MATH 2360 - 121

Review Session 1.

HW 1 Problem 6

Solve the following system of equations.

$$\begin{cases} \frac{1}{2}x + \frac{-3}{4}y = \frac{1}{4} \\ \frac{-1}{2}x + \frac{-1}{6}y = -2 \end{cases}$$

Solution Gaussian elimination.

Get rid of fractions by multiplying the first equation by 4 and the second equation by 6:

$$\begin{cases} 2x - 3y = 1 \\ -3x - y = -12 \end{cases}$$

Multiply the second equation by 3 and subtract the result from the first equation:

$$-9x - 3y = -36$$

$$(2-(-9))x + \overbrace{(-3-(-3))}^{0}y = 1-(-36),$$

$$11x = 37 \qquad x = \frac{37}{11}.$$

Substitute $x = \frac{37}{11}$ into

$$2x - 3y = 1$$

$$2 \cdot \frac{37}{11} - 3y = 1$$

$$2 \cdot \frac{37}{11} - 1 = 3y$$

$$y = \frac{1}{3}\left(2 \cdot \frac{37}{11} - 1\right)$$

$$= \frac{1}{3} \cdot \frac{74 - 11}{11}$$

$$= \frac{63}{3 \cdot 11} = \frac{21}{11}.$$

Answer: $x = \frac{37}{11}$, $y = \frac{21}{11}$.

Verify the answer by substituting x and y back int the original equations:

$$\frac{1}{2} \cdot \frac{37}{11} + \frac{-3}{4} \cdot \frac{21}{11} = \frac{1}{4}$$

$$\frac{2 \cdot 37 - 3 \cdot 21}{4 \cdot 11} = \frac{74 - 63}{44}$$

$$= \frac{11}{44} = \frac{1}{4}.$$

HW 3 Problem 2

If
$$A = \begin{pmatrix} 2 & -4 & -11 \\ -1 & 1 & 3 \\ -1 & 2 & 5 \end{pmatrix}$$

then $A^{-1} = ?$

Given $b = \begin{pmatrix} 0 \\ -1 \\ -3 \end{pmatrix}$, solve $Ax = b$.

Solution Compute A^{-1} using Gaussian elimination

$$\left(\begin{array}{ccc|ccc} 2 & -4 & -11 & 1 & 0 & 0 \\ -1 & 1 & 3 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array}\right)$$

Step 1 Exchange $R1 \longleftrightarrow R2$.

$$\left(\begin{array}{ccc|ccc} -1 & 1 & 3 & 0 & 1 & 0 \\ 2 & -4 & -11 & 1 & 0 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array}\right)$$

Step 2 Set $R2 \leftarrow R2 + 2 \cdot R1$

$R3 \leftarrow R3 - R1$.

$$\left(\begin{array}{ccc|ccc} -1 & 1 & 3 & 0 & 1 & 0 \\ 0 & -2 & -5 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 & -1 & 1 \end{array}\right)$$

Step 3: Exchange $R2 \longleftrightarrow R3$.

$$\left(\begin{array}{ccc|ccc} -1 & 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & -1 & 1 \\ 0 & -2 & -5 & 1 & 2 & 0 \end{array}\right)$$

Step 4: Set $R3 \leftarrow R3 + 2 \cdot R2$.

$$\left(\begin{array}{ccc|ccc} -1 & 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 & 0 & 2 \end{array}\right)$$

Step 5: Set $R1 \leftarrow R1 - R2$

$$\left(\begin{array}{ccc|ccc} -1 & 0 & 1 & 0 & 2 & -1 \\ 0 & 1 & 2 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 & 0 & 2 \end{array}\right)$$

Step 6: Set $R1 \leftarrow R1 + R3$
$R2 \leftarrow R2 + 2 \cdot R3$

$$\left(\begin{array}{ccc|ccc} -1 & 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & 2 & -1 & 5 \\ 0 & 0 & -1 & 1 & 0 & 2 \end{array}\right)$$

Step 7: Set $R1 \leftarrow -R1$
Set $R3 \leftarrow -R3$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -2 & -1 \\ 0 & 1 & 0 & 2 & -1 & 5 \\ 0 & 0 & 1 & -1 & 0 & -2 \end{array}\right)$$

Answer:

$$A^{-1} = \begin{pmatrix} -1 & -2 & -1 \\ 2 & -1 & 5 \\ -1 & 0 & -2 \end{pmatrix}.$$

Verification: Compute

$$A^{-1} \cdot A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Part 2: Given $b = \begin{pmatrix} 0 \\ -1 \\ -3 \end{pmatrix}$,

solve $Ax = b$.

