TOPOLOGY^2

w/ Dr. Pavlov

NOTES JAN. 17

Let X and Y be topological spaces.

[Actually, we are currently only considering X & Y to be simplicial sets, but these results hold for general topological spaces].

From X & Y, we can form the Cartesian product $X \times Y$.

The Cartesian Product of Sets

part 1

set A
set B
form a set $A \times B$
which is defined:

$$A \times B = \{ z \mid \exists a \in A, b \in B : z = (a, b) \}$$

AND

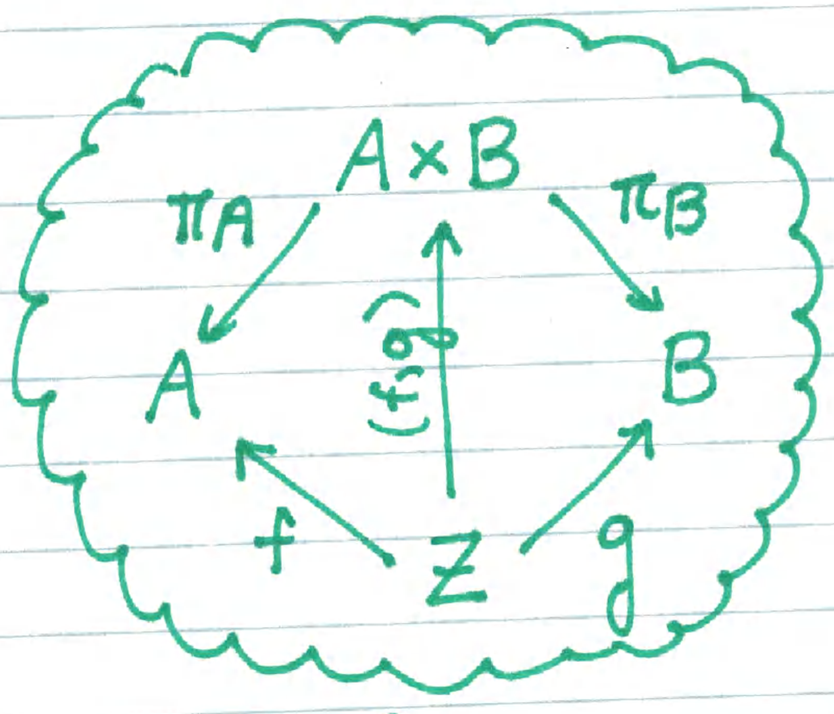
part 2

maps $\pi_A : A \times B \longrightarrow A$ (projection onto A)
$\pi_B : A \times B \longrightarrow B$ (projection onto B)

such that

condition

$\forall$ set Z w/ maps
$f : Z \longrightarrow A$
$g : Z \longrightarrow B$



$\exists!$ map $(f,g) : Z \longrightarrow A \times B$

$\ni$
such that $\pi_A \circ (f,g) = f$ and $\pi_B \circ (f,g) = g$

EILENBERG-ZILBER Theorem

The link between homology groups X and Y and their cartesian product $X \times Y$.
Samuel Eilenberg and J.A. Zilber

$$H(C(X \times Y)) \cong H(C(X) \otimes C(Y))$$

homology — chain complex — tensor product

CHAIN COMPLEXES

$$\cdots \leftarrow C_0 \leftarrow C_1 \leftarrow C_2 \leftarrow \cdots$$

Generally, $C = \{C_n\}_{n \in \mathbb{Z}}$

The chain complex can be represented by a graded family. But we can also express this chain complex in direct sum notation:

$$C = \bigoplus_{i=0}^{\infty} C_i$$

This is similar to the way we can describe polynomials as the direct sum of monomials in each degree:

$$f(x) = x^3 + 2x^2 + 5x - 3$$

x^3: degree 3 monomial; $2x^2$: degree 2 monomial; $5x$: degree 1 monomial; -3: degree 0 monomial

→

→ P_0 : degree 0 monomials
P_1 : degree 1 monomials
P_2 : degree 2 monomials
⋮ etc.

Polynomials : $P = \bigoplus_{i=0}^{\infty} P_i$ ↖

And this categorizes all polynomials!

