group of $(X, *)$ is defined as followed.

The underlying set is the quotient

$$\pi_1(X,*) = \{ [0,1] \xrightarrow[\text{cont.}]{\gamma} X \mid \text{s.t. } \gamma(0) \overset{*}{=} \gamma(1) \} \Big/ \begin{smallmatrix}\text{homotopy}\\ \text{relative}\\ \text{boundry}\end{smallmatrix}$$

homotopy relative boundry is a continuous map

$[0,1] \times [0,1] \xrightarrow{h} X$

$h(0,-) = \gamma_1$

$h(1,-) = \gamma_2$

$h(-,0) = h(-,1) = (t \mapsto *)$

Multiplication:

$$(\gamma_2 \cdot \gamma_1)(t) = \begin{cases} \gamma_1(2t), & 0 \le t \le 1/2 \\ \gamma_2(2t-1), & \frac{1}{2} \le t \le 1 \end{cases}$$

Examples

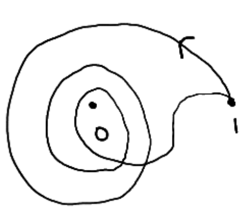
a) $\pi_1(\mathbb{R}^n, 0) = \mathbb{1}$

Any $\gamma : [0,1] \to \mathbb{R}^n$ s.t. $\gamma(0) = \gamma(1)$ is homotopic to $(t \mapsto 0)$ via

$h(s,t) = s \cdot \gamma(t)$ $\quad h(0,t) = 0 \quad h(0,-) = (t \mapsto *)$

$h(1,t) = \gamma(t) \quad h(1,-) = \gamma$

b) $\pi_1(\mathbb{R}^2 \setminus \{0\}, (1,0)) \overset{\text{index}}{\simeq} \mathbb{Z}$



This deforms any loop in $\mathbb{R}^2 \setminus \{0\}$ to a loop in $U(1)$

$\gamma : [0,1] \to \mathbb{R}^2 \setminus \{0\}$

$h(s,t) = (1-s) \cdot \gamma(t) + s \dfrac{\gamma(t)}{\|\gamma(t)\|}$

$h(0,t) = \gamma(t)$

$h(1,t) = \gamma(t) / \|\gamma(t)\|$

The index measures how many times a loop will wrap around the origin

c) $n \geq 3 \quad \pi_1(\mathbb{R}^n \setminus \{0\}, (1,0,0,\dots,0)) \simeq \mathbb{1}$

Remark: Set_*, pointed sets category, objects (X, x)

where X set and $x \in X$,

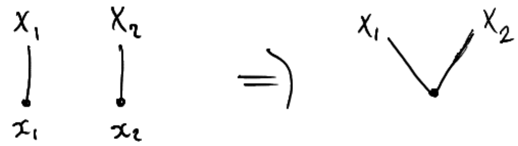
morphisms $(X, x) \to (Y, y)$ are maps of sets s.t.

$f(x) = y$

Prop Set_* admits all small coproducts

$\{(X_i, x_i)\}_{i\in I}$ $\qquad \coprod_{i\in I}(X_i, x_i) = \left(\coprod_{i\in I} X_i\right) / \underbrace{\text{identify all base points.}}_{\text{The equivalence class of all base points}}$

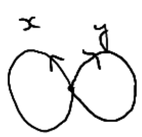
also denoted $\bigvee_{i\in I}(X_i, x_i)$



What is the fundemental Group of $\bigvee_{i\in I}(X_i, x_i)$?

Answer $\pi_1\left(\bigvee_{i\in I}(X_i, x_i)\right) = \coprod_{i\in I} \pi_1(X_i, x_i)$ $\qquad \pi_1$ preserves coproducts

$$\underset{\overset{\in}{\omega}}{\pi_1(\infty)} \cong \pi_1(O) * \pi_1(O) = \mathbb{Z} * \mathbb{Z}$$



$w = x\,x\,y\,x^{-1}x^{-1}\,y\,y^{-1}x$

10-6-20

Let $\mathcal{C}$ be a category, G be a group.

A G-action on an object $X \in \mathcal{C}$ is a homomorphism of groups

$$G \xrightarrow{\alpha} \mathrm{Aut}(X)$$

Important Special Case: Set

A G-action on a set X is a map of sets $\mathcal{U}(G) \times X \xrightarrow{\cdot} X$

s.t. $g \cdot (h \cdot x) = (gh) \cdot x \qquad 1_G \cdot x = x \qquad \forall g, h \in G \quad \forall x \in X$

Proof of Equivalence

The adjoint map $G \to X^X$ should be a homomorphism of monoids

The map

$$\begin{tikzcd} G \arrow[r] \arrow[dr, dashed, "\exists!"'] & X^X \\ & \mathrm{Aut}(X) = \Sigma_X \arrow[u] \end{tikzcd}$$

10hw1

We know that $\{\mathcal{U}(G) \times X \to X\} \cong \{\mathcal{U}(G) \to X^X\} \cong \{G \to \mathrm{Aut}(X)\}$

Example The left regular action. Let $X = \mathcal{U}(G)$

$g \cdot x := gx$

assoc. $$\begin{aligned} g \cdot (h \cdot x) &= g \cdot (hx) \\ &= ghx \\ &= (gh) \cdot x \end{aligned}$$

unitality $\quad 1_G \cdot x = 1_G x = x$

Thm (Cayley) For any group G $\exists$ injective homomorphism $G \to \Sigma_X$ for some set X

Proof Take $X = \mathcal{U}(G)$. $G \to \Sigma_X$ corresponds to the left regular action

if $g \in G$ s.t. $(x \mapsto g \cdot x) = id_X$

take $x = 1_G$ $(1_G \mapsto g \cdot 1_x) = id_X \Rightarrow g = 1$

$\Rightarrow$ injective $\square$

Examples The conjugation action. $X = \mathcal{U}(G)$

$g \cdot x = gxg^{-1}$ ass : $g \cdot (h \cdot x) = g \cdot (hxh^{-1})$

$$= ghxh^{-1}g^{-1}$$
$$= gh\, x (gh)^{-1}$$
$$= (gh) \cdot x$$

unit : $1_G \cdot x = 1_G x 1_G^{-1} = x$

Ex) $G \curvearrowright G/H$

$g \cdot aH := (ga)H$ Need to show this is well defined

Suppose $aH = a'H \Rightarrow \exists!\ h \in H$ s.t. $ah = a'$

$g \cdot a'H = (ga')H = (gah)H = (ga)H = g \cdot aH$

This is clearly ass and unit

A G-equivariant map of G-sets $(X, G\times X \to X) \to (Y, G\times Y \to Y)$
is a map of sets $f: X \to Y$ s.t. $f(g.x) = g.f(x)$

In terms of G-action definition

$$(G \xrightarrow{\alpha} \mathrm{Aut}(X)) \longrightarrow (G \xrightarrow{\beta} \mathrm{Aut}(Y))$$

is a morphism $f: X \to Y$ s.t. $\forall g \in G \quad f \circ \alpha(g) = \beta(g) \circ f$

Proposition G-sets with G-equivariant maps form a category. Set_G

Proof

$$X \xrightarrow{f} Y \xrightarrow{f'} Z$$

$G\times X \to X \quad G\times Y \to Y \quad G\times Z \to Z$

comp in set

$f' \circ f := f' \circ f$

$$(f'\circ f)(g.x) = f'(f(g.x)) = f'(g.f(x)) = g.f'(f(x)) = g.(f'\circ f)(x)$$

Assoc, unitality inherited from sets

10-8-20

Prop Set_G has small products and coproducts.

<u>Proof</u> a) Products $\{X_i\}_{i\in I}$ $G \curvearrowright X_i$

$$\prod_{i\in I} X_i = \prod_{i\in I} \mathcal{U}(X_i) \qquad g\cdot(x_i)_{i\in I} = (g\cdot x_i)_{i\in I}$$

$$\prod_{i\in I} X_i \xrightarrow{p_i} X_i \qquad p_j(x_i)_{i\in I} = x_j$$

$$p_j(g\cdot(x_i)_{i\in I}) = p_j((g\cdot x_i)_{i\in I}) = g\cdot x_j$$

Universal property Given $\{Y \xrightarrow{f_i} X_i\}_{i\in I}$ WTS $\exists!\ Y \xrightarrow{\varphi} \prod_{i\in I} X_i$ $p_i\circ\varphi = f_i$

Uniqueness is inherited from Set

Existence: $\varphi(y) = (f_i(y))_{i\in I}$

$$\varphi(g\cdot y) = (f_i(g\cdot y))_{i\in I} = (g\cdot f_i(y))_{i\in I} = g\cdot(f_i(y))_{i\in I}$$

Coproducts $\coprod_{i\in I} X_i = \coprod_{i\in I} \mathcal{U}(X_i)$ $\qquad g(i, \underset{\in X_i}{x}) = (i, g\cdot x)$

$$\iota_i : X_i \to \coprod_{i\in I} X_i \qquad \iota_i(g\cdot x) = (i, g\cdot x) = g\cdot(i,x)$$

$$x \longmapsto (i,x)$$

Universal Prop: Given $X_i \xrightarrow{f_i} Y$ WTS $\exists!\ \coprod_{i\in I} X_i \xrightarrow{\varphi} Y$ s.t. $\varphi\circ\iota_i = f_i$

Uniqueness is inherited from Set

$\varphi(i,x) = \iota_i(x)$

$$\varphi(g\cdot(i,x)) = \varphi(i, g\cdot x) = \iota_i(g\cdot x) = g\cdot\iota_i(x) \qquad \square$$

A G-set X, is transative (orbit) if

a) $X \neq \emptyset$

b) $\forall x, x' \in X \quad \exists g \in G$ s.t. $x = g.x'$

Proposition $\forall$ G-set X $\quad X = \coprod_{i \in I} X_i$ where X_i is transative (unique up to unique esomorphism.)

Proof Define an equivalence relation on X: $\quad x \sim x' \Leftrightarrow \exists g \in G$ s.t. $g.x = x'$

Take $I = X/\sim$. Take $X_i = q^* \{i\}$ where $X \xrightarrow[q]{} X/\sim$

X_i inherits the G action as $x \in X_i \quad g \in G \quad g.x_i \in X_i$ since $x \sim g.x$

$\Rightarrow q(g.x) = i$

Hence $X \cong \coprod_{i \in I} X_i$ as X_i partition X into desjoint subsets

Now need to show X_i is transative

$X_i \neq \emptyset$ by construction

$\forall x, x' \in X_i \qquad x \sim x' \Rightarrow \exists g \in G$ s.t. $g.x = x'$

$\Rightarrow$ transanative $\square$

Prop Any morphism $\coprod_{i \in I} X_i \xrightarrow{f} \coprod_{j \in J} Y_j \qquad X_i, Y_j$ transative

$\exists! \; I \xrightarrow{\alpha} J \qquad X_i \xrightarrow{s_j} Y_{\alpha(j)}$ s.t. $f = [i_{\alpha(j)} \circ s_j]_{i \in I}$

$f(i, x) = (\alpha(j), s_i(x))$

Proof f respects $\sim \qquad x, x' \in \coprod_{i \in I} X_i \qquad x \sim x' \Leftrightarrow \exists g \in G \;\; g.x = x'$

$f(g.x) = f(x') \qquad \Rightarrow f(x) \sim f(x')$

$\overset{\shortparallel}{g.f(x)}$

$$\begin{tikzcd} X_i \arrow[r, "f|_{X_i}"] \arrow[d, "s_i"'] & \coprod_{j \in J} Y_j \\ Y_{\alpha(i)} \arrow[ur] & \end{tikzcd}$$

Thus f induces a map $\alpha: I \to J$ $\square$

if X is a G-set and $x \in X$, then the stabalizer of x is

the group $\mathrm{Stab}_x X := \{ g \in G \mid g\cdot x = x \}$

Ex) a) $G \curvearrowright \mathcal{U}(G)$ $\quad g\cdot x = gx$

$$\mathrm{stab}_x = \{ g \in G \mid gx = x \} = \{1\}$$

b) $G \curvearrowright \mathcal{U}(G)$ $\quad g\cdot x = gxg^{-1}$

$$\mathrm{stab}_x = \{ g \in G \mid gxg^{-1} = x \} = \{ g\in G \mid gx = xg \} = \underbrace{C_G(x)}_{\text{centralizer of } x}$$

Prop $\forall X$ transative G-set $\forall x \in X$, the canonical map

$G/\mathrm{Stab}_x X \xrightarrow{\varphi} X$ is an isomorphism

<u>Proof</u> $\quad G \to X$

$\quad g \mapsto g\cdot x$

$$\begin{tikzcd} G \arrow[r] \arrow[d] & X \\ G/\mathrm{stab}_x \arrow[ur, "\varphi"'] & \end{tikzcd}$$

$g \sim gh \quad \forall h \in \mathrm{stab}_x$

$gh \mapsto (gh)\cdot x = g\cdot(hx) = g\cdot x \Rightarrow$ well defined

$\varphi(g\,\mathrm{stab}) = g\cdot \mathrm{stab}$

$\varphi(g'\cdot g \cdot \mathrm{stab}) = \varphi((g'g)\mathrm{stab}) = (g'g)\,\mathrm{stab} = g'\cdot(g\,\mathrm{stab})$

$\Rightarrow \varphi$ is G-equivariant

φ is surjective $\quad \forall x' \in X \quad x \sim x' \Rightarrow \exists g$ s.t. $g\cdot x = x'$

$\varphi(x\,\mathrm{stab}) = g\cdot x = x'$

φ is injective as $\varphi(g\,\mathrm{stab}) = \varphi(g'\,\mathrm{stab})$

$$\begin{array}{ccc} \varphi(g\,\mathrm{stab}) & = & \varphi(g'\,\mathrm{stab}) \\ \shortparallel & & \shortparallel \\ g\cdot x & & g'x \end{array}$$

$\Rightarrow x = g^{-1}g'x$

$\Rightarrow gg' \in \mathrm{stab}_x \Rightarrow g\,\mathrm{stab}_x = g'\,\mathrm{stab}$ $\quad \square$

Remark: $Stab_x = g^{-1} \cdot Stab_{x'} \cdot g \quad \forall\, x, x' \in X \quad g \cdot x = x'$

Proposition G a group $H_1 < G$ $H_2 < G$

$$\mathrm{Hom}(G/H_1, G/H_2) \xrightarrow{\cong} \{ \underset{\text{"}gH_2\text{"}}{[g]} \in G/H_2 \mid H_1 < gH_2g^{-1} \} \text{ in } Set_G$$

Proof Pick $\varphi : G/H_1 \to G/H_2$

$gH_2 := \varphi(H_1)$

Let $g'H_1 \in G/H_1 \quad \varphi(g'H_1) = \varphi(g' \cdot H_1) = g' \cdot \varphi(H_1) = g'gH_2$

Define $\varphi(g'H_1) = g'gH_2$

$g'' = g'h_1 \quad h_1 \in H_1 \quad g'gH_2 \stackrel{?}{=} g'h_1 gH_2 \iff g^{-1}h_1^{-1}gH_2 = H_2 \iff g^{-1}h_1^{-1}g \in H_2 \quad \forall h_1 \in H$

$\iff g^{-1}H_1 g \subset H_2$ ✓

$\square$

Remark: $\mathrm{Aut}(G/H) = N_G(H)/H$ The Weyl Group

10-13-20

A functor $F : \mathcal{C} \to \mathcal{D}$ is an arrow

$$\mathrm{Obj}(\mathcal{C}) \xrightarrow{F} \mathrm{Obj}(\mathcal{D})$$

$$\mathrm{Hom}(\mathcal{C}) \xrightarrow{F} \mathrm{Hom}(\mathcal{D})$$

Such that

- if $f : X \to Y$ in $\mathcal{C}$, then $F(f) : F(X) \to F(Y)$
- $F(id_X) = id_{F(X)}$
- $F(g \circ f) = F(g) \circ F(f)$ if $g \circ f$ is defined

Ex) Given monoids $M_1, M_2 \in \mathrm{Monoid}$, then a functor $BM_1 \to BM_2$ is a homomorphism of monoids $M_1 \to M_2$

Ex) $P_1, P_2 \in \mathrm{Poset}$ A functor $P_1 \to P_2$ is a map of sets $P_1 \to P_2$ such that if $p \le p' \Rightarrow F(p) \le F(p')$, i.e., an order-preserving map.

Ex) Forgetful Functors

$$\begin{tikzcd}
\mathrm{Ab} \arrow[d, "\mathcal{U}"'] \arrow[drr, "\mathcal{U}"] & & \\
\mathrm{Group} \arrow[rr, "\mathcal{U}"'] \arrow[dr, "\mathcal{U}"'] & & \mathrm{Set} \\
& \mathrm{Monoids} \arrow[ur, "\mathcal{U}"'] &
\end{tikzcd}$$

Since objects and mophisms in these categories are equal iff the underlying set is equal. So $\mathcal{U}$ is a functor.

Ex) Free Functors   Reverse arrows of forgetful functor

$$\begin{tikzcd}
Ab & \\
Group \arrow[u, "\text{abelianization}"] & Set \arrow[l, "\text{free } \cancel{\text{group}}"'] \arrow[ul, "\text{free abelian group}"'] \arrow[d, "\text{free monoid}"] \\
& Monoid \arrow[ul, "\text{group completion}"]
\end{tikzcd}$$

$$Set \xrightarrow{\text{free group}} Group$$

· $S \xrightarrow{F} \coprod_{s \in S} \mathbb{Z}$

· $S \xrightarrow{f} T$

$$\begin{tikzcd}
\coprod_{s\in S} \mathbb{Z} \arrow[r, "\exists!\, F(f)"] & \coprod_{t \in T} \mathbb{Z} \\
S \arrow[u] \arrow[ur, "\forall"] \arrow[r] & T \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
S \arrow[r] \arrow[dr] & V \\
& T \arrow[u]
\end{tikzcd} \Rightarrow \begin{tikzcd}
F(S) \arrow[r] \arrow[dr] & F(U) \\
& F(T) \arrow[u]
\end{tikzcd}$$

both of these fall from the universal property of free groups

Abelianization : $Group \longrightarrow Ab$

$$G \longrightarrow G/[G,G]$$

we also proved that $\{G \to A\} \cong \{G/[G,G] \to A\}$

Group Completion : $Monoid \longrightarrow Group$

Given $M \longmapsto \text{free group}(\mathcal{U}(M)) \Big/ \text{subgroup generated by} \left(m_1 \cdot m_2 \cdot (m_1 m_2)^{-1} \text{ and } 1_m \right)$

· multiplication in group
multi in monoid

$$\{ \text{Group Completion}(M) \xrightarrow{\text{homo groups}} G \} \cong \{ M \xrightarrow{\text{homo monoids}} \mathcal{U}(G) \}$$

Ex) $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$ group completion of $\mathbb{Z}_{\geq 0}$ is $\mathbb{Z}$

Cat the category of small categories (Ob, Mor are sets)

CAT the category of locally small category (Hom(X,Y) is a set)

Let $\mathcal{C}$ and $\mathcal{D}$ be categories $\mathcal{C} \times \mathcal{D}$ is also a category

Objects are pairs in $\mathrm{Obj}(\mathcal{C}) \times \mathrm{Obj}(\mathcal{D})$

Morphisms are pairs in $\mathrm{Hom}(\mathcal{C}) \times \mathrm{Hom}(\mathcal{D})$

10-15-20

The opposite category $\mathcal{C}^{op}$ contains all the same objects, but the arrows are all switched, i.e., f $\quad A \xrightarrow{f} B \quad B \xrightarrow{f^{op}} A$

Ex) For locally small $\mathcal{C}$

$$\hom : \mathcal{C}^{op} \times \mathcal{C} \to \mathrm{Set}$$

$$(x, y) \longmapsto \{f : x \to y\} = \{f \in \mathrm{Hom}\,\mathcal{C} \mid s(f) = x \quad t(f) = y\}$$

$$\begin{array}{c} (x, y) \\ \downarrow (g, h) \\ (x', y') \end{array}$$

$$\hom(g, h) : \hom(x, y) \to \hom(x', y')$$
$$f \longmapsto h \circ f \circ g$$

$g : x' \to x$ in $\mathcal{C}$

$g^{op} : x \to x'$ in $\mathcal{C}$

$h : y \to y'$ in $\mathcal{C}$

in $\mathcal{C}$

$$\begin{tikzcd} x \arrow[r, "f"] & y \arrow[d, "h"] \\ x' \arrow[u, "g"] \arrow[r, "hfg"'] & y' \end{tikzcd}$$

Terminology: $\quad \mathcal{C}^{op} \to \mathcal{D}$ "contravariant functor $\mathcal{C} \to \mathcal{D}$"

$\mathcal{C} \to \mathcal{D}$ " covariant functor $\mathcal{C} \to \mathcal{D}$"

"hom is contravariant in the first variable and covariant in the second"

Ex) $\mathrm{Hom} : Ab^{op} \times Ab \to Ab$

Observation on Set

$$\{A\times B \to C\} \cong \{A \to \mathrm{Hom}(B,C)\} \cong \{B \to \mathrm{Hom}(A,C)\}$$

in Ab

$$\{A\otimes B \to C\} \cong \{A \to \mathrm{Hom}(B,C)\} \cong \{B \to \mathrm{Hom}(A,C)\}$$

$\otimes$ is not $\times$ as $A=\mathbb{1}$ $\quad \{B\to C\} \not\cong \{0\}$

Prop $\forall A, B, C \in \mathrm{Ab}$ $\quad \{A \to \mathrm{Hom}(B,C)\} \cong \{B \to \mathrm{Hom}(A,C)\} \cong \mathrm{Bilin}(A,B;C)$

Let $A,B,C \in \mathrm{Ab}$. A bilinear map $A,B \to C$ is a map of sets

$\mathcal{U}(A)\times\mathcal{U}(B) \xrightarrow{p} \mathcal{U}(C)$ s.t. $\quad p(a,-): B \to C \quad \forall a$

$\qquad p(-,b): A \to C \quad \forall b$

are homomorphisms of Abelian groups

$$\begin{tikzcd} A\otimes B \arrow[dr, dashed, "g"] & \\ A\times B \arrow[u, "\theta"] \arrow[r, "\omega"'] & C \end{tikzcd}$$

Proof $f: A \to \mathrm{Hom}(B,C)$ $\quad a \mapsto (b \mapsto$

$\qquad B \to \mathrm{Hom}(A,C)$

Universal Property of Tensor Product of Abelian Groups

The tensor product $(A\otimes B, \theta)$ where

$A\otimes B$ is an abelian group

$\theta: A\times B \to A\otimes B$ is a bilinear homomorphism

s.t. for every pair (C,ω) where

C is an abelian group $\quad \omega: A\times B \to C$ is bilinear homo

$\exists!\ g: A\otimes B \to C$ with $\omega = g\circ\theta$

10-20-20

Recall $A, B \in Ab$

Consider a category whose objects are bilinear maps $A, B \longrightarrow F \in Ab$

morphisms are

$$\begin{tikzcd} & F \arrow[dd, "h"', "\text{homomorphism}"] \\ A, B \arrow[ur, "b"] \arrow[dr, "c"'] & \\ & G \end{tikzcd} \qquad \begin{tikzcd} & F \arrow[dd, "\mathcal{U}(h)"] \\ \mathcal{U}(A) \times \mathcal{U}(B) \arrow[ur, "b"] \arrow[dr, "c"'] & \\ & G \end{tikzcd}$$

The tensor product is the initial object in this category.

