Mathematics 5325 (Topology II)

Midterm 1

1. Prove or disprove: for any $n \geq 0$ and any real-valued $n \times n$ matrix A such that for all i and j we have $A_{i,j} > 0$, there is an n-vector b such that $b_k > 0$ for all k and $Ab = \lambda b$ for some $\lambda > 0$.

2. Construct an example of a simplicial set X with a cohomology class $x \in \mathsf{H}^2(X)$ such that $x^2 \neq 0$. (Here $x^2 = x \cup x$ is the cup product of x with itself.) Construct an example of a simplicial set Y with a cohomology class $y \in \mathsf{H}^2(Y)$ such that $y \neq 0$ and $y^2 = 0$, but $\mathsf{H}^4(Y) \neq 0$.

3. Consider the n-sphere with tubes along coordinate axes removed, i.e., the set W of points $x \in \mathbf{R}^{n+1}$ such that $\|x\| \leq 3^n$ and $\|p_i(x)\| \geq 1$ for all i, where $p_i\colon \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ sets the ith coordinate to 0 and leaves the other coordinates unchanged. Compute the singular homology of W as a module over the cohomology ring with coefficients in an arbitrary ring A.

4. The *lens space* $L(p, n)$ is defined as follows: $n > 0$ is an integer and $p \in \mathbf{Z}/n$ is a generator (i.e., $\gcd(p, n) = 1$). There are $n + n$ generating 3-simplices, denoted by c_i and d_i ($i \in \mathbf{Z}/n\mathbf{Z}$). The simplicial set looks like an orange sliced along the equatorial plane, with c_i above and d_i below, the 0th vertices being the center and the 1st vertices being the poles. The relations are as follows: $\mathrm{d}_1(c_i) = \mathrm{d}_1(d_i)$ (gluing along the equatorial plane), $\mathrm{d}_2(c_i) = \mathrm{d}_3(c_{i+1})$ and $\mathrm{d}_2(d_i) = \mathrm{d}_3(d_{i+1})$ (gluing along the lateral sides of carpels). To these relations we add the following: $\mathrm{d}_0(c_i) = \mathrm{d}_0(d_{i+p})$ (the lens space relation). Compute the homology as a module over the cohomology ring with coefficients in an arbitrary ring A.

5. Compute the homology of the group $\mathbf{Z}/2$ as a module over the cohomology ring of the group $\mathbf{Z}/2$ with coefficients in an arbitrary ring A.