# MATH 2360 Digest Week 1

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Hello MATH 2360 section 121, my name is Mason Springfield and I am the TA for this class. Each week I will be preparing a brief digest for all of you to view with common questions from the previous week. Please, feel free to reach out to me if questions you had about the material were not covered here. You can contact me at mason.springfield@ttu.edu and I will respond as soon as I can. I look forward to working with you all.

## 1 Week 1

### 1.1 Homework Questions

4) Here we are asked to solve the linear equation

$$-7x + 4y - 7z = 0$$

and present our solution in the form of a solution set

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + t \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$$

The first thing one might notice is that this equation seems to have multiple solutions, and they would be right! But how do we show this? To start, we may solve the equation in terms of $z$,

$$\begin{aligned} -7x + 4y - 7z &= 0 \\ -7z &= 7x - 4y \\ z &= -x + \frac{4}{7}y \end{aligned}$$

From this we see that the value of $z$ depends on our selection of values $x$ and $y$, so let us select arbitrary values $s$ and $t$ for $x$ and $y$ respectively. Then our substitution will look something like this,

$$\begin{aligned} x &= s \\ y &= t \\ z &= -s + \frac{4}{7}t \end{aligned}$$

Or, alternatively, our substitution could look like this,

$$\begin{aligned} x &= s + 0t \\ y &= 0s + t \\ z &= -s + \frac{4}{7}t \end{aligned}$$

From this it becomes clear that we need the following for our solution set

$$x_1 = 1, x_2 = 0, y_1 = 0, y_2 = 1, z_1 = -1, z_2 = \frac{4}{7}$$

s So our final solution will be

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ \frac{4}{7} \end{bmatrix}$$

Since our selection for $s$ and $t$ can be any real number, the equation has infinite solutions. Thus, the expression above forms a complete solution set for the linear equation $-7x + 4y - 7z = 0$ and thus we have solved the equation, completing the problem. In fact, a similar method is used for problem 10 this week.

# 2 Week 2

## 2.1 Homework Questions

3) For this question we are asked to identify the elementary row operations which define the following transformation,

$$\begin{bmatrix} -7 & 5 & -6 & 8 \\ 7 & -2 & 0 & 7 \\ -6 & 3 & -4 & -5 \\ -7 & -6 & 5 & 4 \end{bmatrix} \to \begin{bmatrix} -7 & 5 & -6 & 8 \\ 7 & -2 & 0 & 7 \\ 15 & -3 & -4 & 16 \\ -7 & -6 & 5 & 4 \end{bmatrix}$$

As such, first we should recall what the elementary row operations are,

- First, we may swap two rows.
- Second, we can multiply a row by a scalar $s$.
- Third, we may add a scalar multiple of one row to another row.

Looking back at our given transformation it is clear that this is neither a transformation of the first or second kind. So, we must be looking at a transformation which is the result of the addition of a scalar multiple of one row to another row. It is clear that the row that has been affected by the transformation is the third row, so which row has been added to it? Observation will eventually lead one to the conclusion that 3 times the second row has been added to the third row. In order to answer this question, one would put the following information into WebWork, maintaining the spacing and style

3 E 2 + E 3
This is the answer to such a question.

4) Here we are asked to reduce the following matrix to reduced row-echelon form,

$$\begin{bmatrix} 3 & 3 & -2 & 0 \\ -1 & 3 & -2 & 16 \\ 1 & -1 & -2 & 0 \end{bmatrix}$$

A reduction is preformed by manipulating the matrix with elementary operations until it is of the form,

$$\begin{bmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \end{bmatrix}$$

Or, more generally, until the pivot points of the matrix (the 1s from above) have only zeros above and below them. So in order to reduce the above matrix, let us start by multiplying the first row by $\frac{1}{3}$,

$$\begin{bmatrix} 3 & 3 & -2 & 0 \\ -1 & 3 & -2 & 16 \\ 1 & -1 & -2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & \frac{-2}{3} & 0 \\ -1 & 3 & -2 & 16 \\ 1 & -1 & -2 & 0 \end{bmatrix}$$

Now, we add the first row to the second row, and subtract the first row from the third row (i.e. add -1 times the first row to the second row),

$$\begin{bmatrix} 1 & 1 & \frac{-2}{3} & 0 \\ -1 & 3 & -2 & 16 \\ 1 & -1 & -2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & \frac{-2}{3} & 0 \\ 0 & 4 & \frac{-8}{3} & 16 \\ 1 & -1 & -2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & \frac{-2}{3} & 0 \\ 0 & 4 & \frac{-8}{3} & 16 \\ 0 & -2 & \frac{-4}{3} & 0 \end{bmatrix}$$

Now, multiply the second row by $\frac{1}{4}$,

$$\begin{bmatrix} 1 & 1 & \frac{-2}{3} & 0 \\ 0 & 4 & \frac{-8}{3} & 16 \\ 0 & -2 & \frac{-4}{3} & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & \frac{-2}{3} & 0 \\ 0 & 1 & \frac{-2}{3} & 4 \\ 0 & -2 & \frac{-4}{3} & 0 \end{bmatrix}$$

Now, subtract the second row from the first row, and add two times the second row to the third row,

$$\begin{bmatrix} 1 & 1 & \frac{-2}{3} & 0 \\ 0 & 1 & \frac{-2}{3} & 4 \\ 0 & -2 & \frac{-4}{3} & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & \frac{-2}{3} & 4 \\ 0 & -2 & \frac{-4}{3} & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & \frac{-2}{3} & 4 \\ 0 & 0 & \frac{-8}{3} & 8 \end{bmatrix}$$

Finally, multiply the third row by $\frac{-3}{8}$,

$$\begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & \frac{-2}{3} & 4 \\ 0 & 0 & \frac{-8}{3} & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & \frac{-2}{3} & 4 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

and add $\frac{2}{3}$ times the third row to the second row,

$$\begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & \frac{-2}{3} & 4 \\ 0 & 0 & 1 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

Thus, we have successfully reduced our matrix to reduced row-echelon form.