Thus it is equiped with $A, B \xrightarrow{\otimes} A \otimes B$

$a, b \longmapsto a \otimes b$

Uniqueness of tensor products follows from theorem of initial objects

<u>Existence</u>: Need a bilinear map $A, B \longrightarrow A \otimes B$

Denote F the free abelian group on the set $\mathcal{U}(A) \times \mathcal{U}(B)$

We have maps of sets $\mathcal{U}(A) \times \mathcal{U}(B) \xrightarrow{h} \mathcal{U}(F)$

This map is not bilinear. But we can force it to be bilinear

$$\left.\begin{array}{l} h(a+a', b) - h(a, b) - h(a', b) \\ h(a, b+b') - h(a, b) - h(a, b') \end{array}\right\} \maltese$$

Denote R the subgroup of F generated by elements of the form $\maltese$ $\forall a, a', b, b' \in A$

$A \otimes B = F / R$ $\qquad$ $\begin{tikzcd} \mathcal{U}(A) \times \mathcal{U}(B) \arrow[r, "h"] \arrow[rr, bend left, "\otimes"] & F \arrow[r] & \mathcal{U}(F/R) \end{tikzcd}$

The universal property:

$$\begin{tikzcd}
 & A\otimes B = F/R \arrow[d, dashed, "p"] & \\
A, B \arrow[ur, "\otimes"] \arrow[r, "b"', "\text{bilinear}"] & Z & F \arrow[ul] \arrow[l, "b'"] & R \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{U}(A)\times\mathcal{U}(B) \arrow[d, "b"] \\
\mathcal{U}(Z)
\end{tikzcd}$$

Since b bilinear b' vanishes on R

$$b'(h(\alpha+\alpha',\beta) - h(\alpha,\beta) - h(\alpha',\beta))$$
$$= b(\alpha+\alpha',\beta) - b(\alpha,\beta) - b(\alpha',\beta) = 0$$

$\Rightarrow P$ exists

uniqueness is from if P and P' make ✡ commute then

$P(a\otimes b) = P'(a\otimes b) \quad \forall a\in A \; b\in B$

Elements of the form $a\otimes b$ generate $A\otimes B$. Thus $P = P'$ $\square$

Properties

1) $A \otimes \bigoplus_{i\in I} B_i \cong \bigoplus_{i\in I} A\otimes B_i$

2) $A\otimes(B/C) \cong (A\otimes B)/(A\otimes C)$

3) $\mathbb{Z}\otimes A \cong A \cong A\otimes \mathbb{Z}$

4) $A\otimes(B\otimes C) \cong (A\otimes B)\otimes C$

5) $A\otimes B \xrightarrow{\cong} B\otimes A$

6) $0\otimes A \cong 0 \cong A\otimes 0$

7) $\bigoplus_{i\in I}\mathbb{Z} \otimes A \cong \bigoplus_{i\in I} A$ $\quad(1,3)$

Proof of (2):

$A\otimes B/C \longrightarrow (A\otimes B)/(A\otimes C)$

$\mathcal{U}(A)\times\mathcal{U}(B/C) \longrightarrow \mathcal{U}(A\otimes B)/\mathcal{U}(A\otimes C)$

$a, bC \longmapsto (a\otimes b)(A\otimes C)$

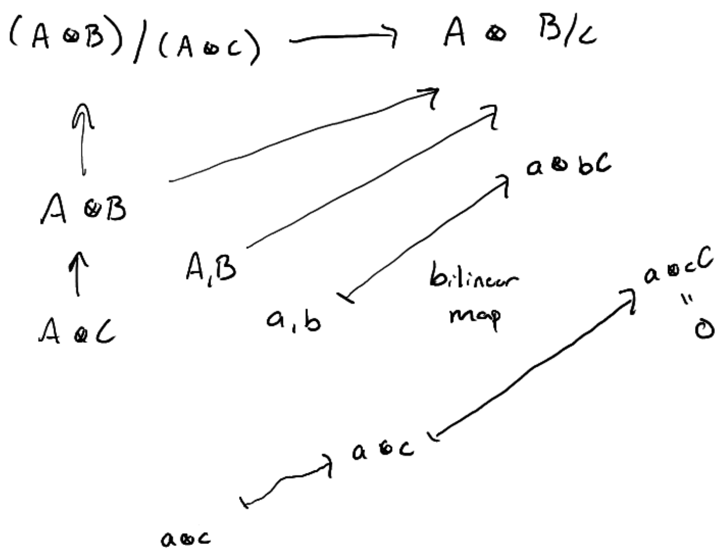
say $bC = b'C \Rightarrow b' = b + c$

$a, bC \mapsto (a\otimes b)(A\otimes C)$

$a, b'C \mapsto (a\otimes b')(A\otimes C) = (a\otimes(b+c))(A\otimes C)$
$= (a\otimes b + a\otimes c)(A\otimes C)$
$= (a\otimes b)(A\otimes C)$

$\Rightarrow$ well defined

Now need to show the map is bilinear, which is easy



$$A \otimes B/C \longrightarrow (A \otimes B)/(A \otimes C) \longrightarrow A \otimes B/C$$

$$a \otimes bC \longmapsto (a \otimes b)(A \otimes C) \longmapsto a \otimes bC$$

$$(A \otimes B)/(A \otimes C) \longrightarrow A \otimes B/C \longrightarrow (A \otimes B)/(A \otimes C)$$

$$(a \otimes b)(A \otimes C) \longmapsto a \otimes bC \longmapsto (a \otimes b)(A \otimes C) \qquad \square$$

10-22-20

Ex) $A \in Ab$, $m \in \mathbb{Z}$

$$A \otimes \mathbb{Z}/m \cong (A \otimes \mathbb{Z}) / (A \otimes m\mathbb{Z}) \cong A/mA$$

Ex) $\mathrm{Tor}(\mathbb{Z}, A) \xrightarrow{\ker} \overset{\cong A}{A \otimes \mathbb{Z}} \longrightarrow A \otimes \mathbb{Q}$

$\quad \overset{\shortparallel}{\{a \in A \mid \exists n \neq 0 : na = 0\}}$

Pick $a \in A$: $na = 0$ for $n \neq 0$

$$\underset{\mathrm{Tor}(\mathbb{Z},A)}{\overset{}{a}} \longmapsto \underset{A \otimes \mathbb{Z}}{a \otimes 1} \longmapsto \underset{A \otimes \mathbb{Q}}{a \otimes 1} = a \otimes (n \cdot \tfrac{1}{n}) = n(a \otimes \tfrac{1}{n}) = na \otimes \tfrac{1}{n} = 0 \otimes \tfrac{1}{n} = 0$$

Suppose that $a \in A$ and $a \otimes 1 = 0 \in A \otimes \mathbb{Q}$.

WTS $\exists n \neq 0$: $na = 0$. Suppose not $\Rightarrow$ $na \neq 0$ $n \neq 0$

Dmitri gave up :(

A concrete description of $A \otimes \mathbb{Q}$

$a \otimes p/q = ap \otimes \frac{1}{q}$ Denote by a/q the element $a \otimes \frac{1}{q}$

Claim $A \otimes \mathbb{Q}$ can be constructed as the abelian group of fractions.

a/q $\quad a \in \mathrm{Tor}(\mathbb{Z}, A)$ $q \in \mathbb{Z}^+$ $\quad \frac{a}{q} \sim \frac{a'}{q'}$ if $aq' = a'q$

The quotient $\big((A / \mathrm{Tor}(\mathbb{Z},A)) \times \mathbb{Z}^+\big) / \sim \; = B$

has an abelian group structure. $\frac{a}{q} + \frac{a'}{q'} = \frac{aq' + a'q}{qq'}$, $-\frac{a}{q} = \frac{-a}{q}$, $0 = \frac{0}{1}$

$A \otimes \mathbb{Q} \rightleftarrows B$ $\quad a \otimes \frac{p}{q} \mapsto \frac{ap}{q}$ $\quad \Rightarrow A \otimes \mathbb{Q} \cong B$

$\quad \frac{a}{q} \longmapsto a \otimes \frac{1}{q}$

Every Equivalence class has a canonical representative a/q where q is as small as possible.

Such a representative is unique because $a/q = a'/q \Rightarrow aq = a'q$

$\Rightarrow (a-a')q = 0$

$\Rightarrow a - a' \in \mathrm{Tor}(\mathbb{Z}, A)$

$\Rightarrow [a] = [a'] \in A/\mathrm{Tor}(\mathbb{Z}, A)$

Ex) $\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/(m,n)\mathbb{Z}$

$\mathbb{Q} \otimes \mathbb{Q} \cong \mathbb{Q}$

$r_1 \otimes r_2 \longmapsto r_1 r_2$

$r \otimes 1 \longmapsfrom r$

Ex) $\mathbb{Q} \otimes \mathbb{Q}/\mathbb{Z} \cong (\mathbb{Q} \otimes \mathbb{Q})/(\mathbb{Q} \otimes \mathbb{Z}) \cong \mathbb{Q}/\mathbb{Q} \cong 0$

$\mathbb{Q}/\mathbb{Z} \otimes \mathbb{Q}/\mathbb{Z} \cong (\mathbb{Q}/\mathbb{Z} \otimes \mathbb{Q})/(\mathbb{Q}/\mathbb{Z} \otimes \mathbb{Z}) \cong 0/\mathbb{Q}/\mathbb{Z} \cong 0$

A ring R is $(A \in Ab,\ A \otimes A \xrightarrow{m} A,\ \mathbb{Z} \xrightarrow{1} A)$

$aa' := a \otimes a'$, $(\mathcal{U}(A), m, 1)$ is a monoid

A homomorphism of rings $(A, m, 1) \to (A', m', 1')$

is a homomorphism of abelian groups $A \to A'$

and a homomorphism of monoids

Remark: Previously rings could be non-unital

Ex) $\mathbb{Z} \rightarrowtail \mathbb{Q} \rightarrowtail \mathbb{R} \rightarrowtail \mathbb{C} \rightarrowtail \mathbb{H}$

A commutative ring if $ab = ba$ $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ are commy rings

Ex) $\mathbb{Z}/m\mathbb{Z}$ is another commy ring

2^X $A, A' \in 2^X$ $A + A := A \oplus A' = (A \setminus A') \cup (A' \setminus A)$

$A \cdot A := A \cap A'$

Ring denotes category of rings

CRing denotes category of commutative rings

Ex) $\{0\}$ zero ring this is terminal object in Ring, CRing

$\mathbb{Z}$ is initial object in Ring and CRing

Ex) $GL_n(R)$ $n \geq 0$ and R a ring not

A matrix is a map of sets $I \times I \longrightarrow U(R)$

where I is the indexing set (finite)

$M, M' \in I \times I \Rightarrow M \cdot M' := \left(i,k \mapsto \sum_{j \in I} M_{i,j} \cdot M'_{j,i} \right)$

noncommy for $n > 1$

10-27-20

Affine Schemes $:= \mathrm{CRing}^{op}$

Lemma: if we have $A \xrightarrow{f} B$ a homomorphism of rings then its $\ker f: K \to A$ (as a homomorphism of abelian) satisfies the following property

if $k \in K$ and $a \in A$, then $k \cdot a \in K$ and $a \cdot k \in K$

Proof $f(k) = 0 \implies f(k \cdot a) = f(k) f(a) = 0 \cdot f(a) = 0$ $\square$

A two sided ideal over a ring A is an abelian subgroup $K < A$ s.t. $\forall a \in A$: $a \cdot k \in K$, $k \cdot a \in K$ for $k \in K$

Remark: Any subset $S \subset \mathcal{U}(A)$ generates an ideal $K < A$

Concretely, $K = \{ \sum_{\substack{\text{finite} \\ i \in I}} a_i s_i a_i' \mid a_i, a_i' \in A,\ s_i \in S \}$

K is the smallest ideal of A containing S.

Remark: Ideals are not subrings

Any subring B of a ring A contains the element $1 \in A$. If K is an ideal of A, and $1 \in K$ then $K = A$ because $\underset{\substack{\cap \\ K}}{1} \cdot \underset{\substack{\cap \\ A}}{a} = a \in K$

Ex) Let A be a ring

a) $\{0\} \subset A$ is an ideal of A

b) $A = A$ is an ideal of A

c) $A = \mathbb{Z}$: $m \in \mathbb{Z}$ $m\mathbb{Z}$ is an ideal of A

d) S a set, R a ring $\{S \xrightarrow{f} \mathcal{U}(R)\} = R^S$ is a ring

with point wise operations

Pick $T \subset S$. Consider $K = \{S \xrightarrow{k} \mathcal{U}(R) \mid k|_T = 0\}$

K is an ideal of R^S.

if $k \in K \Rightarrow k|_T = 0, \Rightarrow (f \cdot k)|_T = 0 \quad \forall f \in R^S$

Quotients of Ideals

Let R be a ring and I an ideal in R.

Then the quotient R/I is a homomorphism

$q : R \longrightarrow R/I$ that satisfies the following universal prop:

$q|_I = 0$ and $\forall r : R \longrightarrow T$ s.t. $r|_I = 0$

$\exists!\ R/I \xrightarrow{t} T$ such that the following diagram commutes

$$\begin{tikzcd} R \arrow[r, "q"] \arrow[d, "r"'] & R/I \arrow[dl, dashed, "\exists! t"] \\ T & \end{tikzcd}$$

Lemma Quotients of rings are unique up to unique isomorphism.

Proof Express quotients as initial objects in the following category:

Objects: homomorphisms of rings $r: R \to T$

$$R \xrightarrow{r} T \quad \text{where} \quad r|_I = 0$$

Morphisms:

$$\begin{tikzcd} R \arrow[r, "r"] \arrow[dr, "r'"'] \arrow[dr, phantom, shift left=3, "\#"] & T \arrow[d, "t"] \\ & T' \end{tikzcd}$$

This is a full subcategory of the slice category $R \mid \text{Ring}$ on objects r s.t. $r|_I = 0$

if quotients exist then they are initial in the constructed category. $\square$

Lemma Quotients of rings exist.

R a ring and I an ideal

Proof Construct the quotient of abelian groups:

$$q : R \to R/I$$

Endow it with the structure of a ring.

$$1 \in R \Rightarrow [1] \in R/I$$

$R/I \otimes R/I \longrightarrow R/I$ the multiplication

$$\begin{array}{ccc} [r_1] \,,\; [r_2] & \longmapsto & [r_1 r_2] \\ \shortparallel \quad\;\; \shortparallel & & \shortparallel \\ r_1 + I \quad r_2 + I & & r_1 r_2 + I \end{array}$$

$$\frac{(R \otimes R/I)}{I \otimes R/I} \cong R/I \otimes R/I \longrightarrow R/I$$ verify $R \otimes R/I$ vanishes on $I \otimes R/I$

$$\frac{R \otimes R}{R \otimes I} \cong R \otimes R/I \longrightarrow R/I$$ verify $R \otimes R$ vanishes on $R \otimes I$

$$\begin{tikzcd} r_1, r_2 \quad R \otimes R \arrow[r] \arrow[d, "\text{multiplication for } R"'] & R/I \\ R \arrow[ur] & {[r_1 r_2]} \\ r_1 \cdot r_2 & \end{tikzcd}$$

Vanishing on $R \otimes I$: $r \otimes i \longmapsto [\underbrace{r \cdot i}_{\in I}] = [0]$

Vanishing on $I \otimes R/I$: $i \otimes [r] \longmapsto [\underbrace{i \cdot r}_{\in I}] = [0]$

This multiplication is also associative and unital.

$\Rightarrow R/I$ is a ring and q homomorphism of rings.

WTS that R/I satisfies the universal property:

Let T be another ring w/ ring homomorphism $r: R \to T$ s.t. $r|_I = 0$.

$$\begin{tikzcd} R \arrow[r, "q"] \arrow[d, "r"'] & R/I \arrow[dl, dashed, "\exists! t"] \\ T & \end{tikzcd}$$

where t is a homomorphism of abelian groups

WTS t is a ring homomorphism

Recall $t([s]) = r(s)$ $s \in R$ from construction of quotient in Ab

$t([s_1] \cdot [s_2]) = t([s_1 s_2]) = r(s_1 s_2) = r(s_1) r(s_2) = t([s_1]) t([s_2])$

$t([1]) = r(1) = 1$

$\Rightarrow t$ is a ring homomorphism $\square$

Typical Application

Given a ring homomorphism $R \xrightarrow{f} T$.

Verify $f|_I = 0$ where I is an ideal of R

Conclude there is a homomorphism $R/I \to T$

Ex) R a ring $R/\{0\} \cong R$ and $R/R \cong \{0\}$

Ex) $\mathbb{Z}$, $m \in \mathbb{Z}$ $\mathbb{Z}/m\mathbb{Z}$ (a ring of residues modulo m)

Ex) R a ring S set $T \subset S$ $I \subset R^S$ if $f \in I \Leftrightarrow f|_T = 0$

an ideal

then $R^S / I \cong R^T$

$$R^S/I \longrightarrow R^T$$

$$[f] \longmapsto f|_T \qquad [f]=[f'] \iff f-f' \in I$$

$$\overset{\shortparallel}{[f']} \longmapsto f'|_T \qquad \iff (f-f')|_T = 0$$

$$\iff f|_T = f'|_T$$

$\Rightarrow$ This is well defined.

$$R^T \longrightarrow R^S/I$$

$$g \longmapsto [g'] \quad \text{where} \quad g'(s) = \begin{cases} g(s) & s \in T \\ 0 & s \notin T \end{cases}$$

These maps give the isomorphism.

10 - 29 - 20

Ring $\longrightarrow$ Group A functor from Ring to Group

$R \longmapsto \{r \in R \mid \exists r' \in R : rr' = r'r = 1\} = R^\times$

$R \to S \longmapsto R^\times \longrightarrow S^\times$

Ex) $\mathbb{Z}^\times = \{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z}$

$\mathbb{Q}^\times = \mathbb{Q} \setminus \{0\}$

$\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$

$\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$

$R \setminus \{0\}$ where R is a ring s.t. $1 \neq 0$ and has no zero divisors

1) A monoid — R is a domain
2) a commutative monoid $\Longrightarrow$ R is an integral domain
3) group — R is a division ring (skew field)
4) abelian group — R is a field

Ex) Fields

- $\mathbb{Q}, \mathbb{R}, \mathbb{C}$
- $\mathbb{Z}/p\mathbb{Z}$ where p is prime

Ex) Division Ring

- $\mathbb{H}$ the quaternions $\mathbb{H} = \langle r \in \mathbb{R}, i, j, k \mid \mathbb{R},\ \overset{\text{"} -ji}{ij = k},\ \overset{\text{"} -kj}{jk = i},\ \overset{\text{"} -ik}{ki = j} \rangle$

$i^2 = j^2 = k^2 = -1$

Ex) Integral domain

- $\mathbb{Z}$
- R^S, $S \in \mathrm{Set}$ $R \in \mathrm{Ring}$ is <u>never</u> a domain when $|S| > 1$
- $\mathbb{Z} + i\mathbb{Z}$ is integral domain

A functor the group-ring constructor

$$\begin{aligned} \mathrm{Group} &\longrightarrow \mathrm{Ring} \\ G &\longmapsto \mathbb{Z}[G] \end{aligned}$$

$\mathbb{Z}[\mathcal{U}(G)]$ is an abelian group under addition $\left(\mathbb{Z}[\mathcal{U}(G)] = \bigoplus_{g \in G} \mathbb{Z} \right)$

$$\begin{tikzcd} \mathbb{Z}[\mathcal{U}(G)] \otimes \mathbb{Z}[\mathcal{U}(G)] \arrow[r] \arrow[d, phantom, "\cong" rotate=-90] & \mathbb{Z}[\mathcal{U}(G)] \\ \mathbb{Z}[\mathcal{U}(G) \times \mathcal{U}(G)] \arrow[ur, bend right] & \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{U}(G) \times \mathcal{U}(G) \arrow[r] \arrow[d, "\cdot"] & \mathcal{U}(\mathbb{Z}[\mathcal{U}(G)]) \\ \mathcal{U}(G) \arrow[ur, bend right] & \end{tikzcd}$$

multiplication in G

A concrete description $\mathbb{Z}[G]$: elements are $\sum_{\substack{g \in G \\ \text{finite}}} \alpha_g \cdot g$ $\alpha_g \in \mathbb{Z}$

$$\left(\sum_{g \in G} \alpha_g \cdot g \right) \left(\sum_{h \in G} \beta_h \cdot h \right) = \sum_{g, h \in G} (\alpha_g \beta_h) \cdot (gh)$$

mult id: $1 \cdot 1_G$

Ex) 1) $G = 1$ $\quad \mathbb{Z}[G] = \mathbb{Z}$ $\quad$ Laurent Polynomials $(\mathbb{Z}[x, x^{-1}])$

2) $G \cong \mathbb{Z}/2\mathbb{Z}$ $\quad \mathbb{Z}[G] = \{ \alpha \cdot [0] + \beta[1] \mid \alpha, \beta \in \mathbb{Z} \}$

$$(\alpha[0] + \beta[1]) \cdot (\alpha'[0] + \beta'[1]) = (\alpha\alpha' + \beta\beta')[0] + (\alpha\beta' + \beta\alpha')[1]$$

3) $\mathbb{Z}[\mathbb{Z}]$ $\quad \sum\limits_{\substack{n \in \mathbb{Z} \\ \text{finite}}} a_n \cdot x^n$ $\quad$ where $\quad x = \underset{\mathbb{Z}}{\underset{\cap}{1}} \cdot \underset{\mathbb{Z}}{\underset{\cap}{1}} \in \mathbb{Z}[\mathbb{Z}]$

The universal property of group rings

$G \in \mathrm{Group}$ and $R \in \mathrm{Ring}$

$$\{\mathbb{Z}[G] \longrightarrow R\} \cong \{G \longrightarrow R^\times\}$$

Prop $\mathbb{Z}[G]$ actually satisfies the group ring universal property

<u>Proof</u> $\{\mathbb{Z}[G] \xrightarrow{f_1} R\} \longrightarrow \{\mathbb{Z}[\mathcal{U}(G)] \xrightarrow{f_2} \mathcal{U}_{Ab}(R)\} \cong \{\mathcal{U}(G) \xrightarrow{f_3} \mathcal{U}(R)\}$

f_1 preserves multiplication $\Longleftrightarrow$ preserves mult. on basis elements

$\Longleftrightarrow$ f_3 preserves mult.

$$\begin{matrix} f_1(g_1 \overset{\mathbb{Z}[G]}{\cdot} g_2) & = & f_1(g_1 \overset{G}{\cdot} g_2) \\ \| & & // \\ f_1(g_1) f_1(g_2) & & \end{matrix} \quad \Longleftrightarrow \quad f_3(g_1 \cdot g_2) = f_3(g_1) f_3(g_2)$$

$$\Rightarrow \{\mathbb{Z}[G] \longrightarrow R\} \longrightarrow \{G \longrightarrow \overset{\text{monoid}}{(R, \cdot, 1)}\} \longrightarrow \{\mathcal{U}(G) \longrightarrow \mathcal{U}(R)\}$$

$$\begin{tikzcd} G \arrow[r] \arrow[dr] & (R, \cdot, 1) \\ & R^\times \arrow[u] \end{tikzcd}$$

Since homomorphisms in $\{\mathbb{Z}[G] \to R\}$ factor into homomorphisms of monoids which factor as $R^\times$ we obtain $\{\mathbb{Z}[G] \longrightarrow R\} \cong \{G \longrightarrow R^\times\}$. $\square$

Generalizations of Group Ring Construction.