Solution: Multiply both sides of $Ax = b$ by A^{-1} on the left:

$$x = A^{-1} \cdot A \cdot x = \underbrace{A^{-1} \cdot b}$$

$$= \begin{pmatrix} -1 & -2 & -1 \\ 2 & -1 & 5 \\ -1 & 0 & -2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \\ -3 \end{pmatrix} = \begin{pmatrix} 5 \\ -14 \\ 6 \end{pmatrix}.$$

Answer: $x = \begin{pmatrix} 5 \\ -14 \\ 6 \end{pmatrix}$

Verification: $A \cdot x = \begin{pmatrix} 2 & -4 & -11 \\ -1 & 1 & 3 \\ -1 & 2 & 5 \end{pmatrix} \begin{pmatrix} 5 \\ -14 \\ 6 \end{pmatrix}$,

$$= \begin{pmatrix} 10 + 4 \cdot 14 - 66 \\ -5 - 14 + 18 \\ -5 - 28 + 30 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -3 \end{pmatrix} = b$$

HW 1 Problem 10

Solve the equation

$$-2x - 3y - 9z = -8.$$

<u>Solution</u> Use Gaussian elimination.

$$-2x - 3y - 9z = -8$$

Divide this equation by -2 to make the coefficient of x equal to 1.

$$x + \frac{3}{2}y + \frac{9}{2}z = 4.$$

Gaussian elimination stops here. The remaining variables y and z are <u>free variables</u>.

We have $x = 4 - \frac{3}{2}y \overset{-}{\cancel{+}} \frac{9}{2}z$.

To express the solution in the form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix} + s \cdot \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix} + t \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

free variables

Take $y = s$ and $z = t$.
(We renamed the free variables.)

We have
$$x = 4 - \frac{3}{2}s \overset{-}{\cancel{+}} \frac{9}{2}t$$
$$y = s$$
$$z = t.$$

That is,

Answer: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -\frac{3}{2} \\ 1 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} -\frac{9}{2} \\ 0 \\ 1 \end{pmatrix}.$

Verification:

$$-2x - 3y - 9z = -8$$

$$\parallel$$

$$-2\left(4 - \frac{3}{2}s \overset{-}{\cancel{+}} \frac{9}{2}t\right) - 3s - 9t$$

$$\parallel$$

$$-8 + \underbrace{3s + 9t - 3s - 9t}_{} = -8 \ \checkmark$$

HW 1 Problem 3

If the linear system

$$\begin{cases} 5x - 6y + 5z = 7 \\ -3x + 5y + 9z = -8 \\ -11x + 16y + h \cdot z = k \end{cases}$$

has infinitely many solutions, then $k = ?$, $h = ?$

Solution Use Gaussian elimination.

$$\begin{pmatrix} 5 & -6 & 5 \\ -3 & 5 & 9 \\ -11 & 16 & h \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ -8 \\ k \end{pmatrix}.$$

Step 1: $R1 \leftarrow R1 \cdot \frac{1}{5}$.

$$\begin{pmatrix} 1 & -\frac{6}{5} & 1 & \frac{7}{5} \\ -3 & 5 & 9 & -8 \\ -11 & 16 & h & k \end{pmatrix}$$ augmented matrix

Step 2 $R2 \leftarrow R2 + 3 \cdot R1$, $R3 \leftarrow R3 + 11 \cdot R1$

$$\begin{pmatrix} 1 & -\frac{6}{5} & 1 & \frac{7}{5} \\ 0 & \frac{7}{5} & 12 & -\frac{19}{5} \\ 0 & \frac{14}{5} & h+11 & k+\frac{77}{5} \end{pmatrix}$$

Step 3: $R2 \leftarrow 5 \cdot R2$ $R3 \leftarrow 5 \cdot R3$

$$\begin{pmatrix} 1 & -\frac{6}{5} & 1 & \frac{7}{5} \\ 0 & 7 & 60 & -19 \\ 0 & 14 & 5h+55 & 5k+77 \end{pmatrix}$$

Step 4: $R3 \leftarrow R3 - 2 \cdot R2$

$$\begin{pmatrix} 1 & -\frac{6}{5} & 1 & \frac{7}{5} \\ 0 & 7 & 60 & -19 \\ 0 & 0 & 5h-65 & 5k+115 \end{pmatrix}$$

Step 5 If $5h - 65 \neq 0$, then the Gaussian elimination process will compute the unique solution to this linear system.

$5h - 65 = 0 \iff h = 13$

If $h \neq 13$, the linear system has a unique solution.

If $h = 13$, then we get

$$\left(\begin{array}{ccc|c} 1 & -\frac{6}{5} & 1 & \frac{7}{5} \\ 0 & 7 & 60 & -19 \\ 0 & 0 & 0 & 5k+115 \end{array}\right)$$

$$\begin{cases} x - \frac{6}{5}y + 3 = \frac{7}{5} \\ 7y + 60z = -19 \\ 0 = 5k+115 \end{cases}$$

If $5k + 115 \neq 0$, i.e., $k \neq 23$, then the system has no solutions.

If $5k + 115 = 0$,
i.e., $k = 23$
then the last equation
becomes $0 = 0$

In this case, the Gaussian elimination terminates with z as a free variable. That is, the linear system has infinitely many solutions.

<u>Answer</u>: $h = 13$, $k = 23$.