Mathematics 5325 (Topology II)

Midterm 2

1. Construct an example of a connected simplicial set X whose fundamental group is not commutative. (Make sure to include a proof that the computed group is indeed noncommutative.)

2. Fix a parameter $n > 0$ and a generator $p \in \mathbf{Z}/n$ of $\mathbf{Z}/n$ (i.e., $\gcd(p, n) = 1$). The *lens space* $L(p, n)$ is defined as follows: There are $n + n$ generating 3-simplices, denoted by c_i and d_i ($i \in \mathbf{Z}/n\mathbf{Z}$). The simplicial set looks like an orange sliced along the equatorial plane, with c_i above and d_i below, the 0th vertices being the center and the 1st vertices being the poles. The relations are as follows: $\mathrm{d}_1(c_i) = \mathrm{d}_1(d_i)$ (gluing along the equatorial plane), $\mathrm{d}_2(c_i) = \mathrm{d}_3(c_{i+1})$ and $\mathrm{d}_2(d_i) = \mathrm{d}_3(d_{i+1})$ (gluing along the lateral sides of carpels). To these relations we add the following: $\mathrm{d}_0(c_i) = \mathrm{d}_0(d_{i+p})$ (the lens space relation). Compute the fundamental group of the lens space, taking the north pole as the basepoint.

3. Compute the fundamental group of the classifying simplicial set $\mathrm{B}\mathbf{Z}/2$ of the group $\mathbf{Z}/2$.

4. Does the real projective plane admit a nontrivial connected covering (meaning a covering $f\colon X \to Y$ such that X is connected and f is not an isomorphism)? (Include a complete proof of your claim.)

5. Fix $n \geq 0$. Consider the functor $\mathsf{U}\colon \mathsf{sSet} \to \mathsf{Set}$ that sends a simplicial set X to the set X_n and a simplicial map $f\colon X \to Y$ to the map of sets $f\colon X_n \to Y_n$.

- Prove or disprove: U has a left adjoint.
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