- $G \rightsquigarrow$ any monoid
- $\mathbb{Z} \rightsquigarrow$ any ring

Result $R[M]$ The monoid algebra

$$\mathrm{Ring} \times \mathrm{Mon} \longrightarrow \mathrm{Ring}$$
$$(R, M) \longmapsto R[M]$$

Let R be a ring. $R[x] := R[\mathbb{Z}_{\geq 0}]$

$$\sum_{\substack{n \geq 0 \\ \text{finite}}} f_n \cdot x^n \qquad f_n \in R$$

$$R[x_1, x_2, \dots, x_n] := R[\mathbb{Z}_{\geq 0}^n]$$

Remark: The universal property of Group Rings can be extended to monoid rings.

Ex) $R[x] \xrightarrow{ev} R^{\mathcal{U}(R)}$ The Evaluation homomorphism

$$\begin{array}{c} f \longmapsto \left(r \mapsto \sum_{n \geq 0} f_n r^n \right) \\ \| \\ \sum_{n \geq 0} f_n x^n \end{array}$$

ev can have a non-trivial kernel.

NSF Topology grant app deadline 11-3-20

Let $A \in Ab$ the endomorphism ring of A is

$End\, A = \{ f : A \to A \} \in Ab$ group w/ pointwise operations

$End\, A \otimes End\, A \xrightarrow{\circ} End\, A$

$id_A \in End\, A$

Ex) a) $End(0) = 0$

b) $End(\mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/m\mathbb{Z}$

c) $End(\mathbb{Z}) = \mathbb{Z}$

d) $End(\mathbb{Z}^n) = Mat_n(\mathbb{Z})$

Let R be a ring. A left R-module M is a homomorphism of rings $R \xrightarrow{\rho} End\, A$, where A is the underlying abelian group of M

A left R-module M is an abelian group A together with

$R \otimes A \xrightarrow{\cdot} A \iff R \xrightarrow{\rho} Hom(A, A) = End\, A$

$r, a \longmapsto r \cdot a$

such that $\cdot$ is Associative: $r_1 \cdot (r_2 \cdot a) = (r_1 \cdot r_2) \cdot a$

Unitality: $1 \cdot a = a$

$\rho(r) = (a \mapsto r \cdot a)$

$\rho(r_1 r_2) = (a \mapsto (r_1 r_2) \cdot a) = (a \mapsto r_1 \cdot a) \circ (a \mapsto r_2 \cdot a)$

A left R-module is an abelian group A with $R \times A \xrightarrow{\cdot} A$ such that $\cdot$ is bilinear, associative, and unital.

A right R-mod

① $R \to (\mathrm{End}\,A)^{op}$

② $A \otimes R \longrightarrow A$

<u>Remark</u> $\mathrm{Ring} \xrightarrow{op} \mathrm{Ring}$

$(A, A\otimes A \xrightarrow{m} A, \mathbb{Z} \xrightarrow{1} A) \longmapsto (A, m\circ br, 1)$

$br: A\otimes A \to A\otimes A$

$a_1 \otimes a_2 \mapsto a_2 \otimes a_1$

<u>Remark</u>: if $R \in \mathrm{CRing}$, then $R \xrightarrow[op]{\cong} R^{op}$

$\Rightarrow$ right R-mod is left R-mod

<u>Ex</u> a) $R = \{0\}$

Any R-mod $\{0\} \xrightarrow{\rho} \mathrm{End}\,A = \{0\} \Rightarrow A = 0$

b) $R = \mathbb{Z}$ $\qquad \mathbb{Z} \xrightarrow[\exists!]{\rho} \mathrm{End}\,A$

$\Rightarrow$ $\mathbb{Z}$-mods $\equiv$ abelian groups

for $a \in A$ $\qquad n\cdot a = \underbrace{a + a + \cdots + a}_{n\text{-times}}$

c) R a division ring $(\mathbb{H}, \mathbb{C}, \mathbb{R}, \mathbb{Q})$

R-mod is just a vector space

Any vector space $V \cong \bigoplus_{i\in I} R$ $\quad I$ is called the basis

Note if R-mod $\cong \bigoplus_{i\in I} R$ then it is called free.

${}_R\mathrm{Mod}$, Mod_R category of left modules (right)

Obj = R-mods

Mor = homomorphisms of R-mods

$f: M_1 \to M_2$ s.t. R-linear $f(r\cdot a) = r\cdot f(a)$

Ex) Let S be a set

R a ring

Then $R^S = \{ S \to \mathcal{U}(R) \}$ is a ring.

Assume S is finite. Any R^S-mod M decomposes as follows

$A = \mathcal{U}_{Ab}(M) \cong \bigoplus_{s\in S} A_s$ where A_s is a R-module

$$(r_s)_{s\in S} \cdot (a_s)_{s\in S} = (r_s \cdot a_s)_{s\in S}$$

$$R \otimes \bigoplus_s A_s \longrightarrow \bigoplus_s A_s$$

$e_s \in R^S$ $\quad e_s(t) = \begin{cases} 1 \in R & s=t \\ 0 \in R & s\neq t \end{cases}$

Take $A_s := e_s \cdot M$.

R acts on A_s : $r\cdot(e_s\cdot m) := e_s \cdot \iota(r)\cdot m$ $\qquad \iota: R \to R^S$, $r \mapsto (s \mapsto r)$

$\Rightarrow A_s$ is an R-mod

$1 \in R^S = \sum_{s\in S} e_s$ $\qquad \underset{\in M}{m} = 1\cdot m = (\sum_{s\in S} e_s)\cdot m = \sum_{s\in S} e_s \cdot m \in \bigoplus_s A_s$ $\qquad \square$

11-5-20

Limits and Colimits

Limits: products, kernels, fiber products (pullbacks), terminal object

Colimits: coproducts, cokernels, pushouts, initial object

Input data: diagram

A **diagram** is a functor $\mathcal{D}: I \to V$ where
I is the indexing category.

Ex) a) $I = \phi$ $\quad \exists!\ \phi \to V$ the empty diagram

limit: terminal $\qquad$ colimit: initial

b) I only has id morphisms. A functor $I \to V$ is a family of objects $\{V_i\}_{i \in I}$

limit: (co) product

c) Walking pair I: $0 \rightrightarrows 1$

A functor $I \to V$ is a diagram $X_0 \underset{g}{\overset{f}{\rightrightarrows}} X_1$

The limit of this diagram are (co) equivalizers

if g is trivial then (co) kernels

d) I:
$$\begin{array}{ccc} & & 2 \\ & & \downarrow \\ 1 & \longrightarrow & 0 \end{array}$$
A functor $I \xrightarrow{\mathcal{D}} V$ is
$$\begin{array}{ccc} & & X_2 \\ & & \downarrow \\ X_1 & \longrightarrow & X_0 \end{array}$$

limits: pullback

e) $0 \to 1 \to 2 \to 3 \to 4 \to \cdots$

A functor $I \to V$ is

$$X_0 \to X_1 \to X_2 \to X_3 \to X_4 \to \cdots$$

cotower

colimit: "direct limit"

e') $0 \leftarrow 1 \leftarrow 2 \leftarrow 3 \leftarrow \cdots$

$$X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow X_3 \leftarrow \cdots$$

tower limit = "inverse limit"

Given $\mathcal{D}: I \to V$. The limit of $\mathcal{D}$ is the terminal object in the category of cones over $\mathcal{D}$

Similarly the colimit of $\mathcal{D}$ is the initial object in the category of cocones under $\mathcal{D}$

Let $\mathcal{D}: I \to V$ is a diagram.

The category of cones over $\mathcal{D}$:

Obj: $(A \in V, \{p_i: A \to \mathcal{D}(i)\}_{i \in I})$

Such that $\forall i \xrightarrow{h} j$ (morphisms in I)

$$\begin{tikzcd} & A \arrow[dl, "p_i"'] \arrow[dr, "p_j"] & \\ \mathcal{D}(i) \arrow[rr, "D(h)"'] \arrow[rr, phantom, shift left=3, "\#"] & & \mathcal{D}(j) \end{tikzcd}$$

commutes

morphisms: $(A, \{p_i\}_{i\in I}) \longrightarrow (A', \{p_i'\}_{i \in I})$

a morphism $a: A \to A'$ in V

s.t. $\forall i \in I$

$$\begin{tikzcd} A \arrow[rr, "a"] \arrow[dr, "p_i"'] \arrow[rr, phantom, shift right=3, "\#"] & & A' \arrow[dl, "p_i'"] \\ & \mathcal{D}(i) & \end{tikzcd}$$

commutes

Cocones:

$$\begin{tikzcd} & A & \\ \mathcal{D}(i) \arrow[ur, "\iota_i"] \arrow[rr, "\mathcal{D}(h)"'] & \# & \mathcal{D}(j) \arrow[ul, "\iota_j"'] \end{tikzcd}$$

and

$$\begin{tikzcd} A \arrow[rr] & \# & A' \\ & \mathcal{D}(i) \arrow[ul, "\iota_i"] \arrow[ur, "\iota_i'"'] & \end{tikzcd}$$

arrows in the opposite directions

Ex) a) $I = \emptyset$ Cone $\equiv$ apex

mor of cone $\equiv$ mor of apex

Cat cones $\equiv$ cat cocones

$\lim \mathcal{D}$ = terminal object of V

$\operatorname{colim} \mathcal{D}$ = initial object of V

b) I has only id morphisms

$\Rightarrow$ $\lim \mathcal{D} = \prod_{i \in I} \mathcal{D}(i)$ product

$\operatorname{colim} \mathcal{D} = \coprod_{i \in I} \mathcal{D}(i)$ coproduct

c) $I: 0 \rightrightarrows 1$

A cone:

$$\begin{tikzcd} D_0 \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] & \# & D_1 \\ & A \arrow[ul, "p_0"] \arrow[ur, "p_1"'] & \end{tikzcd}$$

where $p_1 = \boxed{f \circ p_0 = g \circ p_0}$

redundant can be computed w/ p_0

$$\begin{tikzcd} & A \arrow[dl, "p_0"'] \arrow[dr, "p_1"] \arrow[dd, "a"] & \\ D_0 \arrow[rr, shift left] \arrow[rr, shift right] & & D_1 \\ & A' \arrow[ul, "p_0'"] \arrow[ur, "p_1'"'] & \end{tikzcd}$$

$\lim \mathcal{D} = \text{equalizer}(f, g)$

Prop: $V = \text{Set}$ $\quad eq(f,g) = \{x \in \mathcal{D}_0 \mid f(x) = g(x)\}$

$p_0(x) = x$

$p_1(x) = f(x) = g(x)$

11-10-20

Recall a diagram is a functor $\mathcal{D} : I \to V$

$$I = \left\{ \begin{tikzcd} 0 \arrow[r, "g"] \arrow[d, "f"'] & 2 \\ 1 \end{tikzcd} \right\} \xrightarrow{\mathcal{D}} \begin{tikzcd} D(0) \arrow[r, "D(g)"] \arrow[d, "D(f)"'] & D(2) \\ D(1) \end{tikzcd}$$

The limit is the terminal object in the category of cones over $\mathcal{D}$

Prop $I = \{ 0 \underset{g}{\overset{f}{\rightrightarrows}} 1 \} \xrightarrow{\mathcal{D}} V$

$$\mathcal{D}(0) \underset{D(g)}{\overset{D(f)}{\rightrightarrows}} \mathcal{D}(1)$$

$\lim \mathcal{D} =$ equalizer

if $V = Set$ $\quad eq(\mathcal{D}(f), \mathcal{D}(g)) = \{ x \in \mathcal{D}(0) \mid \mathcal{D}(f)(x) = \mathcal{D}(g)(y) \}$

Proof

$$\begin{tikzcd} & \mathcal{D}_0 \arrow[r, shift left, "\mathcal{D}f"] \arrow[r, shift right, "\mathcal{D}g"'] & \mathcal{D}_1 \\ A \arrow[ur, "q"] \arrow[r, dashed] & eq(\mathcal{D}f, \mathcal{D}g) \arrow[u] & \end{tikzcd}$$

$\exists!$ the corestriction of q

$\forall a \in A \quad \mathcal{D}f(q(a)) = \mathcal{D}g(q(a))$ b/c cones

$\Rightarrow q(a) \in eq(\mathcal{D}f, \mathcal{D}g)$

$\square$

limits and colimits in Set

Prop $\mathcal{D}: I \to \mathrm{Set}$. Assume I is small

$\lim \mathcal{D} = \left\{ x \in \prod_{i \in I} \mathcal{D}(i) \mid \forall f: i \to j : \mathcal{D}f(x_i) = x_j \right\}$

Proof Given any cone A, $p_i' : A \to \mathcal{D}_i$

$$\begin{tikzcd} & A \arrow[dl, "p_i'"'] \arrow[dr, "p_j'"] & \\ D_i \arrow[rr, "\mathcal{D}(f)"'] & & D_j \end{tikzcd}$$

WTS $\exists!$ $A \to \lim \mathcal{D}$ s.t.

$$\begin{tikzcd} A \arrow[r, dashed] \arrow[dr, "p_i'"'] & \lim \mathcal{D} \arrow[d, "p_j"] \arrow[dl, phantom, "\#"] \\ & \mathcal{D}_i \end{tikzcd}$$

Observe that the composition $A \to \lim \mathcal{D} \hookrightarrow \prod \mathcal{D}_i$

must be equal to $(p_i')_{i \in I}$

It now remains to show that the image of $(p_i')_{i \in I}$ is a subset of the lim. this follows from commutivity of cone $\square$

Prop $\mathcal{D}: I \to \mathrm{Set}$. Assume I is small.

$\operatorname{colim} \mathcal{D} = \coprod_{i \in I} \mathcal{D}_i / \sim$ $\qquad \iota_i : \mathcal{D}_i \to \operatorname{colim} \mathcal{D}$ $\quad x_i \mapsto [x_i]$

where $\forall f: i \to j$, $\forall x_i \in \mathcal{D}_i$: $\mathcal{D}f(x_i) \sim x_j$

Verify
$$\begin{tikzcd} D_i \arrow[r] \arrow[dr] & D_j \arrow[d] \arrow[dl, phantom, "*"] \\ & \operatorname{colim} \mathcal{D} \end{tikzcd}$$

$x_i \mapsto \mathcal{D}f(x_i) \mapsto [\mathcal{D}f(x_i)] = [x_i] \mapsfrom x_i$

$\Rightarrow$ this is a cocone

colim satisfies that for any set A and $\iota_i' : \mathcal{D}_i \to A$

then
$$\begin{tikzcd} \mathcal{D}_i \arrow[r] \arrow[d] & A \arrow[dl, phantom, "\#"] \\ \mathcal{D}_j \arrow[ur] & \end{tikzcd}$$
commutes and
$$\begin{tikzcd} \operatorname{colim} \mathcal{D} \arrow[rr, dashed, "\exists!"] & \# & A \\ & \mathcal{D}_i \arrow[ul] \arrow[ur] & \end{tikzcd}$$

A variety of algebras is a category

Obj: Set (finite collection of sets) X

w/ a finite collection of operations $X^k \to X$ $(k \geq 0)$ $\underset{\text{arity}}{k}$

Satisfying identities of the form

---- = ---

Examples

- Set, Monoid, Group, Ab, Ring, CRing, Module, Mod_R, Set_G

Non examples

- Fields, integral domain, division rings, domains

Thm Any variety of algebra V has all small limits and colimits

The forgetful functor $V \to Set^S$ creates

(preserves and reflects limits)

11-12-20

A functor $F : \mathcal{C} \to \mathcal{D}$ preserves limits if the image under F of any limit cone in $\mathcal{C}$ is a limit cone in $\mathcal{D}$.

$$\begin{tikzcd} & X \times Y \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd} \overset{F}{\longmapsto} \begin{tikzcd} & F(X \times Y) \arrow[dl] \arrow[dr] & \\ F(X) & & F(Y) \end{tikzcd}$$

i.e., $F(X \times Y) \cong F(X) \times F(Y)$

$\Psi : I \to \mathcal{C}$

$$F(\lim \Psi) \cong \lim (F \circ \Psi)$$

- reflects limits if any cone in $\mathcal{C}$ whose image in $\mathcal{D}$ is itself a limit cone in $\mathcal{C}$

Proof

a) limits

$\mathcal{D} : I \to \mathcal{V}$ and $\mathcal{V} \xrightarrow{\mathcal{U}} \mathrm{Set}$

Construct $\lim \mathcal{D}$ as follows

$$\mathcal{U}(\lim \mathcal{D}) := \lim(\mathcal{U} \circ \mathcal{D}) = \lim_{i \in I} (\mathcal{U}(\mathcal{D}(i))) \qquad (1)$$

$$= \left\{ x \in \prod_{i \in I} \mathcal{U}(\mathcal{D}(i)) \;\middle|\; \forall f : i \to j \;\; \mathcal{D}f(x_i) = x_j \right\}$$

Operations on $\mathcal{U}(\lim \mathcal{D})$ are defined index wise, e.g., for a binary operation m

$m(x, x') := (m(x_i, x_i'))_{i \in I}$ Recall $\mathcal{D}f$ preserves all operations

$$\mathcal{D}f(m(x_i, x_i')) = m(\mathcal{D}f(x_i), \mathcal{D}f(x_i')) = m(x_j, x_j')$$

Projection morphisms: $p_i : \lim \mathcal{D} \to \mathcal{D}(i)$ (2)

$x \mapsto x_i$

again this preserves all the defined operations.

The universal property:

Let $A \in V$, $q_i : A \to \mathcal{D}(i)$ s.t.
$$\begin{tikzcd} & A \arrow[dl, "q_i"'] \arrow[dr, "q_j"] \arrow[d, phantom, "\#"] & \\ \mathcal{D}(i) \arrow[rr] & & \mathcal{D}(j) \end{tikzcd}$$

Need to show $\exists!\ A \xrightarrow{\alpha} \lim \mathcal{D}$ s.t.
$$\begin{tikzcd} A \arrow[r, dashed, "\alpha"] \arrow[dr, "q_i"'] \arrow[r, phantom, shift right=3, "\#"] & \lim \mathcal{D} \arrow[d, "p_i"] \\ & \mathcal{D}(i) \end{tikzcd}$$

This condition implies that $\alpha(a) = (q_i(a))_{i \in I}$

this is the only choice to make the diagram commute.

Since $\mathcal{D}f(q_i(a)) = q_j(a)$ q_i lands in $\lim \mathcal{D}$

α preserves all operations

Preservation of limits (under $\mathcal{U}$): (1) and (2) by construction

Reflection of limits under $\mathcal{U}$:

$A \in \mathcal{V}$ $p_i : A \to \mathcal{D}(i)$ is a cone in $\mathcal{V}$

$\mathcal{U}(A) \in \mathrm{Set}$ $\mathcal{U}(A) \longrightarrow \mathcal{U}(\mathcal{D}(i))$ is a limit cone in Set.

Consider the morphism $A \to \lim \mathcal{D}$

Applying $\mathcal{U}$: $\mathcal{U}(A) \xrightarrow{\cong} \mathcal{U}(\lim \mathcal{D}) \overset{\text{pres lim}}{=} \lim \mathcal{U} \circ \mathcal{D}$

since these are terminal cones over Set

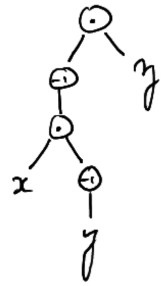
$\Rightarrow$ $A \to \lim \mathcal{D}$ is an isomorphism.

Colimits: $\mathcal{D}: I \to \mathcal{V}$

To construct underlying set $\mathcal{U}(\operatorname{colim} \mathcal{D})$

Consider $\{\text{trees on } \mathcal{V}\} / \sim$

tree : $(x \cdot y^{-1})^{-1} \cdot y \Rightarrow$



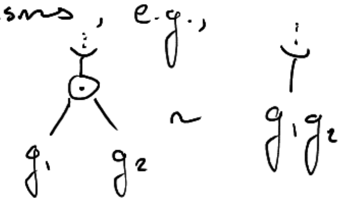
leafs are $x \in \coprod_{i \in I} \mathcal{U}(\mathcal{V})$

nodes are opperations in $\mathcal{V}$

$\sim$ is the equivalence relation generated by

- the variety's identities, e.g., in Group $x \cdot (y \cdot z) = (x \cdot y) \cdot z$
- relations that force the injective maps $\mathcal{U}(\mathcal{D}(i)) \longrightarrow Q$ to be homomorphisms, e.g.,

$g_1, g_2 \in \mathcal{D}(i)$



11-17-20

An equivalence of categories is a functor $F: \mathcal{C} \to \mathcal{D}$ together with an inverse functor $G: \mathcal{D} \to \mathcal{C}$ and isomorphisms

$\eta_c : c \xrightarrow{\cong} G(F(c)) \ \forall c \in \mathcal{C}$ $\quad \varepsilon_d : F(G(d)) \xrightarrow{\cong} d \ \forall d \in \mathcal{D}$

such that $\forall c \xrightarrow{f} c'$
$\forall d \xrightarrow{g} d'$

$$\begin{tikzcd} c \arrow[r, "f"] \arrow[d, "\eta_c"'] \arrow[dr, phantom, "\#"] & c' \arrow[d, "\eta_{c'}"] \\ G(F(c)) \arrow[r, "G(F(f))"'] & G(F(c')) \end{tikzcd} \qquad \begin{tikzcd} d \arrow[r, "g"] \arrow[dr, phantom, "\#"] & d' \\ F(G(d)) \arrow[u, "\varepsilon_d"] \arrow[r, "F(G(g))"'] & F(G(d')) \arrow[u, "\varepsilon_{d'}"'] \end{tikzcd}$$

η is the natural isomorphism $id_{\mathcal{C}} \to G \circ F$

ε is the natural isomorphism $F \circ G \to id_{\mathcal{D}}$

Remark: For any equivalence of categories $F: \mathcal{C} \to \mathcal{D}$

$\forall c, c' \in \mathcal{C}$ $\qquad \hom_{\mathcal{C}}(c, c') \xrightarrow{\cong} \hom_{\mathcal{D}}(F(c), F(c'))$

An algebra (associative) is a morphism of rings

$k \longrightarrow A$

k: ring of coefficients (commutative) and $im(k) \subset Z(A)$

A: algebra

Ex) $\mathbb{C}$ and $\mathbb{H}$ are real algebras.

Ex) k a commutative ring and S a set

k^S is a k algebra

$$\mathrm{Set}^{op} \longrightarrow \mathrm{Alg}_k$$

$$S \longmapsto k^S \qquad \text{Let } x \in k^{S'} \quad x : \begin{array}{c} S' \\ \downarrow \\ k \end{array} \longmapsto x \circ f$$

$$\begin{array}{c} S \\ f \downarrow \\ S' \end{array} \longmapsto \begin{array}{c} k^{S'} \\ \downarrow \\ k^S \end{array}$$

Let Ψ be a full subcategory of Alg_k given by the essential image of $\mathrm{FinSet} \longrightarrow \mathrm{Alg}_k$

i.e., algebras $A \in \mathrm{Alg}_k$ that are isomorphic to k^S for a finite $S \in \mathrm{Set}$

Prop $A \in \mathrm{Alg}_k$ $A \in \Psi$ iff $\exists$ finite collection of idempotent elements $e_i \in A$: $e_i^2 = e_i$ s.t.