# 3 Week 3

## 3.1 Homework Questions

7) Here we are asked to give a 4 x 4 elementary matrix that corresponds to the operation $R_3 - 4R_1 \to R_3$, or the elementary matrix which corresponds to subtracting 4 times row 1 from row 3. So, how do we construct such an elementary matrix? First we will start with the 4 x 4 identity matrix,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This matrix corresponds with doing nothing to our original matrix, as such it acts as a base for all elementary matrices! Now, the elementary matrix which corresponds to our operation is,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Notice how we have put a -4 in the first column of the third row, this directly corresponds to subtracting four times row one from row three! To verify this, observe the following matrix multiplication,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -4 & 3 \\ -4 & 4 \\ 4 & -1 \\ 4 & -3 \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ -4 & 4 \\ 20 & -13 \\ 4 & -3 \end{bmatrix}$$

As you can see, multiplying the given matrix on the right by our elementary matrix corresponded directly to a subtraction of row 3 by 4 times row 1! Thus we have constructed the desired elementary matrix.

8) Here we are asked to find the LU factorization of a given 3x3 matrix, but what does that entail? We are looking for two matrices, $L$ and $U$ with the following characteristics,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_1 & 1 & 0 \\ l_2 & l_3 & 1 \end{bmatrix}, U = \begin{bmatrix} u_1 & u_2 & u_3 \\ 0 & u_4 & u_5 \\ 0 & 0 & u_6 \end{bmatrix}, and, A = LU$$

Looking at what we need, it seems that a good place to start might be Gauss-Jordan elimination. So given,

$$A = \begin{bmatrix} 4 & 4 & -1 \\ -16 & -12 & 1 \\ 12 & -4 & 11 \end{bmatrix}$$

We will start by finding $U$. Observe that,

$$\begin{bmatrix} 4 & 4 & -1 \\ -16 & -12 & 1 \\ 12 & -4 & 11 \end{bmatrix} \to \begin{bmatrix} 4 & 4 & -1 \\ 0 & 4 & -3 \\ 12 & -4 & 11 \end{bmatrix} \to \begin{bmatrix} 4 & 4 & -1 \\ 0 & 4 & -3 \\ 0 & -16 & 14 \end{bmatrix} \to \begin{bmatrix} 4 & 4 & -1 \\ 0 & 4 & -3 \\ 0 & 0 & 2 \end{bmatrix}$$

So, we have that,

$$U = \begin{bmatrix} 4 & 4 & -1 \\ 0 & 4 & -3 \\ 0 & 0 & 2 \end{bmatrix}$$

Now, to find $l_1, l_2, l_3$ we observe the reductions from our finding of $U$. Much like we'll see in problem 10, this can be done by taking inverses of the elementary matrices that correspond to the actions taken above, since

$$E_3E_2E_1A = U \implies A = E_1^{-1}E_2^{-1}E_3^{-1}U$$

So we have that,

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

So,

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}, E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix}$$

And,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 3 & -4 & 1 \end{bmatrix}$$

Thus we have found our answers as,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 3 & -4 & 1 \end{bmatrix}, U = \begin{bmatrix} 4 & 4 & -1 \\ 0 & 4 & -3 \\ 0 & 0 & 2 \end{bmatrix}$$

One should verify that $A = LU$.

10) Here we are asked to find a collection of elementary matrices, $E_1, E_2, E_3, E_4$, which correspond to a reduction of a given matrix $A$ to row-echelon form. So we need,

$$E_4E_3E_2E_1A = I_n$$

Now, we are given the following,

$$\underbrace{\begin{bmatrix} 1 & 0 & 5 \\ -6 & 0 & -29 \\ 0 & -7 & 0 \end{bmatrix}}_{A} \to \underbrace{\begin{bmatrix} 1 & 0 & 5 \\ 0 & 0 & 1 \\ 0 & -7 & 0 \end{bmatrix}}_{E_1A} \to \underbrace{\begin{bmatrix} 1 & 0 & 5 \\ 0 & -7 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_2E_1A} \to \underbrace{\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_3E_2E_1A} \to \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_4E_3E_2E_1A}$$

Let us start by analyzing what row operations are being performed in each step. $E_1$ seems to add 6 times row one to row two. $E_2$ seems to swap rows two and three. $E_3$ seems to multiply row 2 by $\frac{-1}{7}$. Finally, $E_4$ subtracts 5 times row 3 from row one. So our resulting matrices are,

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-1}{7} & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_4 = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Please, take the time to analyze these matrices to see why they correspond to the described operation. Now, we wish to take the inverses of these matrices that we have,

$$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}$$

Thankfully, finding the inverse of an elementary matrix is a fairly simple task, and we have that,

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -6 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_4^{-1} = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Verify for yourself that, $A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}$. Also, can you see how we took the inverse of the elementary matrices? A hint is that we preform the inverse elementary operation, so addition becomes subtraction and etc. Thus, the problem is solved.