Math 2360 - 121

Review Session 2.

Please ask your questions in chat.

Week 7 Problem 3.

Determine whether the given set S is a subspace of the vector space V.

A) $V = C^2 I = \{ f: I \longrightarrow \mathbb{R} \mid f', f''$ exists and f'' is continuous$\}$.

$S = \{ y \in V \mid y'' - 4y' + 3y = 0 \}$.

Recall: If V is a vector space, then a subset S of V is a vector subspace if S is closed under the operations of addition and multiplication by a number (and S is nonempty).
This means: a) $0 \in S$

b) If $y_1 \in S$ and $y_2 \in S$, then $y_1 + y_2 \in S$.

c) If $y \in S$ and $r \in \mathbb{R}$, then $r \cdot y \in S$.

$S = \{ y \in V \mid y'' - 4y' + 3y = 0 \}$.

a) $0 \in S$. $y = 0$, i.e., $y(t) = 0$ must belong to S.

Indeed, $y = y' = y'' = 0$, so
$y'' - 4y' + 3y = 0$.

b) If $y_1 \in S$ and $y_2 \in S$, then $y_1 + y_2 \in S$.

We know: $y_1, y_2 \in V$ such that

$$y_1'' - 4y_1' + 3y_1 = 0$$

and

$$y_2'' - 4y_2' + 3y_2 = 0.$$

We want: $(y_1 + y_2)'' - 4(y_1 + y_2)' + 3(y_1 + y_2) = 0$.

$$\begin{aligned} &\| \\ &y_1'' + y_2'' - 4y_1' - 4y_2 + 3y_1 + 3y_2 \\ &= (y_1'' - 4y_1' + 3y_1) + (y_2'' - 4y_2 + 3y_2) = 0 + 0 = 0. \end{aligned}$$

c) If $y \in S$, $r \in \mathbb{R}$, then $r \cdot y \in S$.

We know: $y \in V$, $y'' - 4y' + 3y = 0$.

We want: $(r \cdot y)'' - 4(r \cdot y)' + 3(r \cdot y) = 0$

$$\begin{aligned} &\| \\ &r \cdot (y'') - 4r \cdot (y') + 3r \cdot y \\ &= r \cdot (y'' - 4y' + 3y) = r \cdot 0 = 0. \end{aligned}$$

Thus, S is a subspace of V. ▨

Week 9 Problem 3

Consider the inner product

$$\langle p, q \rangle := \int_0^1 p(x) \cdot q(x)\, dx.$$

where $p, q \in V = \{ a \cdot x + b \mid a, b \in \mathbb{R} \}$.

$\|$ the vector space of real polynomials of degree less than 2

The linear functional $f : V \to \mathbb{R}$ is defined by

$p \in V$ $\quad f(p) := p'(18) + 4\,p(3).$

$a_0 + a_1 \cdot x + \cdots + a_n \cdot x^n$

degree := the largest k such that $a_k \neq 0$.

degree $< 2 \iff a_k = 0$ forall $k \geqslant 2$.

(Such q exists by the Riesz representation theorem.) Find $q \in V$ such that $f(p) = \langle p, q \rangle$ for all $p \in V$.

Recall: any $p \in V$ has the form $p(x) = ax + b$, for some $a, b \in \mathbb{R}$.

We compute $f(p) = p'(18) + 4\,p(3)$
$= a + 4(3a + b) = 13a + 4b.$

$p'(x) = a$

Given $q \in V$, $q(x) = a' \cdot x + b'$, we compute $\langle p, q \rangle = \int_0^1 p(x)\, q(x)\, dx$

$$= \int_0^1 (ax + b)(a' x + b')\, dx$$

$$= \int_0^1 \left(a a' x^2 + (ab' + a'b)\, x + bb' \right) dx$$

$$= \left(a a' \frac{x^3}{3} + (ab' + a'b) \frac{x^2}{2} + bb' x \right) \Bigg|_{x=0}^{x=1}$$

$$= \frac{1}{3} a a' + \frac{1}{2}(ab' + a'b) + bb'.$$

Riesz = (Reese)

Find $q \in V$ such that

$$\underset{\|}{f(p)} = \underset{\|}{\langle p, q \rangle} \quad \text{for all } p \in V.$$

$$13a + 4b = \frac{1}{3} a a' + \frac{1}{2}(ab' + a'b) + bb'$$

$q(x) = a'x + b'$

Collect coefficients of a & b and move everything to one side:

$$(\star) \quad \left(\frac{1}{3} a' + \frac{1}{2} b' - 13 \right) \cdot a + \left(\frac{1}{2} a' + b' - 4 \right) \cdot b = 0$$

This relation must hold for all $p \in V$, $p(x) = ax + b$, i.e., for all a and b.

$(\star)$ vanishes for all a and b if the coefficients of a and b vanish.

Thus,

$$\begin{cases} \frac{1}{3} a' + \frac{1}{2} b' = 13 \\ \frac{1}{2} a' + b' = 4. \end{cases}$$

$$\underset{\underset{\frac{1}{12}}{\|}}{\left(\frac{1}{3} - \frac{1}{4} \right)} a' = 13 - \frac{1}{2} \cdot 4 = 11$$

$a' = 11 \cdot 12 = 132.$

$b' = 4 - \frac{1}{2} a' = 4 - 11 \cdot 6 = -62.$

Answer: $q(x) = 132\, x - 62.$ ∎