- $\sum_i e_i = 1$
- $e_i e_j = 0$ if $i \neq j$
- $A \cdot e_i \cong k$ as a module over k.

Proof

Pick $S \in \mathrm{FinSet}$. $A = k^S$ and $e_i \in A$: $e_i(s) = \begin{cases} 0 & s \neq i \\ 1 & s = i \end{cases}$

$\therefore$ $e_i^2 = e_i$ $\wedge$ $e_i e_j = 0$ if $i \neq j$ $\wedge$ $\sum_i e_i = 1$

also $A \cdot e_i = \{ x \in k^S \mid x(j) = 0 \ \forall j \neq i \} \cong k$

Now let $\{e_i\}_{i \in I}$ (I finite) let $S = I$

$A \longrightarrow k^S \qquad a \longmapsto (s \longmapsto a \cdot e_s)$

$k^S \longrightarrow A \qquad x \longmapsto \sum_{s \in S} x_s \cdot e_s$ $\square$

$$\Psi \longrightarrow \mathrm{FinSet}^{op}$$

$$A \longmapsto \hom_{\mathrm{Alg}_k}(A, k)$$

$$\begin{array}{ccccc} A & & \hom_{\mathrm{Alg}_k}(A', k) & & g \\ f \downarrow & \longmapsto & \downarrow & & \downarrow \\ A' & & \hom_{\mathrm{Alg}_k}(A, k) & & g \circ f \end{array}$$

11-19-20

Last time, for k a field,

$$\Psi \rightleftarrows \mathrm{FinSet} \qquad k^S \longleftarrow S \qquad A \longmapsto \hom(A,k)$$

$$\begin{array}{c}\Psi\\ \cap \\ \mathrm{Alg}_k\end{array} \qquad \begin{array}{c} k^S \\ \uparrow \\ k^{S'}\end{array} \longleftarrow \begin{array}{c} S \\ \downarrow \\ S'\end{array} \qquad \begin{array}{c} A \\ \downarrow \\ A'\end{array} \longmapsto \begin{array}{c} \hom(A',k) \\ \downarrow \\ \hom(A,k)\end{array}$$

dictionary

geometry	algebra
• finite set S	• k-algebra in Ψ
• map of finite sets $S \to S'$	• homomorphisms of k-alg $A' \to A$
• $\emptyset$	• 0
• $\{*\}$	• k
• $\{*\} \to S$	• $A \to k$
• $S' \rightarrowtail S$	• $A \twoheadrightarrow A'$ or $\overset{\text{ideal}}{I} < A$
$\emptyset \rightarrowtail S$	$I = A \longrightarrow A \to 0$
$S \rightarrowtail S$	$I = 0 \longrightarrow A \to A$
• $\{V_s\}_{s \in S}$ and V_s is a vector space over k, "vector bundle"	• A module M over A

Today we want to generalize this equivalence to

$$\mathrm{CRing}^{op} \rightleftarrows \text{Affine Schemes}$$

$$\mathrm{CRing}^{op} \xrightarrow[\text{Zariski Spectrum}]{} \text{Topological Space}$$

(1914, Hausdorff)

Def A topological space is a pair $(X, \mathcal{U})$

X : Set and $\mathcal{U} \subset 2^X$ (open sets) s.t.

$\mathcal{U}$ is closed under arbitrary unions

and finite intersections

$\therefore\ \emptyset, X \in \mathcal{U}$

Ex) Let (X, d) be a metric space

i.e., $(X, \mathcal{U})$ $V \in \mathcal{U}$ if V can be represented as a union of open balls in (X, d)

Def A continuous map $(X, \mathcal{U}) \longrightarrow (X', \mathcal{U}')$ is a map of sets $X \to X'$ s.t.

$\forall V' \in \mathcal{U}' : f^* V' \in \mathcal{U}$ (f^* is pre image)

Ex) Met $\longrightarrow$ Top

$(X, d) \longmapsto (X, \mathcal{U})$

$$\begin{tikzcd} (X, d) \arrow[d] & \longmapsto & (X, \mathcal{U}) \arrow[d] \\ (X', d) & & (X', \mathcal{U}') \end{tikzcd}$$

Crash course in general topology!

Def A **frame** is a poset F that has all finite infima (meets $\wedge$), arbitrary suprema (joins $\vee$), and for any element $a \in F$

$$\begin{aligned} F &\longrightarrow F \\ b &\longmapsto a \wedge b \end{aligned}$$ preserves suprema

$$\left(\bigvee_{i \in I} b_i\right) \wedge a = \bigvee_{i \in I} (b_i \wedge a)$$

Def A **homomorphism of frames** is a map of posets that preserves that preserves finite infima and arbitrary suprema.

(Loc)

Def **Locale** is the opposite category of Frm

Ex) Top $\longrightarrow$ Loc

$(X, \mathcal{U}) \longmapsto (\mathcal{U}, \subset)$ ordering on $\mathcal{U}$

in this case $\wedge = \cap$ (finite)

$\vee = \cup$ (arbitrary)

not quite an equivalence (almost though!)

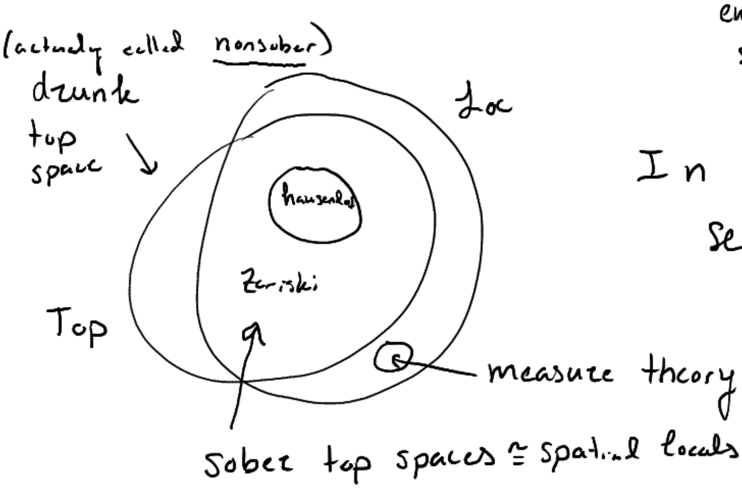
Def $\mathcal{L}oc \xrightarrow{Sp} Top$

$$\mathcal{L} \longmapsto (\hom_{\mathcal{L}oc}(\mathbb{1}, \mathcal{L}), \mathcal{U})$$

($\mathcal{U}$ can be defined but we are not gonna do it lol)

$\mathcal{L}oc$ has a terminal object $\mathbb{1} = \{0 < 1\}$

0 ← empty set, 1 ← singleton



In sober top space you don't see double hence the name

There is basically no use for "drunk" top spaces we typically throw them out

Recall

$$\begin{tikzcd} CRing^{op} \arrow[rr, "\text{Zariski Spectrum}"'] \arrow[dr] & & Top \\ & \mathcal{L}oc \arrow[ur, "Sp"'] & \end{tikzcd}$$

Def $CRing \xrightarrow[\text{Zariski Spectrum}]{Zar} Frm$

$$A \longmapsto \overset{Zar(A)}{\overbrace{\{\text{radical ideals of } A, \subset\}}}$$

$$\begin{array}{ccccc} A & & Zar(A) & & I \\ f \downarrow & \longmapsto & \downarrow & & \downarrow \\ A' & & Zar(A') & & Rad(f_*(I)) \text{ also denoted } \sqrt{f_*(I)} \end{array}$$

Def The radical is $\mathrm{Rad}(I)$ or $\sqrt{I}$ of an ideal I of $A \in \mathrm{Ring}$ is $\mathrm{Rad}(I) := \{ a \in A \mid \exists n > 0 : a^n \in I \}$

I is called a radical ideal if $I = \mathrm{Rad}(I)$

Proof that $\mathrm{Zar}(A)$ is a frame:

I_1, I_2 are radical ideals of A

$I_1 \cap I_2$ is an ideal. WTS it is radical ideal

WTS if $a \in A : a^n \in I_1 \cap I_2$, then $a \in I_1 \cap I_2$

$\therefore$ $a^n \in I_1$ and $a^n \in I_2$ which are radical

$\therefore$ $a \in I_1$ and $a \in I_2$

$\therefore$ $a \in I_1 \cap I_2$. So $I_1 \cap I_2$ is radical ideal.

Claim: $\bigvee_{k \in K} I_k = \mathrm{Rad}\left(\sum_k I_k \right)$

This is a radical ideal and it is smallest one that contains I_k.

$\therefore$ it is a join

WTS $I \cap \bigvee_{k \in K} J_k = \bigvee_{k \in K} (I \cap J_k)$

We know that $I \cap \bigvee_{k \in K} J_k \supseteq \bigvee_{k \in K} (I \cap J_k)$ always true

for any join and meet

$$I \cap \bigvee_{k \in K} J_k \subset \bigvee_{k \in K} (I \cap J_k)$$

Let $a \in I$ $\therefore$ $a^n = \sum_k j_k \quad j_k \in J_k$

WTS $\exists m > 0 : a^m = \sum_k j'_k \quad j'_k \in I \cap J_k$

Let $m = n+1$. $\therefore$ $a^{n+1} = a \sum_k j_k = \sum_k \overbrace{\underset{\in I}{a} \cdot \underset{\in J_k}{j_k}}^{\therefore j'_k} \in I \cap J_k$ $\square$

11-24-20

Dictionary

geometry		algebra
• "Space" X		• ring of functions on X
• closed subspace $F \subset X$		• ideal I of R consisting of all functions vanishing on F
$F \subset F'$	$\Rightarrow$	$I' \subset I$
$\emptyset, X$	$\Rightarrow$	$R, 0$
the zero locus $V(I) = \{p \in X \mid f(p) = 0 \ \forall f \in I\}$	$\Leftarrow$	I

open subsets are encoded thru their complements

$\mathcal{U} \subset \mathcal{U}'$ open

$\therefore$ $I < I'$ ideals

Gelfand Duality

Compact Hausdorff topological space — Commutative C^*-algebra

$$X \longmapsto C(X, \mathbb{C})$$ (continuous functions valued over $\mathbb{C}$)

$$\hom(A, \mathbb{C}) \xleftarrow[\text{Spec}]{} A$$

* This is an equivalence of categories *

$\mathcal{D} : H \longrightarrow H$ a linear operator

$f : \mathbb{C} \longrightarrow \mathbb{C}$ continuous

What is $f(\mathcal{D})$?

composition

if $f(z) = z^3 + 2z$, then $f(\mathcal{D}) = \mathcal{D}^3 + 2\mathcal{D}$

if $f(z) = \sin(z)$, what is $\sin(\mathcal{D})$?

---

if R is a commutative ring, then what are the points of spec R?

$\operatorname{Spec} R \in \mathcal{L}ocale$

Suppose $\mathcal{L}$ is a locale and $p : \mathbb{1} \longrightarrow \mathcal{L}$ (a point in $\mathcal{L}$), where $\mathbb{1} = \{0 < 1\}$ (a frame)

Pass to frames $\quad F \xrightarrow{p^*} \{0 < 1\}$

$$U \longmapsto \begin{cases} 0 & \text{if } p \notin U \\ 1 & \text{if } p \in U \end{cases}$$

What does it mean $p \in \mathcal{U}$? $\Rightarrow$

$$\mathcal{U} = \{V \in \mathcal{L} \mid V \leq U\}$$

$$\begin{tikzcd} & \mathcal{U} \arrow[dr] & \\ \mathbb{1} \arrow[ur, "\exists"] \arrow[rr, "p"] & & \mathcal{L} \end{tikzcd}$$

$p \in U$

Take $\mathcal{W} = \bigvee_{p^*\mathcal{U}=0} \mathcal{U}$.

$\mathcal{W}$ is the maximal open in $\mathcal{L}$ that doesn't contain p.

The homomorphism p^* can be reconstructed from $\mathcal{W}$ as follows:

$$\mathcal{U} \longmapsto \begin{cases} 0, & \text{if } \mathcal{U} \le \mathcal{W} \\ 1, & \text{otherwise} \end{cases}$$

Prop This correspondence is a bijection between

$$\mathbb{1} \xrightarrow{p} \mathcal{L}$$

and meet-irreducible opens $\mathcal{W}$ of $\mathcal{L}$

i.e., $\mathcal{W} \neq 1 \in \mathcal{L}$ and $\forall \mathcal{U}, V$ opens if $\mathcal{U} \wedge V \le \mathcal{W}$, then $\mathcal{U} \le \mathcal{W}$ or $V \le \mathcal{W}$

$(X, \mathcal{U})$ where $\mathcal{U} \subset 2^X$ $\qquad$ $(\mathcal{U}, \le)$ where $\le$ is $\subset$

$\exists$ finite infima $\wedge \overset{=}{} \cap$ (meets)

$\exists$ arbitrary suprema $\vee \overset{=}{} \cup$ (joins)

$$\mathcal{U} \cap \bigcup_i V_i = \bigcup_i (\mathcal{U} \cap V_i)$$

Prop The meet-irreducible opens in $Zar(R) = Spec(R)$ are precisely the prime ideals of R, i.e., $I < R$ s.t. R/I is an integral domain

A prime ideal is always radical

equivalently $I < R$. $\forall r_1, r_2 \in R$ if $r_1 r_2 \in I \Rightarrow r_1 \in I$ or $r_2 \in I$

Proof: pick radical ideal $W < I$ (if $r^n \in I$ for some $n \in \mathbb{Z}^+$ $\forall r \in R$ then $r \in I$)

W is meet-irreducible if $\forall U, V \leq_r R$

if $U \cap V \subset W$, then $U \subset W$ or $V \subset W$

To verify W is a prime ideal, pick $r_1, r_2 \in R$:

$r_1 r_2 \in W$. or equivalently $\sqrt{(r_1 r_2)} \subset W$

WTS: $r_1 \in W$ or $r_2 \in W$ i.e., $\sqrt{(r_1)} \subset W$ or $\sqrt{(r_2)} \subset W$

Claim: $\sqrt{(r_1 r_2)} = \sqrt{(r_1)} \cap \sqrt{(r_2)}$.

(x, y, z) ideal gen by x, y, z

$\subset$: is trivial.

$\supset$: $r \in \sqrt{(r_1)} \cap \sqrt{(r_2)}$ $\therefore$ $r^{n_1} = r_1 \cdot s_1$ $\quad s_1, s_2 \in R$

$r^{n_2} = r_2 \cdot s_2$

$r^{n_1+n_2} = r_1 r_2 s_1 s_2 \in (r_1 r_2)$ $\therefore$ $r \in \sqrt{(r_1 r_2)}$

Recall $\sqrt{I} = \{ r \in R \mid \exists n > 0 : r^n \in I \} = Rad(I)$ ← I like this notation

$I = \sqrt{I} \iff I$ is radical

Other direction is similar. $\square$

$\operatorname{Spec}\mathbb{Z}$ $\quad I < \mathbb{Z}$ is a PID $\quad I = (m)$

I is radical $\iff$ $m = \prod_i p_i$ $\quad p_i$ prime, $p_i \neq p_j$ for $i \neq j$

$p_i \notin (p_i^2)$ but $p_i^2 \in (p_i^2)$ $\therefore$ (p_i^2) not radical

I is prime iff $I = (p)$ p prime or $I = (0)$

$$\operatorname{Spec}\mathbb{Z} = \{\, (2),\ (3),\ (5),\ (7),\ \dots,\ (0) \,\}$$

generic point

Closed subsets are finite collection of points as well as entire space

open subsets are cofinite and $\emptyset$

1-21-21

Adjuntions

Def $\mathcal{C}, \mathcal{D}$ are categories with functors

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "L"] & \mathcal{D} \arrow[l, shift left, "R"] \end{tikzcd}$$

An adjunction between L and R is a natural homomorphism $\hom_{\mathcal{D}}(L(-), -) \xrightarrow[\cong]{} \hom_{\mathcal{C}}(-, R(-))$ of functors $\hom_{\mathcal{D}}(L(-),-), \hom_{\mathcal{C}}(-,R(-)) : \mathcal{C}^{op} \times \mathcal{D} \to Set$

natural

$\forall$ $c' \xrightarrow{f} c$, $d \xrightarrow{g} d'$ we have

$$\begin{tikzcd} \hom_{\mathcal{D}}(L(c), d) \arrow[r, "\cong"] \arrow[d, "\hom_{\mathcal{D}}(L(f), g)"] & \hom_{\mathcal{C}}(c, R(d)) \arrow[d, "\hom_{\mathcal{C}}(f, R(g))"] \\ \hom_{\mathcal{D}}(L(c'), d') \arrow[r, "\cong"] & \hom_{\mathcal{C}}(c', R(d')) \end{tikzcd}$$

Ex)
$$\begin{tikzcd} Set \arrow[r, shift left, "Free"] & Group \arrow[l, shift left, "\mathcal{U}"] \end{tikzcd}$$

naturality for $\overset{F}{\overset{\shortparallel}{Free}} \dashv \mathcal{U}$

$$\begin{tikzcd} & & G' \\ & F(S) \arrow[r, "p"] & G \arrow[u, "g"] \\ F(S') \arrow[ur, "F(f)"] \arrow[uurr, bend right] & & \end{tikzcd} \qquad \begin{tikzcd} & & \mathcal{U}(G') \\ & S \arrow[r, "q"] & \mathcal{U}(S) \arrow[u, "\mathcal{U}(g)"] \\ S' \arrow[ur, "f"] \arrow[uurr, bend right] & & \end{tikzcd}$$

$$\begin{tikzcd} p \arrow[r, mapsto] \arrow[d, mapsto] & q \arrow[d, mapsto] \\ g \circ p \circ F(f) \arrow[r, mapsto] & U(g) \circ q \circ f \end{tikzcd}$$

$q = p|_{S \subset U(F(S))}$

好きじゃない

naturality: $\left(g \circ p \circ F(f)\right)\big|_{S'} = U(g) \circ q \circ f$

$$\shortparallel$$

$$U(g \circ p) \circ f$$

Def $L \dashv R$

The **unit** of $L \dashv R$ is a natural transformation $id_C \to R \circ L$ whose components are morphisms $c \to R(L(c))$ adjoint to $L(c) \xrightarrow[id_{L(c)}]{} L(c)$

Counit $LR \to id_D$

$L(R(d)) \to d$

$R(d) \xrightarrow[id_{R(d)}]{} R(d)$

Ex) $F \dashv U$ $\quad$ $Set \underset{U}{\overset{F}{\underset{\perp}{\rightleftarrows}}} Group$

Unit: pick $S \in Set$

$F(S) \xrightarrow{id} F(S)$

$S \longrightarrow U(F(S))$

the inclusion of gen as words of len 1

Counit: pick $G \in Group$ きらい

$$\mathcal{U}(G) \xrightarrow{id} \mathcal{U}(G)$$

$$F(\mathcal{U}(G)) \xrightarrow[\text{evaluation}]{} G$$

Ex) $R \in CRing$

$$F(S) = \bigoplus_{s \in S} R$$ the free R-Mod w/ basis S

$$Set \underset{\mathcal{U}}{\overset{F}{\rightleftarrows}} Mod_R$$

$F(S) \longrightarrow M$

unit: $S \to \mathcal{U}(F(S))$ is inclusion

$S \longrightarrow \mathcal{U}(M)$

counit: $F(\mathcal{U}(M)) \longrightarrow M$ is eval

Ex) $Set \underset{\mathcal{U}}{\overset{F}{\rightleftarrows}} CRing$

$F(S)$ = polynomials with $\mathbb{Z}$ coef w/ variables being elements of S

i.e., $F(\{x, y, z\}) = \mathbb{Z}[x, y, z]$

$F(S) \longrightarrow R \in CRing$

$S \longrightarrow \mathcal{U}(R)$

Ex)

$$\begin{tikzcd} \mathrm{Group} \arrow[r, shift left, "\mathbb{Z}[-]"] & \mathrm{Ring} \arrow[l, shift left, "(-)^\times"] \end{tikzcd}$$

$$\mathrm{Hom}_{\mathrm{Ring}}(\mathbb{Z}[G], R) \cong \mathrm{Hom}_{\mathrm{Group}}(G, R^\times)$$

$$f \longmapsto f|_G$$

unit: $G \to (\mathbb{Z}[G])^\times$ inclusion of basis elements

counit: $\mathbb{Z}[R^\times] \longrightarrow R$ evaluation

Ex)

$$\begin{tikzcd} \mathrm{Set} \arrow[r, shift left, "L"] \arrow[r, phantom, "\perp"] & \mathrm{Top} \arrow[l, shift left, "\mathcal{U}"] \end{tikzcd}$$

$S \in \mathrm{Set}$, $T \in \mathrm{Top}$ $\qquad \mathrm{Hom}_{\mathrm{Top}}(L(S), T) \cong \mathrm{Hom}_{\mathrm{Set}}(S, \mathcal{U}(T))$

If L exists, $\{L(S) \longrightarrow T\} \cong \{S \longrightarrow \mathcal{U}(T)\}$

Ansatz: $\mathcal{U}(L(S)) = S$ Equip S w/ discrete topology

So L equips a set with the discrete topology

$$\begin{tikzcd} \mathrm{Set} \arrow[r, shift right, "R"'] \arrow[r, phantom, "\perp"] & \mathrm{Top} \arrow[l, shift right, "\mathcal{U}"'] \end{tikzcd}$$

If R exists, $\{\mathcal{U}(T) \to S\} \cong \{T \to R(S)\}$

Ansatz: $\mathcal{U}(R(S)) = S$

Equip S with the antidiscrete topology

So R equips a set w/ antidiscrete topology.

Here L does not have a left adjoint

Assume it does exist let it be P

$\therefore$ $\mathrm{Set} \underset{L}{\overset{P}{\rightleftarrows}} \mathrm{Top}$ ($\perp$) : $\{P(T) \to S\} \cong \{T \xrightarrow[g]{} L(S)\}$

$\forall s \in L(S)$ $\{s\} \subset L(S)$ is open

$\therefore$ $g^{-1}\{s\} \subset T$ is open

$\therefore$ $\{g^{-1}\{s\}\}_{s \in S}$ is a partition of T into open subsets

If T is connected, then $g^{-1}\{s\} = \emptyset$ for all $s \in S$ except 1

Thus, $\{T \to L(S)\}$ has same no. of elements as S

$$\begin{gathered}\{T \to L(S)\} \\ \cong \\ \{P(T) \to S\}\end{gathered}$$

This is bad!

$\therefore$ $P(T) = \{*\}$

Thm If L is a left adjoint functor, then L preserves colimits

$L : \mathcal{C} \to \mathcal{D}$ $\quad \forall \mathcal{F} : \mathcal{I} \to \mathcal{C}$

$$\operatorname{colim}(L \circ F) \xrightarrow{\cong} L(\operatorname{colim} F)$$

$$\operatorname{colim}(L \circ F) = \operatorname*{colim}_{i \in I}(L(F(i))) \xrightarrow{[L\iota_i]_{i \in I}} L(\operatorname{colim} F)$$

Ex) binary coproducts

$$L(A) \sqcup L(B) \xrightarrow{[L(\iota_A), L(\iota_B)]} L(A \sqcup B)$$

Proof. The inverse map

$$L(\operatorname{colim} F) \longrightarrow \operatorname{colim}(L \circ F)$$

is adjoint to $\operatorname{colim} F \xrightarrow{[R(\iota_i) \circ \varepsilon_{F_i}]_{i \in I}} R\operatorname{colim}(L \circ F)$

$$\begin{tikzcd}
F_i \arrow[r] \arrow[d, "\varepsilon_{F_i}", "\text{unit map}"'] & R(\operatorname*{colim}_{i \in I}(L(F(i)))) \\
R(L(F(i))) \arrow[ur, "R(\iota_{L(F(i))})"'] &
\end{tikzcd}$$

Ex) $L(A \sqcup B) \longrightarrow L(A) \sqcup L(B)$

$$A \sqcup B \longrightarrow R(L(A \sqcup B))$$

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\varepsilon_A"] & R[L(A) \sqcup L(B)] \\
R(L(A)) \arrow[ur, "R(\iota_{L(A)})"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
\operatorname{colim}(L \circ F) \arrow[r] \arrow[rr, bend left, "1"] & L(\operatorname{colim} F) \arrow[r] & \operatorname{colim}(L \circ F)
\end{tikzcd}$$

$$\forall i \in I \quad \begin{tikzcd}
L(F(i)) \arrow[r, "L(\iota_{F(i)})"'] \arrow[rr, bend left, "\iota_{L(F(i))}"] \arrow[rr, bend right, "\text{adjoint to } R(\iota_{L(F(i))}) \circ \varepsilon_{F(i)}"'] & L(\operatorname{colim} F) \arrow[r] & \operatorname{colim}(L \circ F)
\end{tikzcd}$$

$$\begin{tikzcd}
F(i) \arrow[r] \arrow[d] & R\left(\operatorname*{colim}_{i \in I} L(F(i))\right) \\
R(L(F(i))) \arrow[ur, "R(\iota_{L(F(i))})"'] &
\end{tikzcd}$$

1-28-21

Cor Right adjoint functors preserve limits

$$\begin{tikzcd} \mathrm{Top} \arrow[r, "U"'] & \mathrm{Set} \arrow[l, bend right, "\text{discrete}"'] \arrow[l, bend left, "\text{indiscrete}"] \end{tikzcd}$$

$\perp$

Discrete does not preserve infinite products

Disc$\left(\prod_{\mathbb{Z}^+} \{0,1\}\right)$ is a discrete space,

but $\prod_{\mathbb{Z}^+}$ Disc$(\{0,1\})$ is not a discrete space

Indiscrete does not preserve binary products

Indisc$(\{0\} \sqcup \{1\})$ is an indiscrete space,

but $\bigsqcup_{i \in \{0,1\}}$ Indisc$(\{i\})$ is not indiscrete.

Multilinear Algebra

The three legs of multilinear algebra ····· divided power alg

tensor algebra symmetric alg exterior alg

Two variations

- graded
- ungraded

Def G is a monoid. A **G-graded abelian group** is a family $\{A_g\}_{g\in G}$, $A_g \in Ab$

Morphisms $\{A_g\}_{g\in G} \longrightarrow \{B_g\}_{g\in G}$

are $\{A_g \xrightarrow{f_g} B_g\}$, f_g are homomorphisms

Most graded rings are $G = \mathbb{Z}$

Def The **graded tensor product**

$$\{A_g\}_{g\in G} \otimes \{A_h\}_{h\in G} := \left\{ \bigoplus_{\substack{g,h\in G \\ g\cdot h = k}} A_g \otimes B_h \right\}_{k\in G}$$

Def $G = \mathbb{Z}$ The braiding isomorphism

$$\{A_g\}_{g\in G} \otimes \{B_h\} \xrightarrow[\text{gbr}]{\cong} \{B_h\}_{h\in G} \otimes \{A_g\}_{g\in G}$$

induced by $A_g \otimes B_h \xrightarrow{(-1)^{gh}} B_h \otimes A_g$

Def Koszul Sign Rule

$$(-1)^{gh} = \det br_{\mathbb{R}^g, \mathbb{R}^h}$$

$$\begin{array}{ccc} br_{\mathbb{R}^g,\mathbb{R}^h} : \mathbb{R}^g \oplus \mathbb{R}^h & \longrightarrow & \mathbb{R}^h \oplus \mathbb{R}^g \\ \parallel & & \\ \mathbb{R}^{g+h} & \longrightarrow & \mathbb{R}^{g+h} \end{array}$$

$$\begin{array}{ccc} & & \\ h & \multicolumn{2}{c}{\left(\begin{array}{c|c} 0 & id \\ \hline id & 0 \end{array}\right)} \\ g & & \\ & g & h \end{array}$$

Def A G-graded algebra (over a $\overset{\text{commutative}}{\wedge}$ ring k) is

$A = \{A_g\}_{g\in G} \in \mathrm{GrMod}_k^G$ together w/ a multiplication

$$A \otimes A \xrightarrow{m} A : \qquad \mathbb{1} \xrightarrow{u} A$$

$$\begin{tikzcd} (A\otimes A)\otimes A \arrow[r] \arrow[d, "m\otimes id_A"'] \arrow[dr, phantom, "\#"] & A\otimes(A\otimes A) \arrow[dl, bend left, "id_A\otimes m"] \\ A\otimes A & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r] \arrow[dr, phantom, "\#"] & \mathbb{1}\otimes A \arrow[d, "u\otimes id_A"] \\ & A\otimes A \arrow[ul, "m"] \end{tikzcd}$$

Def The monoidal unit of graded $\otimes$ is $\mathbb{1}$, where

$$\mathbb{1}_g = \begin{cases} \mathbb{Z}, & g = e \in G \\ 0, & g \neq e \in G \end{cases}$$

Lemma $\mathbb{1} \otimes A \cong A \otimes \mathbb{1} \cong A$

Proof. $(A \otimes \mathbb{1})_k = \bigoplus_{\substack{g,h \in G \\ gh = k}} A_g \otimes \mathbb{1}_h = A_k \otimes \mathbb{Z} \cong A_k$ $\square$

Concretely; u_k $\quad k = e \quad \mathbb{Z} \longrightarrow A_e$, ie an element $1 \in A_e$

$k \neq e \quad \exists!\ 0 \longrightarrow A_k$

$$m \text{ is } A_g \otimes A_h \xrightarrow{\cdot} A_{gh} \quad \forall g, h \in G$$

or a bilinear map

$$A_g, A_h \xrightarrow{\cdot} A_{gh}$$

Def A **G-graded k-algebra** is

- commutative if $m \circ b_r = m$ $\qquad x \cdot y = y \cdot x$
- graded-commutative if $m \circ_g b_r = m$ $\qquad x \cdot y = (-1)^{gh} y \cdot x$

Ex) Polynomials in n-variables over a commutative ring k

degree d $\equiv$ the k-mod of degree d polynomials

Ex) The graded group algebra of a group G over $k \in \mathrm{CRing}$

is a G-graded alg $k[G]$

Def

$$\begin{aligned} \mathrm{GrAlg}_k^G &\longrightarrow \mathrm{Alg}_k \\ A &\longmapsto \left(\bigoplus_{g\in G} A_g, \ \cdot\ , 1 \right) \end{aligned}$$

k

Ex) $k[G]_{\text{graded}} \longrightarrow k[G]$

Def

$$\begin{tikzcd} \mathrm{Mod}_k \arrow[r, shift left, "T"] \arrow[r, phantom, "\perp"] & \mathrm{GrAlg}_k^{\mathbb{Z}} \arrow[l, shift left, "(-)_1"] \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Mod}_k \arrow[r, shift left, "\mathrm{Sym}"] \arrow[r, phantom, "\perp"] & \mathrm{CGrAlg}_k^{\mathbb{Z}} \arrow[l, shift left, "(-)_1"] \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Mod}_k \arrow[r, shift left, "\Lambda"] & \mathrm{GCGrAlg}_k^{\mathbb{Z}} \arrow[l, shift left, "(-)_1"] \end{tikzcd}$$

2-2-21

Recall $R \in CRing$

$$\begin{tikzcd} Mod_R \arrow[r, shift left, "T"] \arrow[r, phantom, "\perp"] & GAlg_R \arrow[l, shift left, "{[-]_1}"] \end{tikzcd}$$

Prop T exists

$M \in Mod_R$ $\qquad T(M) \longrightarrow A$

$\qquad M \longrightarrow A_1$ $\qquad \forall A \in GAlg_R$

$T(M)_1 = M$ $\qquad T(M)_1 \otimes T(M)_1 \longrightarrow T(M)_2$

$$M \otimes_R M \xrightarrow{\cong} T(M)_2$$

$$T(M)_n = M \otimes_R M \otimes_R \cdots \otimes_R M = M^{\otimes n}$$

Define $T(M)_n := M^{\otimes n}$

$$T(M)_n \otimes T(M)_m \longrightarrow T(M)_{m+n}$$

$$M^{\otimes m} \otimes M^{\otimes n} \xrightarrow{\cong} M^{\otimes (m+n)}$$

$$R \longrightarrow T(M)_0 = R \quad \text{is multi id}$$

The adjunction property. Given $M \xrightarrow{\varphi} A_1$ for some $A \in GAlg_R$

Construct $T(M) \longrightarrow A$ as follows

$$T(M)_n \longrightarrow A_n$$

$$\begin{tikzcd} M^{\otimes n} \arrow[r] \arrow[rr, bend right, "\varphi^{\otimes n}"'] & A_n & A_1^{\otimes n} \arrow[l, hook', "\text{mult in } A"'] \end{tikzcd}$$

Given $T(M) \longrightarrow A_1$

Construct $M \to A_1$ by restricting to degree 1

Arnold Classical Mechanics

2-4-21

$\dim((TM)_n) = \dim(M^{\otimes n}) = \dim((R^d)^{\otimes n}) = R^{d^n}$

Prop $\mathrm{Mod}_R \underset{[-]_1}{\overset{\exists\,\mathrm{Sym}}{\rightleftarrows}} \mathrm{CGAlg}_R$ ($\perp$)

Construction similar to TM, but $\Sigma_n \curvearrowright M^{\otimes n}$ must become trivial in $\mathrm{Sym}^n M$

$M^{\otimes n} \longrightarrow \mathrm{Sym}^n M$ must be invariant under the action of Σ_n

Take $\mathrm{Sym}^n M$ to be the universal target for Σ_n-invariant maps $M^{\otimes n} \longrightarrow \mathrm{Sym}^n M$

That is $\mathrm{Sym}^n M = \operatorname{colim}_{B\Sigma_n} M^{\otimes n} = (M^{\otimes n})_{\Sigma_n}$

$B\Sigma_n \longrightarrow \mathrm{Mod}_R$ "orbits of the Σ_n-action of $M^{\otimes n}$"

$* \longmapsto M^{\otimes n}$

$\sigma : * \to * \longmapsto M^{\otimes n} \to M^{\otimes n}$

The multiplication map

$$\mathrm{Sym}^m M \otimes_R \mathrm{Sym}^n M \xrightarrow{\cdot} \mathrm{Sym}^{m+n} M$$

$$\left(\operatorname{colim}_{B\Sigma_m} M^{\otimes m}\right) \otimes_R \left(\operatorname{colim}_{B\Sigma_n} M^{\otimes n}\right) \longrightarrow \mathrm{Sym}^{m+n} M$$

$$\operatorname*{colim}_{\substack{B\Sigma_m \times B\Sigma_n \\ \shortparallel \\ B(\Sigma_m \times \Sigma_n)}} \left(M^{\otimes m} \otimes_R M^{\otimes n} \right) \longrightarrow \mathrm{Sym}^{m+n} M$$

$$\shortparallel$$

$$\operatorname*{colim}_{\substack{B(\Sigma_m \times \Sigma_n) \\ \Sigma_{m+n}}} M^{\otimes(m+n)} \longrightarrow \mathrm{Sym}^{m+n} M$$

$$M^{\otimes(m+n)} \longrightarrow \mathrm{Sym}^{m+n} M = \left(M^{\otimes(m+n)}\right)_{\Sigma_{m+n}}$$

This is commutative

The universal : $A \in CGAlg_R$

$$\mathrm{Sym}\, M \xrightarrow{h} A \qquad \Big\downarrow (-)_1 \qquad \Big\uparrow$$

$$M \xrightarrow[\varphi]{} A_1$$

Given φ we want to construct $\mathrm{Sym}\, M \to A$

In degree n: $\mathrm{Sym}^n M \to A_n$

$$\shortparallel \nearrow$$

$$\operatorname*{colim}_{B\Sigma_n} M^{\otimes n}$$

むずかしい

We need to construct $M^{\otimes n} \longrightarrow A_n$ and verify Σ_n-invariance

$$\begin{array}{ccc} M^{\otimes n} & \longrightarrow & A_n \\ \big\downarrow \varphi^{\otimes n} & \nearrow & \\ A_1^{\otimes n} & & \end{array}$$

multi. on A $\leftarrow$ commutative

$\therefore$ Σ_n-invariant

$h \mapsto \varphi \mapsto h'$ WTS $h = h'$

By construction $h_1 = h_1'$

$\mathrm{Sym}^1 M$ generates $\mathrm{Sym}\, M$

Ex) $M = \bigoplus_n R$

$$\mathrm{Sym}\, M = \mathrm{Sym}\left(\bigoplus_n R\right) = \coprod_n \mathrm{Sym}\, R = \bigotimes_n \mathrm{Sym}\, R$$

$$\mathrm{Sym}^n R = \left(R^{\otimes_R n}\right)_{\Sigma_n} = (R)_{\Sigma_n} = R$$

$$\therefore \mathrm{Sym}\, R = R[x]$$

$$\mathrm{Sym}\, M = R[x_1, \dots, x_n]$$

2 - 9 - 21

Prop $R \in CRing$ $V \in Mod_R$. (free and $\cong R^n$)

$Sym V^* \cong \{$ polynomial fn $V \to R\} :=$ closure of linear fn. $V \to R$ under ring operations

if R is ID and infinite

Proof. Construct $Sym V^* \xrightarrow{q} Poly(V, R)$

using universal prop of

$$\begin{array}{ccc} V^* & \longrightarrow & Poly(V,R) \\ \| & \nearrow & \\ hom_R(V,R) & & \end{array}$$

q is a surjection by construction

q is injective $V \cong \bigoplus_S V$ for some $S \in Set$

$V^* \cong \prod_S R$

$Sym V^* \cong R[x_1, \ldots, x_n]$

take $f \in R(x_1, \ldots, x_n] = (R[x_2, \ldots, x_n])[x_1]$

Induction on n

$n = 0$ $R \xrightarrow{\cong} R$ ✓

if $n > 0$, take $p \cdot x_1^k$, the highest power of x_1 w/ nonzero coef

by inductive assumption we choose $x_2, \ldots, x_n$ st. $p(x_2, \ldots, x_n) \neq 0$

We get a polyn in x_1 w/ nonzero coef. before x_1^k

We have $r(x_1) = r_1(x_1) \prod_i (x_1 - a_i)$ $\quad r_1(x_1) \neq 0 \ \ \forall x_1 \in R$

Choose $x_1 \neq a_i$ (finite a_i infinite R)

$r(x_1) \neq 0$ as R ID $\square$

<u>Remark</u> 1) $R = \mathbb{Z}/p\mathbb{Z}$ $\quad x^p - x$ vanishes on R

2) $R = \mathbb{Z} \oplus A$, $A \in Ab$

$$(n_1, a_1) \cdot (n_2, a_2) = (n_1 n_2, 0)$$

Prop $R \in CRing$

$$\begin{tikzcd} Mod_R \arrow[r, shift left, "\Lambda"] \arrow[r, phantom, "\perp"] & GCGAlg \arrow[l, shift left, "{[-]_1}"] \end{tikzcd}$$

Proof. $\Lambda(M)_n := (M^{\otimes n})/\Sigma_n$ $\qquad (\Sigma_n \curvearrowright M^{\otimes n})$

$$\sigma(m_1 \otimes \cdots \otimes m_n) = \mathrm{sign}\,\sigma \cdot m_{\sigma 1} \otimes \cdots \otimes m_{\sigma n}$$

graded commutativity:

$$[[m_1 \otimes \cdots \otimes m_a] \otimes [m_1' \otimes \cdots \otimes m_b']]$$

$$= [m_1 \otimes \cdots \otimes m_a \otimes m_1' \otimes \cdots \otimes m_b'] = A$$

$$[[m_1' \otimes \cdots \otimes m_b'] \otimes [m_1 \otimes \cdots \otimes m_a]] = [m_1' \otimes \cdots \otimes m_b' \otimes m_1 \otimes \cdots \otimes m_a] = B$$

$A = (-1)^{ab} B$

Ex) $M = \coprod_n R \quad n \geq 0$

$T(M) = \coprod_n T(R) = R[x_1] * \cdots * R[x_n] = R\langle x_1, \ldots, x_n \rangle$

$\mathrm{Sym}(M) = \coprod_n \mathrm{Sym}(R) = R[x_1] \otimes \cdots \otimes R[x_n] = R[x_1, \ldots, x_n]$

$\Lambda(R) = \coprod_n \Lambda(R) = \frac{R[x_1]}{\langle x_1^2 \rangle} \otimes \cdots \otimes \frac{R[x_n]}{\langle x_n^2 \rangle} = \frac{R[x_1, \ldots, x_n]}{\langle x_1^2, \ldots, x_n^2, x_i x_j + x_j x_i \rangle}$

$\Lambda(R)_n = (R^{\otimes n}) / \Sigma_n = R / \Sigma_n = \begin{cases} R & \text{if } n < 2 \\ R/2R & n \geq 2 \end{cases}$

Typically $2 \in R^\times \Rightarrow R/2R \cong 0 \quad \therefore \Lambda(R) \cong R[x]/\langle x^2 \rangle$ $\mathrm{char}\, R \neq 2$

A basis for $\Lambda(M)$ are subsets $I \subset \{1, \ldots, n\}$

$I \to \prod_{i \in I} x_i$

$\dim(\Lambda(M)) = 2^n$

2-23-21

Notation: $\mathrm{Vect}_k^\times$: denotes the category of finite dim vector spaces and isomorphisms

$\mathrm{Line}_k^\times$: is a subcategory of $\mathrm{Vect}^\times$ with dim 1

$$\mathrm{Vect}^\times \underset{\det}{\overset{\mathrm{vol}}{\underset{\mathrm{or}}{\rightrightarrows}}} \mathrm{Line}^\times$$

Let $F = \det, \mathrm{or}, \mathrm{vol}$

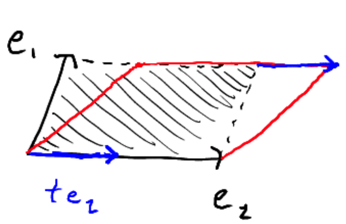
$F(V) = \mathrm{Free}(\mathrm{Base}(V))/\sim$

$\mathrm{vol}(V)$: elements are "units of measure" for volumes

if $e_1, \dots, e_n$ is basis of V, then the parallelpiped $\{t_1 e_1 + \dots + t_n e_n \mid 0 \le t_i \le 1 \ \forall i\}$ (k must be ordered) has volume 1.

The equivalence relation identifies parallelepipeds with the same volume.

① $[e_1, e_2, \dots, e_n] = [e_1 + t\cdot e_2, e_2, \dots, e_n]$ $t \in k$



② $[e_1, \dots, e_i, \dots, e_j, \dots, e_n] = [e_1, \dots, e_j, \dots, e_i, \dots, e_n]$

③ $[e_1, \dots, te_i, \dots, e_n] = |t| \cdot [e_1, \dots, e_i, \dots, e_n] \quad t \neq 0$

Prop $\dim(\mathrm{vol}(V)) = 1$

Proof. Pick any basis $e_1, \dots, e_n$.

Then $[f_1, \dots, f_n] = \lambda \cdot [e_1, \dots, e_n]$ using (1-3)

$\therefore \dim(\mathrm{vol}(V)) = 1$ $\square$

Recall (Non linearly) Measure from real analysis

- a measure μ (on a real vector space V) is a countably addative map

$$M(V) \longrightarrow [0, \infty]$$

that vanish on $\mathcal{N}(V)$

$$\mu\left(\bigcup_{i \in I} B_i\right) = \sum_{i \in I} \mu(B_i)$$

$B_i \in M(V)$

$B_i \cap B_j = \emptyset \quad i \neq j$

I countable.

(Measures that are absolutely cont. w.r.t. Lebesgue measure on V)

$M(V)$ = measureable subsets of V

$\mathcal{N}(V)$ = neglegable subsets of V

$A \in N(V)$ if $\forall \varepsilon > 0$ A can be covered by countably many par. whose total sum is $< \varepsilon$

$B \in M(V)$ iff $B = C \oplus \underset{N(V)}{\underset{\cap}{A}}$ where C is borel subset of V

$\mathrm{Borel}(V)$ = smallest collection of subsets of V closed under complements, countable unions, and contains all par.

Def. A measure is translation invariant (Haar measure) iff $\mu(B) = \mu(B+v)$ $\forall v \in V$

Theorem (Lebesgue, Haar)

$\{$translation invariant measures on $V\} \cong [0, \infty]$ (as a communative monoid)

Def. $\mathrm{Haar}(V)$:= the closure of translation invariant measures on V (except ∞) under subtraction

$\therefore$ $\mathrm{Haar}(V) \in \mathrm{Line}$

Prop $\mathrm{Haar}(V) \underset{\varphi}{\cong} \mathrm{Vol}(V)^* = \hom(\mathrm{Vol}(V), \mathbb{R}) \quad (V \in \mathrm{Vect}_{\mathbb{R}})$

Proof. $\mathrm{Haar}(V) \otimes_{\mathbb{R}} \mathrm{Vol}(V) \longrightarrow \mathbb{R}$

$$\mathrm{Haar}(V),\ \mathrm{Vol}(V) \longrightarrow \mathbb{R}$$

$$\mu,\ [e_1, \dots, e_n] \longmapsto \mu(\text{par. of } e_1, \dots, e_n)$$

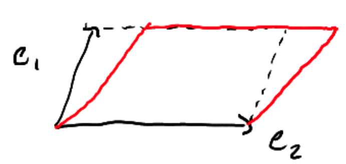
This pairing is nonzero. $\therefore$ φ is iso $\square$

Def. $\mathrm{Or} : \mathrm{Vect}^\times \longrightarrow \mathrm{Line}^\times$

$$\mathrm{Or}(V) = \mathrm{Free}(\mathrm{Bases}(V)) / \sim$$

"Cont. deforming a basis does not change the orientation"

① $[e_1 + t e_2, e_2, \dots, e_n] = [e_1, e_2, \dots, e_n] \quad t \in k$



② $[e_1, \dots, e_i, \dots, e_j, \dots, e_n] = -[e_1, \dots, e_j, \dots, e_i, \dots, e_n]$

③ $[e_1, \dots, t e_i, \dots, e_n] = \mathrm{sign}(t) [e_1, \dots, e_i, \dots, e_n]$

Prop $\dim \mathrm{Or}(V) = 1$

Proof. $[f_1, \dots, f_n] = \pm [e_1, \dots, e_n]$

2-25-21

Def. **$\det V$** $:= \mathrm{or}(V) \otimes \mathrm{vol}(V)$

$\det : \mathrm{Vect}^\times \longrightarrow \mathrm{Line}^\times$

EO	vol	or	det (or vol)
$v_i \leftarrow v_i + t v_j$	1	1	1
$v_i \leftarrow v_j \quad i \neq j$	1	-1	-1
$v_i \leftarrow t v_i \quad t \neq 0$	$\lvert t \rvert$	$\frac{t}{\lvert t \rvert}$	t

Prop $e_1, \dots, e_n \longmapsto [e_1, \dots, e_n] \in \det V$

is multilinear unlike or or vol.

(extend zero to linearly dependent $e_1, \dots, e_n$)

or just relax assumption that $e_1, \dots, e_n$ is a basis

What happens when $k = \mathbb{C}$

Instead of the $\mathbb{Z}/2\mathbb{Z}$-torsor of orientations we get a $U(1)$-torsor of "phases".

Def. An orientation on a finite dimensional vector space V over an ordered field. is a choice of an element in the $\mathbb{Z}/2\mathbb{Z}$-torsor of orientations.

Equivalently, a choice of a half line in either $\operatorname{or} V$ or $\det V$.

Ex) $V = \{0\}$ has a canonical orientation

$\operatorname{or} V \cong \mathbb{Z}/2\mathbb{Z} = \{[\,], -[\,]\}$

if V has dim 1, $\det V = 1$

$\operatorname{or} V = (V \setminus \{0\}) / k^+$

Recall $\dim(\mathrm{vol}\, V) = 1$

half lines: $\{[e_1, \dots, e_n]\}$ $\leftarrow$ canonical orientation

$\{-[e_1, \dots, e_n]\}$

Ex) $V = \{0\}$ $\quad \mathrm{vol}(V) \cong \mathbb{R}$

$\dim V = 1$ $\quad \mathrm{vol}(V)$ has canonical orientation

$\{u, v\}: \ u + v = 0$

Summary:

	vol V	Or V	det V	$\mathbb{R}$
Canonical orientation	Yes $\{[e_1, \dots, e_n]\}$	No	No	yes, $\mathbb{R}_{>0}$
Canonical volume element	No	Yes $\{\pm[e_1, \dots, e_n]\}$	No	Yes, $\{\pm 1\}$
canonical o.v.e.	No	No	No	Yes 1

Remark: Or, vol can be reconstructed from the line det

Def. $\mathrm{Line}^\times \underset{\mathrm{vol}}{\overset{\mathrm{or}}{\rightrightarrows}} \mathrm{Line}^\times$

むずかしい

$$L \longmapsto \mathrm{Free}(L \setminus \{0\})/\sim$$

$\mathrm{vol}: \quad [r \cdot v] = |r| \cdot [v] \qquad v \in L \setminus \{0\} \quad r \in k^\times$

$\mathrm{or}: \quad [r \cdot v] = \frac{r}{|r|}[v] \qquad v \in L \setminus \{0\} \quad r \in k^\times$

Prop $\mathrm{or}(\det V) \cong \mathrm{or}\, V$

$\mathrm{vol}(\det V) \cong \mathrm{vol}\, V$

Def. $\mathrm{Line}^\times \xrightarrow{\mathrm{Dens}_p} \mathrm{Line}^\times$

$\mathrm{Dens}_p(V) \qquad p\text{-density} \qquad p \in \mathbb{R}\ (\mathbb{C})$

$$L \longmapsto \mathrm{Free}(L \setminus \{0\})/\sim$$

$$[r \cdot v] = |r|^p [v]$$

$$\begin{tikzcd}
\mathrm{Vect}^\times \arrow[r] \arrow[d, "\det"'] & \mathrm{Line}^\times \\
\mathrm{Line}^\times \arrow[ur, "\mathrm{Dens}_p"'] &
\end{tikzcd}$$

$$\mathrm{Dens}_p(\det V) =: \mathrm{Dens}_p(V)$$

$\mathrm{Dens}_p^+ V$ has canonical orientation

$\{\pm[e_1, \dots, e_n]\}$

b) $q \in \mathbb{R}$ $p \geq 0$ $\quad \mathrm{Dens}_p^+(V) \xrightarrow{(-)^q} \mathrm{Dens}_{pq}(V)$

$$t\,[v] \longmapsto t^q\,[v]$$

this is not id! the $\sim$ are different

$$[v] \longmapsto [v]$$

$$|\lambda|^{-p} \cdot [\lambda \cdot v] \longmapsto |\lambda|^{-pq}[\lambda \cdot v] = |\lambda|^{-pq}|\lambda|^{pq}[v] = [v]$$

if $q > 0$ $\quad \mathrm{Dens}_p^+ V \cong \mathrm{Dens}_{pq}^+ V$

In particular, $\mathrm{Dens}_p^+ V = (\mathrm{Dens}_1^+ V)^p$

<u>Ex</u>) $\mathcal{L}_p^+ = (\mathcal{L}_1^+)^p$

$\mathcal{L}_1$ are finite measures $\quad M \to [0, \infty)$

3-2-21

Q: What is a physical unit?

A: It's an element of a 1-dim vector space (real)

If $A \in \mathrm{Line}_{\mathbb{R}}$ $\quad A \otimes A^* \cong \mathbb{R}$

$$a, u \longmapsto u(a)$$

Given $L, M, T \in \mathrm{Line}_{\mathbb{R}}$

$$v = \frac{l}{t} \in L \otimes T^*$$

$p \in \mathbb{R}$ $(p \in \mathbb{C}) \Rightarrow t^p \in \mathrm{Dens}_p T$ $\quad$ ($t > 0$ $\therefore$ need T be orientated)

See Ramanan Global Calculus

Prop / HW $\quad$ or, vol, and det are strong monoidal functors

$$\mathrm{FinVect}_k^{\oplus} \longrightarrow \mathrm{Line}_k^{\otimes}$$

$$\det(k) \cong k \quad , \quad \det(V \oplus W) \cong \det(V) \otimes \det(W)$$

Proof Sketch $\quad \det V \otimes \det W \to \det(V \oplus W)$

$$[e_1, \dots, e_{\dim V}], [f_1, \dots, f_{\dim W}] \longmapsto [e_1, \dots, e_{\dim V}, f_1, \dots, f_{\dim W}]$$

Q: $V \in \mathrm{FinVect}_k \qquad A \in \mathrm{End}(V), \qquad \det A \in k$?

$$\begin{array}{ccc} V & & \det V \\ A \downarrow & \overset{\det}{\longmapsto} & \downarrow \det A \\ V & & \det V \end{array} \qquad \det A \in \mathrm{Hom}(\det V, \det V) \cong k$$

Q: $\det AB = \det A \det B$ where $A: V \to V \quad B: V \to V$

$$\det A: \det V \xrightarrow{m} \det V$$
$$\det B: \det V \xrightarrow{n} \det V$$
$$\det AB: \det V \xrightarrow{m} \det V \xrightarrow{n} \det V$$

Q: $\det A$ multilinear and anti-symmetric

$$\underset{\underset{\det V}{\uparrow}}{[e_1, \dots, e_n]} \longmapsto [\underbrace{Ae_1}, \dots, \underbrace{Ae_n}]$$

col vect of A in basis e

3-4-21

Def. Let k be a field

$$\mathrm{Vect}_k \xrightarrow{\wedge} \mathrm{GCAlg}_k \qquad \downarrow \text{as graded algebra}$$

$$X \longmapsto \Big\langle \overbrace{(V, v)}^{\text{degree} = \dim V}, \ (1), (2), (3) \Big\rangle$$

V finite-dim subspace

$v \in \det V$

(1) $(V, v) \wedge (V', v') = \begin{cases} 0 & \text{if } V \cap V' \neq \{0\} \\ (V + V', v \otimes v'), & V \cap V' = \{0\} \end{cases}$

(2) $t \in k \quad t \cdot (V, v) = (V, t \cdot v)$

(3) $\dim V = \dim V' = 1$, then

$v \in V, \ v' \in V' \qquad (V, v) + (V', v') = (\mathrm{Span}(v + v'), \ v + v')$

Lemma ΛV is graded commutative.

Proof. $(V, v) \wedge (V', v') = (V+V', v \otimes v')$

$(V', v') \wedge (V, v) = (V'+V, v' \otimes v)$

Note $V+V' = V'+V$ and $v \otimes v' = (-1)^{\dim V \dim V'} v' \otimes v$ $\square$

Lemma $\Lambda^0 X \cong k$ $\quad (\{0\}, \underset{\det\{0\} \cong k}{\overset{\in}{v}})$

$\Lambda^1 X \cong X$ $\quad (V, v) = (\mathrm{Span}(v), v)$

if $\dim X < \infty$ $\Lambda^{\dim V} X \cong \det X$ $\quad (V, v)$, $V \subset X$ $\dim V = \dim X \Rightarrow V = X$

$v \in \det V = \det X$

Prop $\Lambda_a X \cong \Lambda_g X$

Proof. $\Lambda_a X \longrightarrow \Lambda_g X$

$X \xrightarrow{id} (\Lambda_g X)_1 \cong X$

$\Lambda_g X \to \Lambda_a X$

$(V, v) \longmapsto$

$$\Lambda_a X \longrightarrow \Lambda_g X \longrightarrow \Lambda_a X$$

$$X \longrightarrow (\Lambda_a X)_1 = X \quad \therefore \quad \Lambda_a X \xrightarrow{id} \Lambda_a X$$

$$\Lambda_g X \to \Lambda_a X \to \Lambda_g X$$

$$(V, \nu) \longmapsto \underset{\det V}{\overset{\cup}{\nu}} \longmapsto e_1 \wedge e_2 \cdots \wedge e_{\dim V} = (\mathrm{Span}\{e_1, \dots, e_{\dim V}\}, e_1 \otimes \cdots \otimes e_{\dim V})$$

$\nu = [e_1, \dots, e_{\dim V}]$

Observation $e_1, \dots, e_n$ are linearly independent iff.

$e_1 \wedge \cdots \wedge e_n \neq 0$

See Grueb Multilinear algebra

Bourbaki Alg Ch 3

Prop

$$\Lambda(X^*) \cong \left\langle (\underbrace{X \twoheadrightarrow Q}_{\deg = \dim Q}, q \in \det(Q^*)) \,\middle|\, (1), (2), (3) \right\rangle$$

$$\cong \left\langle (\underbrace{A \hookrightarrow X}_{\deg = \operatorname{codim} A = \dim X - \dim A}, a \in \det A \otimes \det(X^*)) \,\middle|\, (1), (2), (3) \right\rangle$$

$$A = \ker u \hookrightarrow X \xrightarrow{u} k$$

$$A \hookrightarrow X \twoheadrightarrow X/A \qquad \uparrow \cong \alpha$$

$$\alpha \in \hom(X/A, k) = (X/A)^* \cong \det((X/A)^*)$$

3-9-21

Last time we saw a geometric interuptation of $\wedge V^*$ where $V \in Vect_k$

$$\wedge V^* = \langle (V \twoheadrightarrow Q,\ q \in \det Q^*) \mid (1), (2), (3) \rangle$$
$$= \langle (U \rightarrowtail V,\ q \in \det V^* \otimes \det U \mid (1)\ (2)\ (3) \rangle$$

(1) $(Q,q) \wedge (Q',q') = (Q'', q \otimes q')$

$$\begin{tikzcd} & Q \arrow[dr, two heads] & \\ V \arrow[ur, two heads] \arrow[dr, two heads] & & Q'' \\ & Q' \arrow[ur, two heads] & \end{tikzcd}$$
if $\dim Q'' = \dim Q + \dim Q'$

otherwise 0

$$\begin{tikzcd} & U \arrow[dr, tail] & \\ U'' \arrow[ur, tail] \arrow[dr, tail] & & V \\ & U' \arrow[ur, tail] & \end{tikzcd}$$
if $\operatorname{codim} U'' = \operatorname{codim} U + \operatorname{codim} U'$

$$0 \rightarrowtail V_1 \rightarrowtail V_2 \twoheadrightarrow V_3 \twoheadrightarrow 0 \qquad \det V_2 \cong \det V_1 \otimes \det V_3$$

$$\begin{tikzcd} & & W \arrow[dr, two heads] & & \\ & U' \arrow[ur, tail] \arrow[dr, two heads] & & Q \arrow[dr, two heads] & \\ U'' \arrow[ur, tail] \arrow[dr, tail] & & V \arrow[ur, tail] \arrow[dr, two heads] & & Q'' \\ & U \arrow[ur, tail] \arrow[dr, two heads] & & Q' \arrow[ur, two heads] & \\ & & W' \arrow[ur, tail] & & \end{tikzcd}$$

$$\det Q''^* \cong \det W \otimes \det Q^* \cong \det U' \otimes \det U''^* \otimes \det Q^*$$
$$\cong \det W' \otimes \det Q'^* \cong \det$$

$$\det Q \otimes \det Q' \cong (\det V \otimes \det U^*) \otimes (\det V \otimes \det U'^*)$$

$$\det Q'' \cong (\det Q \otimes \det W^*)$$

$$\cong \det Q \otimes \det U'^* \otimes \det U''$$

$$\cong \det Q \otimes \det V^* \otimes \det Q' \otimes \det U''$$

$$\cong \det Q \otimes \det Q' \otimes \det U'' \otimes \det V^*$$

$$Q = Q'' + W$$

$$Q' = Q'' + W'$$

$$Q + Q' = Q'' + Q'' + W + W'$$

$$= Q'' + Q'' + U' - U'' + U - U''$$

This is bad

Choose a splitting

$$U = U'' + A \qquad Q \cong B$$

$$U' = U'' + AB \qquad Q' \cong A$$

$$V = U'' + A + B$$

$$Q'' = A + V$$

$$\det Q'' \cong \det Q \otimes \det Q'$$

(2) $t \cdot (Q, q) = (Q, t \cdot q)$

(3) $(Q, q) + (Q', q')$ $\qquad \dim Q = \dim Q' = 1$

The canonical pairing

$$\bigwedge V \otimes \underbrace{\bigwedge V^*}_{\substack{\text{negative} \\ \text{deg}}} \longrightarrow \underbrace{k}_{\deg 0}$$

The only nonzero components are

$$\bigwedge^k V \otimes \bigwedge^k V^* \longrightarrow k$$

$$\Big(\underset{\dim S = k}{S \overset{i}{\rightarrowtail} V}, s \in \det S\Big), \Big(\underset{\dim Q = k}{V \overset{\pi}{\twoheadrightarrow} Q}, q \in \det Q^*\Big) \longrightarrow q\big(\det(\pi i)(s)\big) \in k$$

$$\begin{tikzcd} S \arrow[r, tail, "i"] \arrow[rr, bend right, "\pi"'] & V \arrow[r, two heads, "\pi"] & Q \end{tikzcd},$$

Recall: If A is a graded algebra over k, then $\hom(A,k)$ is an A-A-bimodule (graded)

if $f \in \hom(A,k)$ $\quad a.f = f(- \cdot a)$ i.e. $(af)(b) = f(ba)$

$\quad f.a = f(a \cdot -)$ i.e., $(f.a)(b) = f(ab)$

In particular, $(\bigwedge V)^* \cong \bigwedge V^*$ is a $\bigwedge V$-$\bigwedge V$ graded bimodule.

Notation: $\quad a.f.b = a \lrcorner f \llcorner b$

$(Q, q) \in \bigwedge^k V^*$

$$(\underset{\shortparallel}{Q}, q) \llcorner (\underset{\substack{Y \\ \vee}}{Y}, \underset{\in \det Y}{y}) \underset{\in \bigwedge^l V}{} = \begin{cases} (Q/Y, q \otimes y) \in \bigwedge^{k-l} V^* \\ 0 \quad \text{otherwise} \end{cases}$$

3-16-21

Integration of Densities

$\dim P = 1$

Note $\cdot$ $\mathrm{vol}\, P \cong \det P \otimes \mathrm{or}\, P^* \cong \det P \otimes \mathrm{or}\, P$

$\mathrm{or}\, P = \langle e, f \mid e+f=0 \rangle \cong \mathrm{or}\, P^* = \langle e^*, f^* \mid e^*+f^*=0 \rangle$

$\cdot$ $\mathrm{or}(\mathrm{vol}\, P) \cong \mathbb{R}$

Prop $V \in \mathrm{Vect}_{\mathbb{R}}$ $\quad W = V \times \underset{\text{dens}_1 V^*}{\underset{\parallel}{\mathrm{vol}(V^*)}}$

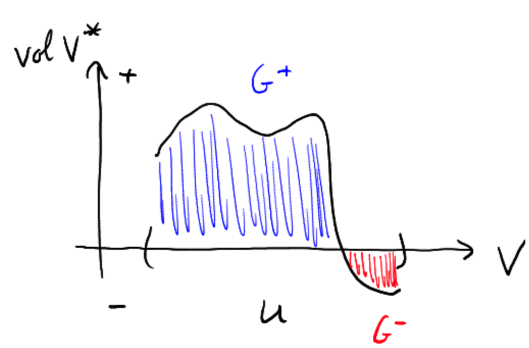
W has a canonical measure μ

$$\begin{aligned}\mathrm{measures}(W) &\cong (\mathrm{vol}\, W)^* \cong (\mathrm{vol}\, V \otimes \mathrm{vol}(\mathrm{vol}(V^*)))^* \\ &\cong (\mathrm{vol}\, V \otimes \mathrm{vol}\, V^* \otimes \mathrm{or}(\mathrm{vol}\, V^*))^* \\ &\cong (\mathbb{R} \otimes \mathbb{R})^* \\ &\cong \mathbb{R}\end{aligned}$$

These are all canonical isos so we have a canonical measure.

Def. $V \in \mathrm{Vect}_{\mathbb{R}}$ $\quad \dim V < \infty$ $\quad U \subset V$ is measurerable

$f : U \to \mathrm{vol}\, V^*$



$$G_+ := \left\{ (u,y) \in W = V \times \mathrm{vol}\, V^* \;\middle|\; \begin{matrix} u \in U \\ 0 \le y \le f(u) \end{matrix} \right\}$$

$$G_- := \left\{ (u,y) \in W \;\middle|\; \begin{matrix} u \in U \\ f(u) \le y \le 0 \end{matrix} \right\}$$

f is called measureable if for any interval $P \subset \mathrm{vol}\, V^*$ the set $f^{-1}(P)$ is also Lebesgue measureable

$\Longrightarrow$ G_+ and G_- are measureable subsets of W.

$$\int_U f := \mu G_+ - \mu G_-$$

This is functorial.

Thus if $g : M \xrightarrow{\cong} N$ is a defeomorphism, then

$$\int_N f = \int_M g^* f \qquad \text{where } g^* \text{ is the pullback}$$

$$g^* f = |\det Df| \cdot f$$

Def $SVect_k$ is the symmetric monoidal category of super vector spaces.

$\mathbb{Z}/2$ - graded vector spaces over k

$$V \otimes W \xrightarrow{\cong} W \otimes V$$

$$v \otimes w \longmapsto \begin{cases} w \otimes v & \text{, if } v \text{ or } w \text{ is of even degree} \\ -w \otimes v & \text{, if } v \text{ and } w \text{ are odd degree} \end{cases}$$

Super commutative means graded commutative

ΛV is super comm.

$\mathrm{Sym} V$ is not super comm

Def. $SAlg_k$ the category of super algebras over k

$A \in SVect_k$ $\quad A \otimes A \xrightarrow{\mu} A \quad k \xrightarrow{1} A$

associative and unital.

Def. A Lie Algebra (not actually an algebra)

$A \in Vect_k \quad A \otimes A \xrightarrow{[-,-]} A$

Jacobi Identity: $\quad [x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$

antisymmetry: $\quad [x,y] = -[y,x]$

This is typically not associative or unital

Ex) Let B be an algebra

Then $A = B$, with $[x,y] = xy - yx$

is a Lie algebra.

This is the universal example

Ex) $gl(V) := End(V)$

Def. $SLie_k$ the category of super Lie algebras over k

$A \in SVect_k$ $\quad A \otimes A \xrightarrow{[-,-]} A$

Ex) $SAlg_k \longrightarrow SLie_k$ $\quad [x,y] = xy - (-1)^{|x|\cdot|y|} yx$

Centrally pointed super Lie Alg $\mathfrak{g}$ over k

pointed $\underset{[-,-]=0}{k[0]} \xrightarrow{1_\mathfrak{g}} \mathfrak{g}$ $\quad k[0] = (k, \{0\})$

centrally pointed :
$$\begin{tikzcd} k[0] \arrow[r] \arrow[dr] \arrow[r, phantom, shift right=4, "*"] & \mathfrak{g} \\ & Z(\mathfrak{g}) \arrow[u] \end{tikzcd}$$
$Z(\mathfrak{g}) = \{ x \in \mathfrak{g} \mid \forall y \in \mathfrak{g},\ [x,y] = 0 \}$

$$\mathrm{CenPointedSLieAlg}_k \underset{\mathcal{U}}{\overset{F}{\underset{\perp}{\rightleftarrows}}} \mathrm{SAlg}_k$$

$\mathcal{U}(A) = (A,\ [x,y] = xy - (-1)^{|x||y|} yx,\ 1 \in A_0)$

$F(\mathfrak{g}) = \dfrac{T\mathfrak{g}}{J} = \dfrac{\bigoplus_{n \geq 0} \mathfrak{g}^{\otimes n}}{J}$

$J = \left\langle [x,y]_\mathfrak{g} - x \otimes y + (-1)^{|x||y|} y \otimes x,\ 1_\mathfrak{g} - 1 \right\rangle$ two sided

Without base points F is known as the universal enveloping super alg of a super lie alg.

$\mathrm{Lie}_k \longrightarrow \mathrm{SLie}_k$

$(\mathfrak{g}, [-,-]) \longrightarrow ((\overset{\oplus k}{\mathfrak{g}}, \{0\}), [-,-]) \overset{F}{\longmapsto} \underbrace{\mathcal{U}\mathfrak{g}}$ universal enveloping alg. $\quad 1 \in k$

$$\mathrm{Vect}_k \underset{[1]}{\overset{[0]}{\rightrightarrows}} \mathrm{SVect}_k$$

$$V \xmapsto{[0]} ((V^{\oplus k}, \{0\}), 0, 1 \in k) \xmapsto{F} TV/\langle -x \otimes y + y \otimes x \rangle \cong \mathrm{Sym}\, V$$

$$\xmapsto{[1]} ((k, V), 0, 1 \in k) \xmapsto{F} TV/\langle -x \otimes y - y \otimes x \rangle \cong \Lambda V$$

$V \in \mathrm{Vect}_k$ $\qquad \omega : \Lambda^2 V \to k$

$V \otimes V \to k$ that is antisymmetric

$$((\overset{0}{V} \oplus \overset{1}{k}, \{0\}), [v_1 \oplus t_1, v_2 \oplus t_2] = \omega(v_1, v_2), 1 \in k) \qquad (*)$$

This is called the Heisenberg Lie alg of (V, ω)

clf $V = \underset{\substack{1 \\ \shortparallel \\ p}}{\mathbb{R}} \oplus \underset{\substack{1 \\ \shortparallel \\ q}}{\mathbb{R}}$ $\qquad p = 1 \oplus 0$

$q = 0 \oplus 1$

$$[p, q] = [(1 \oplus 0) \oplus 0, (0 \oplus 1) \oplus 0] = \omega(p, q) = i\hbar$$

$$* \xmapsto{F} TV/\langle x \otimes y - y \otimes x - \omega(x, y) \rangle \quad \text{Weyl algebra of } (V, \omega)$$

Ex) $V = \mathbb{R}^2 \quad [p, q] = -1$

$$T\mathbb{R}^2/\langle pq - qp - 1 \rangle = \langle p^a q^b \mid pq - qp + 1 \rangle$$

$p \rightsquigarrow x \quad q \rightsquigarrow \partial_x$

$$\Rightarrow \langle x^a \partial_x^b \mid x\partial_x - \partial_x x + 1 \rangle$$

$$x\partial_x f - \partial_x(x.f) + f$$
$$= x\partial_x f - f - x\partial_x f + f = 0$$

Ex) $V \in \mathrm{Vect}_k$ $\quad Q : S^2 V \longrightarrow k$ $\qquad t \in k \quad v \in V$

$\left(Q : V \underset{\mathrm{Sym}}{\otimes} V \to k\right)$ or $\left(Q : V \to k \quad Q(tv) = t^2 Q(v)\right)$

$Q(x+y+z) - Q(x+y) - Q(y+z) - Q(x+z) + Q(x) + Q(y) + Q(z) = 0$

Q is known as a quadratic map

$$k \to k$$
$$t \mapsto t^2$$

$$Q(v,v) = Q(v)$$

but if you want to go the other way you need char $\neq 2$

$$Q(v,w) = \tfrac{1}{2}\left(Q(v+w) - Q(v) - Q(w)\right)$$

$$\left((\overset{0}{k}, \overset{1}{V}),\ [v_1, v_2] = Q(v_1, v_2) \in k,\ 1 \in k\right) \quad *$$

Aside $\quad F\left((V_0 \oplus k, V_1), [-,-] = 0, 1 \in k\right) = \mathrm{Sym}\, V_0 \otimes \Lambda V_1$

$$F(*) = TV \big/ \langle v_1 \otimes v_2 - v_2 \otimes v_1 - Q(v_1, v_2)\rangle$$
or $(v \otimes v - Q(v))$

Clifford Alg $\quad Cl(V, Q)$

3-25-21

Clifford Algebra a geometric def.

(V, q) quadratic vector space.

$C\ell(V,q)$ is generated by (A, a) $A \subset V$ and $a \in \det A$

$q : V \to k$ quadratic

$q : S^2 V \to k$ linear

$q : V \otimes V \to k$ sym. linear

$q : V \to V^*$ (using tensor-hom adjunction)

The generators are subject to the relations

1) $t \cdot (A, a) = (A, t \cdot a)$

2) $(A_1, a_1) + (A_2, a_2) = (\mathrm{Span}(a_1 + a_2), a_1 + a_2)$

3a) if $\overset{A_1 \cap A_2 = 0}{A_1 \perp A_2}$ $(q(a_1, a_2) = 0 \ \forall a_1 \in A_1, a_2 \in A_2)$, then

$$(A_1, a_1) \cdot (A_2, a_2) = (A_1 + A_2, a_1 \otimes a_2)$$

3b) if $\overset{A_1 \cap A_2 = 0}{A_1 \perp A_2}$, then $(A_1 + A_2, a) \cdot (A_2, a_2) = (A_1, \underbrace{a \llcorner a_2}_{a \otimes a_2^*})$

Prop. $Cl_a(V,q) \longrightarrow Cl_g(V,q)$

$$\left((\overset{0}{k}, \overset{1}{V}), [-,-]=q \right) \longrightarrow Cl_g(V,q)$$

$$\begin{array}{ccc} \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ 1,0 & \longmapsto & 1 \\ & v \longmapsto (\mathrm{span}\, v, v) & \\ V & \xrightarrow{\iota} & Cl_g(V,q)_1 \end{array}$$

$$(\iota(v))^2 = q(v) \cdot 1$$

$$[v_1, v_2] = q(v_1, v_2) \in k$$

$$\begin{array}{ccc} [\iota(v_1), \iota(v_2)] & = & q(v_1, v_2) \cdot 1 \\ \parallel & & \parallel \\ \iota(v_1)\iota(v_2) + \iota(v_2)\iota(v_1) & & \end{array}$$

Def. $Cl_g(V,q) \to Cl_a(V,q)$

$\dim A_i = 1$ $\quad (A,a) \overset{(3a)}{\cong} (A_1, a_1) \cdots (A_{\dim A}, a_{\dim A})$

$$\longmapsto a_1 \cdots a_{\dim A} \in Cl_a(V,q)$$

Classification of Clifford alg over $\mathbb{R}$ and $\mathbb{C}$

If $k = \mathbb{C}$ and q is nondegenerate, then

$$Cl(\mathbb{C}^n, q) \cong Cl(\mathbb{C}, q)^{\otimes n} \cong (\underset{0}{\mathbb{C}} \oplus \underset{1}{\mathbb{C}})^{\otimes n} \cong \begin{cases} \mathrm{End}(\mathbb{C}^{2^{n/2}}) & \text{even} \\ (\mathbb{C} \oplus \mathbb{C}) \otimes \mathrm{End}(\mathbb{C}^{2^{(n-1)/2}}) & \text{odd} \end{cases}$$

$- := -t_1^2 - t_2^2 - \cdots - t_n^2$

The Clifford clock $n \longmapsto Cl(\mathbb{R}^n, -)$

$(\mathbb{R}^0 \oplus \mathbb{R}^1)$

$\parallel$

$$Cl(\mathbb{R}, -) = T\mathbb{R}/(\ldots) = \mathbb{R}[x]/(x^2+1) = \mathbb{C}_{\text{graded}}$$

$$Cl(\mathbb{R}^2, -) = T\mathbb{R}^2/(x^2+1,\ y^2+1,\ xy=-yx) \qquad T\mathbb{R} = \mathbb{R}\langle x, y\rangle$$

$$= \mathbb{H}_{\text{graded}}$$

$Cl(\mathbb{R}^n, -)$

8 $M_{16}(\mathbb{R})$

0 $\mathbb{R}_g$

1 $\mathbb{C}_g$

2 $\mathbb{H}_g$

3 $\mathbb{H}_g \oplus \mathbb{H}_g$

4 $M_2(\mathbb{H}_g)$

5 $M_4(\mathbb{C}_g)$

6 $M_8(\mathbb{R})$

7 $M_8(\mathbb{R}) \oplus M_8(\mathbb{R})$

Def.

$Cl(V,q) \quad \Lambda V \quad Cl(V,q)$

$$Cl(V,q) \longrightarrow End(\Lambda V)$$

$$V \longrightarrow End(\Lambda V)_{odd} \qquad V \xrightarrow{q} V^*,\quad v \longmapsto v^*$$

$$v \longmapsto (v \wedge -) + (v^* \lrcorner -) = \beta_v$$

Verify $\beta_v^2 = (q(v) \cdot -)$

The Symbol map $\quad Cl(V,q) \longrightarrow \Lambda V$

$$a \longmapsto a \cdot 1$$

This is an isomorphism of $\mathbb{Z}/2$-graded vector spaces but does not preserve multiplication in the algebra.

3-25-21

$$Cl(V,q) \underset{\text{quantization}}{\overset{\text{symbol}}{\rightleftarrows}} \Lambda V$$

an isomorphism of filtered super vector spaces.
but not an isomorphism of algebras

Def. a **filtered algebra** over a commutative ring k is $\to \cdots \to A_{-2} \to A_{-1} \to A_0 \to A_1 \to A_2 \to \cdots$
(a functor $(\mathbb{Z}, \leq) \to \mathrm{Vect}_k$) together with $A_i \otimes A_j \to A_{i+j}$ and $k \to A_0$ that are both associative and unital.

- A **filtered G-graded algebra**: $A_i \in$ G-graded Vect_k

Ex) • $\Lambda^{\leq k} V$ is a commutative filtered $\mathbb{Z}$-graded algebra

• $Cl^{\leq k}(V,q)$ is a filtered $\mathbb{Z}/2$-graded algebra

Def. $\text{filt Alg} \xrightarrow[\text{gr}]{\text{associated graded-alg}} \mathbb{Z}\text{-graded Alg}_k$

$$\begin{array}{ccc} \to A_{-1} \to A_0 \to A_1 \to \cdots & & B_i = A_i / A_{i-1} \\ A_i \otimes A_j \to A_{i+j} & \longmapsto & B_i \otimes B_j \to B_{i+j} \\ k \to A_0 & & \end{array}$$

Ex) $gr(\Lambda^{\leq k} V)$

$$k \longrightarrow \Lambda^{\leq k} V / \Lambda^{\leq k-1} V = \underbrace{\Lambda^k V}_{\deg k \bmod 2} \in \mathbb{Z}/2\text{-graded } \mathrm{Vect}_k$$

$$\therefore \; gr(\Lambda^{\leq k} V) = \Lambda V$$

Ex) $gr(Cl^{\leq k}(V,q)) = gr(TV/(v^2 - q(v))) \cong \Lambda V$

induced by symbol map.

Def. The *parity involution*

$$\alpha : Cl(V,q) \longrightarrow Cl(V,q)$$

$$\alpha(x) = \begin{cases} x & , \deg x = 0 \\ -x & , \deg x = 1 \end{cases}$$

The *transposition*

$$t : Cl(V,q) \longrightarrow Cl(V,q)^{op}$$

xy is done reversely as yx

$$V \xrightarrow{\iota} Cl(V,q)^{op}$$

$$(v_1 \cdots v_n) \xmapsto{t} (v_n \cdots v_1)$$

Def. $Cl(V,q)^\times \xrightarrow{A} GL(Cl(V,q))$

$$x \longmapsto A_x = (y \mapsto x \cdot y \cdot \alpha(x)^{-1})$$

$\Gamma(V,q) < Cl(V,q)$ The Clifford group

We have a canonical isomorphism

$$\Gamma(V,q) \xrightarrow[A]{\cong} GL(V)$$
$$x \longmapsto A_x$$

Remark the kernel of A is $\{x \in \Gamma(V,q) \mid y\alpha(x) = xy\}$

$$1 \longrightarrow k^\times \rightarrowtail \Gamma(V,q) \longrightarrow GL(V)$$

A typical element of $\Gamma(V,q)$ has the form

$v_1 \cdots v_n \qquad v_i \in V \setminus \{0\}$

Q: What is A_v for some $v \in V$?

$V = \mathrm{span}\{v\} \oplus \underbrace{v^\perp}_{\{u \in V \mid q(u,v)=0\}}$

$$A_v(t \cdot v + \underset{v^\perp}{w}) = v \cdot (tv + w) \cdot (-v)^{-1}$$

$$= -t \cdot v + v^2 \cdot w \cdot \frac{-1}{q(v)} = -tv + w$$

A: So we reflect across orthogonal complement. $\therefore A_v \in O(V,q)$ orthogonal group.

$$1 \rightarrowtail k^{\times} \rightarrowtail \Gamma(V,q) \twoheadrightarrow \overbrace{O(V,q)}^{\substack{\text{group generated} \\ \text{by reflections in } GL(V)}} \twoheadrightarrow 1$$

Spinor Norm $\quad N(x) = x^{T} x$

If $x \in \Gamma(V,q)$, then $N(x) = t \cdot 1$, $t \in k^{\times}$

$N : \Gamma(V,q) \longrightarrow k^{\times}$ is a homomorphism of groups

$\mathrm{Pin} = \ker N$, $\quad S\Gamma(V,q) = \Gamma(V,q) \cap Cl(V,q)$

$$\mathrm{Spin} = \ker N \big|_{S\Gamma(V,q)}$$

The restriction of the exact seq to the Pin group

$$1 \rightarrowtail \{\pm 1\} \rightarrowtail \mathrm{Pin}(V,q) \twoheadrightarrow O(V,q) \twoheadrightarrow 1$$

if we restrict to even elements we get

$$1 \rightarrowtail k^{\times} \rightarrowtail S\Gamma(V,q) \twoheadrightarrow SO(V,q) \twoheadrightarrow 1$$

$$1 \rightarrowtail \{\pm 1\} \rightarrowtail \mathrm{Spin}(V,q) \twoheadrightarrow SO(V,q) \twoheadrightarrow 1$$

if $V = \mathbb{R}^n \quad q = \sum_{i=1}^{n} x_i^2 \quad n \geq 3$

Reference: Meinrenken: Clifford Algebras and Lie Groups

$C^\infty(\mathbb{R}^n, \mathbb{R}) = A$

if $f \in C^\infty(\mathbb{R}^n, \mathbb{R})$ we may compute its differential

$$f \longmapsto \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n$$

$$f \longmapsto \sum_{i=1}^{n} \frac{\partial^k}{\partial x_{i_1} \cdots \partial x_{i_k}} f$$

What are these expressions algebraically

- The first map is the universal derivation st lands in the module of Kähler differentials of A
- The second map is a differential operator on A.

Def. $A \in \mathrm{CRing}$ (or $A \in \mathrm{CAlg}_k$) and $M \in \mathrm{Mod}_A$

a **derivation on A** with values in M is a k-linear map $\mathcal{D} : A \to M$ such that the Leibniz rule holds

$$\mathcal{D}(ab) = \mathcal{D}(a) \cdot b + a\mathcal{D}(b)$$

A **morphism of derivations** is an A-linear map $M \to M'$ such that

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ M \arrow[rr] & & M' \end{tikzcd}$$

commutes.

Def. The universal derivation of A is the initial object in the category of derivations of A

Its codomain is known as the module of Kähler differential of A

Ex) $V \in Vect_{\mathbb{R}}$ $\quad \dim V < \infty$

$$C^\infty(V, \mathbb{R}) \xrightarrow{D} C^\infty(V, V^*)$$

$$f \longmapsto Df \qquad \text{where } Df(v)(w) = (t \mapsto f(v+tw))'(0)$$

Theorem: The universal derivation exists.

Consider the A module M with generators da, $a \in A$ and relations

$$d(a+b) = d(a) + d(b)$$
$$d(ta) = t\,d(a)$$
$$d(ab) = d(a)b \cdot a\,d(b)$$

$A \to M \quad a \mapsto da$

3-30-21

Last time

$A \in CAlg_k$

$$\begin{tikzcd} A \arrow[r] \arrow[d] & M_A \\ \Omega^1 A \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

k-linear map that satisfies the Leibniz rule

$$\Omega^1 A = \langle da \mid a \in A \quad d(ab) - da \cdot b - a \cdot db, \quad d(a+b) - da - db, \quad d(t \cdot a) - t\, da \rangle$$

$$\mathrm{Der}(A, M) \cong \mathrm{Mod}_A(\Omega^1 A, M)$$

Ex) $A = k[x]$

$$f = \sum_{n \geq 0} a_n x^n \qquad a_n \in k$$

$$df = \sum_{n \geq 0} a_n\, d(x^n) = \sum_{n \geq 0} a_n n x^{n-1}\, dx = \left(\sum_{n \geq 0} a_n n x^{n-1} \right) dx := f' dx$$

$\therefore \quad \Omega^1 A = A \cdot dx$

$A = k[x_1, \ldots, x_n] \qquad f = \sum_{\alpha \in \mathbb{Z}_{\geq 0}} a_n x^\alpha \longmapsto df = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_m} dx_m$

$$\Omega^1 A = \bigoplus_i A\, dx_i$$

$V \in \mathrm{Vect}_k \qquad A = \mathrm{Sym}\, V \qquad V^* = \{ V \xrightarrow{\text{poly}} \mathbb{R} \}$

$\Omega' A = (\mathrm{Sym} V) \otimes_k V \implies \Omega'(\mathrm{Sym} V^*) = \mathrm{Sym} V^* \otimes_k V^*$

Warning: $A = C^\infty(V, \mathbb{R})$

$\Omega'(A) \overset{?}{=} A \otimes_k V^* = C^\infty(V, V^*)$ not true unfortunately

Ex) $d(e^x) \neq e^x dx$ in $\uparrow$

To fix the problem

Def. A C^∞-ring is a product preserving functor

$F : \mathrm{Cart} \to \mathrm{Set}$ $\qquad$ Cart : category $\{\mathbb{R}^n \mid n \geq 0\}$

$\mathrm{Mor}(\mathbb{R}^m, \mathbb{R}^n) = C^\infty(\mathbb{R}^m, \mathbb{R}^n)$

Ex) $U \subset \mathbb{R}^a$ open $\qquad W \in \mathrm{Cart}$

$F(W) = C^\infty(U, W)$

$$\begin{array}{ccc} W_1 & & F(W_1) = C^\infty(U, W_1) \\ \downarrow & \longmapsto & \downarrow \qquad\quad \downarrow \\ W_2 & & F(W_2) = C^\infty(U, W_2) \end{array}$$

$F(W_1 \times W_2) \longrightarrow F(W_1) \times F(W_2)$

$\quad \shortparallel$

$C^\infty(U, W_1 \times W_2) \xrightarrow[\cong]{} C^\infty(U, W_1) \times C^\infty(U, W_2)$

by universal prop of products

Remark: replacing C^∞ with poly maps yields $CAlg_{\mathbb{R}}$

$A = F(\mathbb{R}^1) \in Set$ is the underlying set.

$A \times A \xrightarrow{+} A$

$F(\mathbb{R}^2) = F(\mathbb{R}^1) \times F(\mathbb{R}^1) \longrightarrow F(\mathbb{R}^1)$

$F(\mathbb{R}^2 \xrightarrow{x_1+x_2} \mathbb{R})$

Def. A C^∞-derivation of a C^∞-ring A is a $\mathbb{R}$ linear map $A \xrightarrow{\partial} M_A$ such that $\forall f: \mathbb{R}^m \to \mathbb{R}$ $\forall a_1, \dots, a_m \in A$ we have

$$\partial(f(a_1, \dots, a_m)) = \sum_{1 \le i \le m} \frac{\partial f}{\partial x_i}(a_1, \dots, a_m)\, \partial(a_i)$$

Thm $\exists!$ C^∞-derivation universal

$$A \longrightarrow \Omega^1_{C^\infty} A$$

Prop $\Omega^1 \underset{\shortparallel}{C^\infty}(\mathbb{R}^n) \cong \bigoplus_i C^\infty(\mathbb{R}^n)\, dx_i$

$\Omega^1 \; C^\infty(V, \mathbb{R}^n) \cong C^\infty(V, V^*) \cong C^\infty(V, \mathbb{R}) \otimes_{\mathbb{R}} V^*$

4-1-21

Def. a **chain complex** over a $k \in CRing$ is a $\mathbb{Z}$-graded abelian group (k-mod) $\mathcal{C}$, equipped with $d: \mathcal{C} \to \mathcal{C}[1]$ such that $d^2 = 0$ (cochain if $d: \mathcal{C}_n \to \mathcal{C}_{n+1}$)

Def. a **chain map** $\mathcal{C} \to \mathcal{D}$ is a mapping of $\mathbb{Z}$-graded abelians groups that commutes with d.

$$\begin{array}{ccccccccc}
\cdots & \to & C_{n+1} & \xrightarrow{d^C_{n+1}} & C_n & \xrightarrow{d^C_n} & C_{n-1} & \to & \cdots \\
 & & \downarrow f_{n+1} & \circledast & \downarrow f_n & \circledast & \downarrow f_{n-1} & & \\
\cdots & \to & \mathcal{D}_{n+1} & \xrightarrow[d^D_{n+1}]{} & \mathcal{D}_n & \xrightarrow[d^D_n]{} & \mathcal{D}_{n-1} & \to & \cdots
\end{array}$$

Def. Suppose $\mathcal{C}, \mathcal{D} \in Ch_k$

$$(\mathcal{C} \otimes \mathcal{D})_n = \bigoplus_{a+b=n} C_a \otimes D_b \qquad d(x \otimes y) = dx \otimes y + (-1)^{\overset{\text{degree of } x}{|x|}} x \otimes dy$$

Def. $\mathcal{C} \otimes \mathcal{D} \longrightarrow \mathcal{D} \otimes \mathcal{C}$

$$x \otimes y \longmapsto (-1)^{|x| \cdot |y|} y \otimes x$$

Def. a **differential graded algebra** over $k \in CRing$ is $A \in Ch_k$, $A \otimes A \xrightarrow{\mu} A$, $k \xrightarrow{e} A$ that is associative, unital

Def. A commutative DGA over k is a DGA over k such that

$$xy = (-1)^{|x|\cdot|y|} yx$$

Def.
$$\begin{tikzcd} \underset{\text{(cochain)}}{CDGA_k} \arrow[r, bend right, "u"'] \arrow[r, phantom, "\perp"] & CAlg_k \arrow[l, bend right, "\Omega"'] \end{tikzcd}$$

$$A \overset{u}{\longmapsto} A_0$$

ΩA is the algebra of differential forms on $\operatorname{Spec} A$

Prop Ω exists

1) $A \xrightarrow{id} A[0]_0$ for $A \in CAlg_k$

$$\Omega A \longrightarrow A[0]$$

Claim this map is an isomorphism

$$(\Omega A)_0 \overset{?}{=} A$$

$$(\Omega A)_0 \longrightarrow A$$

The unit map $A \longrightarrow (\Omega A)_0$ is the inverse

$$(\Omega A)_0 \longrightarrow A \longrightarrow (\Omega A)_0$$

Observe that for any $\mathcal{E} \in CAlg_k$, we have that

$$\{ \Omega A \to \mathcal{E}[0] \} \cong \{ A \to \mathcal{E} \}$$
$$\cong \{ (\Omega A)_0 \to \mathcal{E} \}$$

$\therefore$ by Yoneda lemma $\quad (\Omega A)_0 \cong A$

2) $(\Omega A)_1 \cong \Omega^1 A$ the kähler differentials

$$A = (\Omega A_0) \xrightarrow{d} (\Omega A)_1$$

$$d(a \cdot b) = da \cdot b + a \cdot db \quad \text{since } \deg(a) = 0$$

$$\begin{tikzcd} A \arrow[r, "d"] \arrow[dr, "d'"'] \arrow[dr, phantom, "\#", near start, shift left=3] & \Omega^1 A \arrow[d, dashed, "\exists!\ k\text{-linear}"] \\ & M \end{tikzcd} \qquad B = \begin{matrix} {}_1\ M \\ d' \uparrow \\ {}_0\ A \end{matrix} \in CDGA_k$$

$$\therefore A \cong B_0 \implies \Omega A \to B \qquad (\Omega A)_1 \to B_1 = M$$

3) To show Ω exists take

$$(\Omega A)_n = \Lambda^n_A \Omega^1 A$$

define d using the Leibniz rule

$$d(p_1 \wedge \cdots \wedge p_n) = dp_1 \wedge p_2 \wedge \cdots \wedge p_n + (-1)^{|p_1|} p_1 \wedge dp_2 \wedge \cdots \wedge p_n + \cdots + \qquad \square$$

4-6-21

Modifications for Differential Geometry (C^∞-rings)

Def. A **C^∞ CDGA** is a

- $CDGA_{\mathbb{R}}$, A
- a structure of a C^∞-ring on A_0
- If $f \in C^\infty(\mathbb{R}^n)$, $a_1, \dots, a_n \in A_0$, then

$$d(f(a_1,\dots,a_n)) = \partial_1 f(a_1,\dots,a_n)\,da_1 + \dots + \partial_n f(a_1,\dots,a_n)\cdot da_n$$

Thus $A_0 \xrightarrow{d} A_1$ is a C^∞-derivation.

Def Ω is the left adjoint to $C^\infty DGA \xrightarrow{(-)_0} C^\infty Ring$

Prop Ω exists

Ex) $\Omega\, C^\infty(V,\mathbb{R}) = C^\infty(V,\mathbb{R}) \otimes_{\mathbb{R}} \Lambda V^*$

If $V = \langle e_1, \dots, e_n \rangle$ and $V^* = \langle x_1, \dots, x_n \rangle$

then $\Omega\, C^\infty(V,\mathbb{R}) = C^\infty(\mathbb{R}^n,\mathbb{R})\langle \underbrace{dx_1, \dots, dx_n}_{\deg = 1} \rangle \Big/ \begin{array}{l} dx_i \wedge dx_j = -dx_j \wedge dx_i \\ dx_i \wedge dx_i = 0 \end{array}$

Def. M a smooth manifold (e.g. $M \subset \mathbb{R}^n$)

The de Rham cohomology of M is the cohomology of the C^∞CDGA $\Omega C^\infty(M)$

Lemma $C^\infty\mathrm{CDGA} \xrightarrow{H^*} \mathrm{GCGA}_{\mathbb{R}}$

Remark: $\mathbb{R}$ can be exchanged for $\mathbb{C}$ and everything works

Ex) Let $M = \mathbb{C}$

$\mathbb{C} \cong \mathbb{R} \times \mathbb{R}$

$z = x + iy$

$dz = dx + i\,dy$

What is $\Omega C^\infty(M, \mathbb{C})$?

$$\Omega C^\infty(M,\mathbb{C}) = C^\infty(M,\mathbb{C})\langle dx, dy\rangle \Big/ \begin{matrix} dx^2 = 0 \\ dy^2 = 0 \end{matrix} \quad dx \wedge dy = -dy \wedge dx$$

Alternative Basis for $\Omega^1 C^\infty M$

$\bar{z} : M \longrightarrow \mathbb{C} \qquad \bar{z} = x - iy \qquad d\bar{z} = dx - i\,dy$

$$dx = \frac{dz + d\bar{z}}{2} \qquad dy = \frac{dz - d\bar{z}}{2i}$$

$$dz \wedge d\bar{z} = (dx + i dy) \wedge (dx - i dy) = -i dx \wedge dy + i dy \wedge dx = -i dx \wedge dy$$
$$= - d\bar{z} \wedge dz$$

$$\Omega C^\infty (M, \mathbb{C}) = C^\infty (M, \mathbb{C}) \langle dz, d\bar{z} \rangle \Big/ \begin{matrix} dz^2 = 0 \\ d\bar{z}^2 = 0 \end{matrix} \quad dz \wedge d\bar{z} = - d\bar{z} \wedge dz$$

Ex) $M = \mathbb{C}^\times$

$$\omega = z^{-1} dz \implies d\omega = d(z^{-1}) dz + z^{-1} d(dz)$$
$$= -z^{-2} dz \wedge dz + 0$$
$$= 0$$

$\omega \in Z^1$ ω is a cocycle, but not a coboundry.

$d\omega = 0 \implies \omega \in Z^1$

Thm (Stokes')

$$\int_c d\psi = 0$$

$$\int_{S^1} z^{-1} dz = \int_{[0,2\pi]} (\cos t + i \sin t)^{-1} d(\cos t + i \sin t)$$

$$= \int_{[0,2\pi]} (\cos t - i \sin t)(-\sin t \, dt + i \cos t \, dt)$$

$$= \int_{[0,2\pi]} i \cos^2 t + i \sin^2 t \; dt = \int_{[0,2\pi]} i \, dt = 2\pi i \neq 0$$

$\therefore \; \omega \notin B^1$

$\therefore \; H^1 \neq 0$ and $[\omega] \neq 0$

$\therefore \; H^1 \cong \mathbb{C}$

4 - 8 - 21

Differential Operators

$C^\infty\mathbb{R} \to C^\infty\mathbb{R}$ $\quad$ $C^\infty\mathbb{R} \to C^\infty\mathbb{R}$ $\quad$ $C^\infty\mathbb{R}^n \to C^\infty\mathbb{R}^n$

$f \mapsto \frac{df}{dx}$ $\quad$ $f \mapsto \frac{d^n f}{dx^n}$ $\quad$ $f \overset{\Delta}{\mapsto} \sum \frac{\partial f}{\partial x_i}$

$d : A \to \Omega^1 A$ $\quad$ $\Omega^n \xrightarrow{d} \Omega^{n+1} A$ $\quad$ really any derivation $d: M \to N$

Def. $k \in \mathrm{CRing}$ $\quad A \in \mathrm{CAlg}_k$ $\quad M, N \in \mathrm{Mod}_A$

$$\mathrm{Diff}^{\le m}(M,N) \subset \hom_k(M,N)$$

$m=0$: $\mathrm{Diff}^{\le 0} = \{0\}$

$m>0$: $\mathrm{Diff}^{\le m} = \left\{ f : M \underset{k\text{-lin}}{\to} N \;\middle|\; \begin{array}{l} \forall a \in A \quad [f;a] \in \mathrm{Diff}^{\le m-1} \\ [f,a] = (\underset{\overset{\cap}{M}}{m} \mapsto f(am) - af(m)) \end{array} \right\}$

Ex) $\mathrm{Diff}^{\le 0} = \{0\}$

$\mathrm{Diff}^{\le 1} = \mathrm{Hom}_A(M,N)$

Def. Fix k, A. The category Diff_A has A modules as objects and morphisms $M \to N$ are elements of $\mathrm{Diff}^{<m}(M, N)$

$$\mathbb{F}_9 := \frac{\mathbb{F}_3[x]}{\langle x^2+1 \rangle} = \frac{\frac{\mathbb{Z}}{\langle 3 \rangle}[x]}{\langle x^2+1 \rangle} = \frac{\mathbb{Z}[x]}{\langle 3, x^2+1 \rangle}$$

$$\mathbb{F}_9[t] := \frac{\mathbb{Z}(x)}{\langle 3, x^2+1 \rangle}[t] \qquad \mathbb{F}_9[t] = \mathbb{Z}/_3(i)[t]$$

$$t^2 + 1 = (t - x)(t - 2x)$$

char k = 0 $\qquad$ 4-13-21

Prop $\mathrm{Diff}(\underbrace{\mathrm{Sym}\,V^*}_{A}) \cong T\big(\overbrace{(\mathrm{Sym}\,V^*)\otimes_k V}^{(\Omega^1_k(\mathrm{Sym}\,V^*))^* \;\cong\; \mathrm{Der}(A,A)}\big) \Big/ \begin{matrix} \theta\otimes f - f\otimes\theta - \theta(f) \\ \theta\otimes\theta' - \theta'\otimes\theta - [\theta,\theta'] \end{matrix}$

dif op of poly fns

Cor The associated graded alg of $\mathrm{Diff}_k(A)$ is

$\mathrm{Sym}_k\,\mathrm{Der}_k(A,A) \cong A\otimes_k \mathrm{Sym}\,V$

In coords $\quad V = \langle \partial_1,\dots,\partial_n\rangle \quad V^* = \langle x_1,\dots,x_n\rangle$

$A = \mathrm{Sym}\,V^* = k[x_1,\dots,x_n]$

$\mathrm{Diff}\,A = k[x_1,\dots,x_n]\langle\partial_1,\dots,\partial_n\rangle \Big/ \begin{matrix} \partial_i x_j - x_j\partial_i = 0 \\ \partial_i x_i - x_i\partial_i = 1 \\ \partial_j\partial_i - \partial_i\partial_j = 0 \end{matrix}$

$$\begin{aligned} \mathrm{gr}(\mathrm{Diff}\,A) = A\otimes_k \mathrm{Sym}\,V &= A[\partial_1,\dots,\partial_n] \\ &= k[x_1,\dots,x_n][\partial_1,\dots,\partial_n] \end{aligned}$$

Ex) The Laplacian is $\Delta = \partial_1^2 + \dots + \partial_n^2$

References: Jet Nestruev: Smooth manifolds and Observables

Moerdijk, Reyes: Models for Smooth infinitesimal Analysis

Def. Let C be a category and $X \in C$ we have

$$\begin{aligned} \hom(-, X) : C^{op} &\longrightarrow Set \\ Y &\longmapsto \hom(Y, X) \end{aligned}$$

$$\begin{tikzcd} Y_1 \arrow[d, "f"'] & & \hom(Y_2, X) \arrow[d, "(-)\circ f"] \\ Y_2 & \longmapsto & \hom(Y_1, X) \end{tikzcd}$$

The functor represented by X

$\hom(X, -) : C \to Set$ The functor corep. by X

Def. A functor $F : C^{op} \longrightarrow Set$ is representable if it is isomorphic to $\hom(-, X)$ for some X.

This means $\forall Y \in C$ $\hom(Y, X) \xrightarrow[t_Y]{\cong} F(Y)$

and $\forall f : Y_1 \to Y_2$

$$\begin{tikzcd} \hom(Y_2, X) \arrow[r, "(-)\circ f"] \arrow[d, "t_{Y_2}"', "\cong"] & \hom(Y_1, X) \arrow[d, "\cong"', "t_{Y_1}"] \\ F(Y_2) \arrow[r, "F(f)"'] & F(Y_1) \end{tikzcd}$$

Def. $C^{op} \xrightarrow{F} Set$. The category of elements of F $\int F$ has objects pairs (Y, y) $Y \in C$ and $y \in F(Y)$ and morphisms $(Y_1, y_1) \to (Y_2, y_2)$ are $f : Y_1 \to Y_2$ such that $y_1 = F(f)(y_2)$

Ex) $F = \hom(-, X)$

Ob are $y : Y \to X$

Mor
$$\begin{tikzcd} & X & \\ Y_1 \arrow[ur, "y_1"] \arrow[rr, "f"'] & \# & Y_2 \arrow[ul, "y_2"'] \end{tikzcd}$$
are commuting triangles

Prop $F : C^{op} \to Set$ is representable iff $\int F$ has a terminal object (Y, y)

In this case Y represents F and Y is unique up to a unique isomorphism

Ex) $\mathcal{D} : I \to C$ $\quad \lim \mathcal{D}$ represents the functor

$F : C^{op} \to Set \qquad F(Y) = \{\text{cones over } \mathcal{D} \text{ w/ apex } Y\}$

$$\begin{tikzcd} Y_1 \arrow[d, "f"'] \\ Y_2 \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} F(Y_2) \arrow[d] \\ F(Y_2) \end{tikzcd} \qquad Y_2 \to Y_1$$

4-15-21

Ex) $L: C \to D$ $\perp$ $R: D \to C$ $\quad \hom_D(Lc, d) \xrightarrow{\cong} \hom_C(c, Rd)$

The functor $\hom_C(-, Rd)$ is isomorphic to $\hom_D(L, d)$

Prop. A functor $F: C^{op} \to Set$ is representable iff it has a left adjoint $L: Set \to C^{op}$ in this case $L(\{*\})$ is the object that represents F.

Proof. $\hom_{C^{op}}(LS, X) \xrightarrow{\cong} \hom_{Set}(S, FX)$ $\quad S \in Set \quad X \in C^{op}$

$$\hom_{C^{op}}(LS, X) = \hom_C(X, LS)$$

Let $S = \{*\}$ $\quad \hom_C(X, L\{*\}) \xrightarrow{\cong} \hom_{Set}(\{*\}, FX) \cong FX$

$\therefore$ F is represented by $L\{*\}$. $\square$

Prop. $F: C \to Set$ is corep. iff it has a left adjoint $L: Set \to C$ with $L\{*\}$ corep. F.

Q: When does a functor $R: D \to C$ have a left adjoint

A: necsary condition: R preserves all small limits

Prop. Suppose a category $\mathcal{C}$ has products indexed by $\mathrm{Mor}\,\mathcal{C}$. Then for and obj $X, Y \in \mathcal{C}$ and any $f, g : X \rightrightarrows Y$, $f = g$.

Thm. (General adjoint theorem)

For a functor $R : D \to C$, D and C have small limits, R preserves small limits, and R satisfies

$\forall X \in \mathcal{C}$ $\exists$ a set $\{d_i\}_{i \in I}$ in D and morph. $\{X \to R d_i\}_{i \in I}$ st.

$\forall d \in D$ $\forall X \to Rd$ $\exists i \in I : \exists X \to R d_i :$
$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & \# & Rd \\ & Rd_i \arrow[ur] & \end{tikzcd}$$

Ex) $U : \mathrm{Group} \longrightarrow \mathrm{Set}$ has a left adjoint

Group, Set have small limits

U preserves small limits

$\forall X \in \mathrm{Set}$

4-20-21

General adjoint functor thm

If $\mathcal{U}: A \to S$ is a cont. functor,

A is complete,

$\mathcal{U}$ satisfies the solution set condition:

$\forall s \ \exists$ set $\Phi = \{ s \xrightarrow{f_i} \mathcal{U}(a_i) \}$: $\forall f: s \to \mathcal{U}a \ \exists i \ \exists g: a \to a_i$:
$\begin{tikzcd} s \arrow[r] \arrow[d, "f_i"'] \arrow[dr, phantom, "*"] & \mathcal{U}a \\ \mathcal{U}a_i \arrow[ur, "\mathcal{U}g"'] & \end{tikzcd}$

Def. a **Banach Space** is a $\mathbb{R}$ (or $\mathbb{C}$) linear space V equiped with a norm $\|-\|: V \to [0, \infty)$:

$$\|x+y\| \leq \|x\| + \|y\| \quad \forall x, y \in V$$
$$\|t \cdot x\| = |t| \cdot \|x\| \quad \forall t \in \mathbb{R} \ \forall x \in V$$
$$\|x\| \geq 0 \quad \forall x \in V \quad \text{equality if } x = 0.$$

The metric on V: $d(x,y) = \|x-y\|$ must be complete.

Ex) a) $C([a,b], \mathbb{R})$, $\|f\| = \sup_{[a,b]} |f|$

b) $C^k([a,b], \mathbb{R})$, $\|f\| = \sum_{0 \leq i \leq k} \sup |f^{(i)}|$
$0 \leq k < \infty$

c) $\ell^p \quad 1 \leq p \leq \infty$

Def. A morphism $f : V \to V'$ of Banach spaces is a contractive linear map. i.e. $\|f(v)\| \leq \|v\| \ \forall v \in V$

product in Banach spaces w/ contractive maps

$$\prod_{i \in I} V_i = \{ (v_i)_{i \in I} \mid v_i \in V_i \quad i \mapsto \|v_i\| \text{ is bounded} \}$$

$$\|v\| = \sup_i \|v_i\|$$

Prop: The forgetfull functor

$$\mathcal{U} : \mathrm{Ban} \longrightarrow \mathrm{Set}$$
$$V \longmapsto \{ v \in V \mid \|v\| \leq 1 \}$$

$\mathcal{U}$ preserves small limits and has a left adjoint.

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$\mathcal{U} : \mathrm{Ban}_1 \longrightarrow \mathrm{Set}$ where Ban_1 are banach spaces w/ contractive maps $\|-\| \leq 1$

This is the unit ball functor.

$\mathcal{U}X = \{ x \in X \mid \|x\| \leq 1 \}$ $\quad$ $\mathcal{U}$ preserves small limits

Prop. $\mathcal{U}$ has a left adjoint

Proof. WTS $\mathcal{U}$ satisfies the solution set condition

$\forall S \in \mathrm{Set}\ \exists \{ S \xrightarrow{f_i} \mathcal{U}X_i \}_{i \in I}$ $\quad \forall g : S \to \mathcal{U}Y\ \exists i\ \exists X_i \xrightarrow{h} Y$:

$$\begin{tikzcd} S \arrow[rr, "g"] \arrow[dr, "f_i"'] & \arrow[d, phantom, "\#"] & \mathcal{U}Y \\ & \mathcal{U}X_i \arrow[ur, "\mathcal{U}h"'] & \end{tikzcd}$$

The cardinality of the Banach space X generated by the image of sum map $S \xrightarrow{p} \mathcal{U}X$ is bounded by

$\max(\#\mathbb{R}, \#S)^{\#\mathbb{N}} = \max(\#\mathbb{R}, \#S)$ (a set) $\square$

Prop. $LS = \ell^1 S = \{ S \xrightarrow{f} \mathbb{R} \mid \sum_{s \in S} f_s \text{ exists} \}$ $\quad \|f\| = \sum_{s \in S} |f_s|$

Note we say $\sum_{s \in S} f_s = A \in \mathbb{R}$ provided
for all nbd $\mathcal{U}$ of A, there is a finite subset $T \subset S$:
$\forall$ finite subset $T \subset Q \subset S$ $\sum_{s \in Q} f_s \in \mathcal{U}$.

Lemma. If $\sum_{s\in S} f_s$ exists, then

1) $f \to 0$, i.e.,

$\forall$ nbd U of 0, $S \setminus \{s \in S \mid f_s \in U\}$ is finite

2) $\operatorname{supp} f = \{s \in S \mid f_s \neq 0\}$ is countable

Ex) $\begin{tikzcd} \mathrm{Set} \arrow[r, dashed, shift left, "?\,L"] \arrow[r, phantom, "\perp"] & \mathrm{CompHaus} \arrow[l, shift left, "U"] \end{tikzcd}$

U preserves small limits. (Tychonoff theorem)
german

WTS U satisfies the solution set condition

We will need the special adjoint functor thm

Thm. Suppose $U : A \to S$ is a functor,

A is complete, well-powered, and has a cogen. set of objects

U preserves small limits, then

U has a left adjoint.

Def. A category is well-powered if any object has a set of isomorphism class of $\underbrace{\text{subobjects}}_{\text{monomorphism}}$ of X form a set

C a category and G a set of objects, G cogens C if

$\forall\ f, g : X \rightrightarrows Y$ $f \neq g$ $\exists\, h : Y \to B \in G$: $hf \neq hg$

Ex] $\mathrm{Set} \xleftarrow{U} \mathrm{Mod}_R$

U preserves small limits and satisfies the solution set condition. Thus U is representable

We want $M \in \mathrm{Mod}_R$ such that $\hom(M, X) \cong U(X)$

maps $M \to X$ pick out a single element in X

M is the free R-mod on one generator

Ex] $\mathrm{Set} \xleftarrow{\text{path}} \mathrm{Top}$

$\mathrm{Path}(T) = \hom([0,1], T)$ with the compact open topology

subbase : $B_{C,U} = \{ f : K \to Y \mid f|_C \subset U \}$

$C \subset K$ compact subspace of K
$U \subset Y$ open subspace of Y.

Ex] Man $\qquad C^\infty \mathrm{Ring}^{op}$

$$M \longmapsto C^\infty(M, \mathbb{R})$$

$$\hom(A, \mathbb{R}) \longleftarrow A$$

4-29-21

$$\mathrm{CRing} \underset{\mathrm{Spec}\,R}{\overset{\mathcal{O}(S)}{\rightleftarrows}} \mathrm{AffScheme}$$

$$\left\{\begin{matrix}\text{radical}\\ \text{ideals}\end{matrix}\right\} \underset{I \mapsto \{s \in S \mid \forall f \in I\ f(s) = 0\}}{\overset{\{f \in \mathcal{O}(S) \mid f|_{U^c} = 0\} \leftarrow U}{\rightleftarrows}} \left\{\begin{matrix}\text{open}\\ \text{subsets}\end{matrix}\right\}$$

or closed as closed $\overset{\cong}{\longleftrightarrow}$ open

Def. Let $R \in \mathrm{CRing}$ $A \subset R$, then the localization $R[A^{-1}] \in \mathrm{CRing}$

$$\hom(R[A^{-1}], T) \cong \hom(R,T)_A = \{R \xrightarrow{h} T \mid hA \subset T^\times\}$$

Construction $R[A^{-1}] = \{(r,a) = \frac{r}{a} \mid r \in R, a \in A\} / \sim$

$$\frac{r}{a} \sim \frac{r'}{a'} \iff (ra' - ar')b = 0 \text{ for some } b \in A$$

Ex) 1) $S = \mathbb{C}$ $\qquad \mathcal{O}(S) = \mathbb{C}[x] = R$

$$\mathrm{Spec}_{\mathbb{C}} R = \{R \to \mathbb{C}\} = \mathbb{C} \sqcup \{(0)\}$$

$U \subset \mathrm{Spec}\, R$

$I \triangleleft R \qquad I = (f)$

$$R \to R[A^{-1}] \qquad A = \{g \in \mathbb{C}[x] \mid (f) \subset \sqrt{(g)}\}$$
$$= \{g \in \mathbb{C}[x] \mid$$

$$R[A^{-1}] = \left\{\frac{h}{g} \,\middle|\, g = \prod_{j \in J}(x - \lambda_{pj})\right\} \qquad f = \prod_{i \in I}(x - \lambda_i) \quad \lambda_i \text{ distinct}$$

$p : J \to I$

Def. A presheaf on a topological space T (or locale) w/ values in cat C is a functor

$$\mathrm{Open}(T)^{op} \to C$$

5-4-21

The glueing property

Suppose X is a topological space,

then the presheaf $\mathrm{Open}(X)^{op} \longrightarrow CAlg_{\mathbb{R}}$ $U \mapsto C(U, \mathbb{R})$

is a sheaf provided

$\forall$ open cover $\{U_i\}_{i \in I}$ of some open $V \subset X$ the fork

$$C(V, \mathbb{R}) \longrightarrow \prod_{i \in I} C(U_i, \mathbb{R}) \underset{(-)|_{U_j \cap U_k} \circ \pi_k}{\overset{(-)|_{U_j \cap U_k} \circ \pi_j}{\rightrightarrows}} \prod_{j,k} C(U_j \cap U_k, \mathbb{R})$$

is an equalizer fork.

Informally: a cont. $f: V \to \mathbb{R}$ can be equiv specified as

$\{U_i \longrightarrow \mathbb{R}\}_{i \in I}$ s.t. $\forall j, k \in I$ $g_j|_{U_j \cap U_k} = g_k|_{U_j \cap U_k}$

Ex) If $T \xrightarrow{p} X$ is cont, then $\mathcal{F}\ (\mathrm{Open}X)^{op} \to \mathrm{Set}$

$\mathcal{F}(U) = \{ s : U \to T \mid ps : U \hookrightarrow X \}$

Theorem: For any $R \in CRing$, we have $\mathrm{Spec}R \in \mathrm{Top}$

and $\mathcal{O} : \mathrm{Open}(\mathrm{Spec}R)^{op} \longrightarrow CRing$ $\in Sh(\mathrm{Spec}R, CRing)$ (affine Scheme)

Def. a scheme is a ringed space $(X, \mathcal{O})$ that is locally isomorphic to an affine scheme.

i.e., $\exists$ open cover $\{U_i\}_{i \in I}$ of X : $\forall_{i \in I}$ $(U_i \mid \mathcal{O}|_{\mathrm{open}(U_i)^{op}})$ is iso to the Zariski spec of a CRing