DEVELOPMENT:

So... let $C = \bigoplus_{i=0}^{\infty} C_i$ (a chain complex)

$E = \bigoplus_{i=0}^{\infty} E_i$ (a chain complex)

and we define

$$C \otimes E = \bigoplus_{i,j=0}^{\infty} (C_i \otimes E_j)$$

let $d_{Ci} : C_i \longrightarrow C_{i-1}$

$d_{Ei} : E_i \longrightarrow E_{i-1}$

(homology "boundary maps")

then we define

$$d_{Cj \otimes Ej} = d_{Cj} \otimes {}_j d_{Ej} + {}_j d_{Cj} \otimes d_{Ej}$$

FOR EXAMPLE:

✳ $d_{C \otimes E} = d_C(a) \otimes b + (-1)^a a \otimes d_E(b)$

Resolution:

$d_{C\otimes E} = d_C \otimes id_E + id_C \otimes d_E$

$d^2_{C\otimes E} = 0$

and the Koszul sign rule $(-1)^{ij}$

$d(c\otimes e) = dc\otimes e + (-1)^{|c|} c\otimes de$

We proceed to verify the property: $d^2(c\otimes e)=0$

$d^2(c\otimes e) = d(d(c\otimes e)) = d(dc\otimes e + (-1)^{|c|} c\otimes de)$

$= d(dc\otimes e) + (-1)^{|c|} d(c\otimes de)$

$= [d^2c\otimes e + (-1)^{|dc|}(dc\otimes de)] + (-1)^{|c|} d(c\otimes de)$

$= [d^2c\otimes e] + (-1)^{|dc|}(dc\otimes de)$

$+(-1)^{|c|}[dc\otimes de + (-1)^{|c|} c\otimes d^2e]$

$= [d^2c\otimes e] + (-1)^{|dc|}(dc\otimes de)$

$+(-1)^{|c|}(dc\otimes de) + (-1)^{2|c|}[c\otimes d^2e]$

$= (dc\otimes de)[(-1)^{|dc|} + (-1)^{|c|}] = 0$

$d^2c = 0$, $d^2e = 0$, so the tensor product $0\otimes k = 0$, $k\otimes 0 = 0$

The degree of the boundary is one less than the simplex.

$|dc| + 1 = |c|$

So, these terms will cancel, that is, if

$|dc| = 1$, $|c| = -1$,

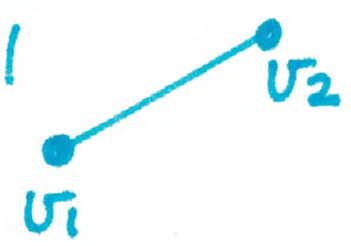
$(-1)^1 + (-1)^{-1} = 0$ and vice-versa!

So, $d^2(c\otimes e) = 0$.

$\cdots \to C_m \xrightarrow{d} C_{m+1} \to \cdots$

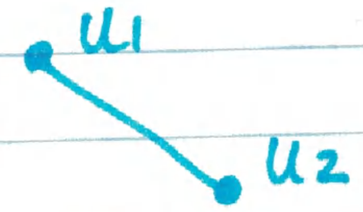
Ψ_c $\quad$ Ψ_{dc}

Ex: I_1: the unit interval



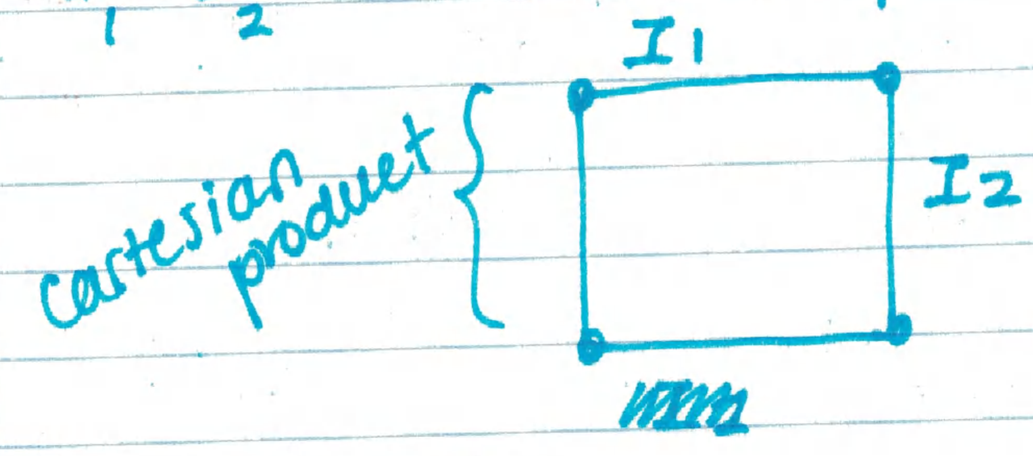
$\partial(I_1) = \{v_1, v_2\}$

I_2: the unit interval



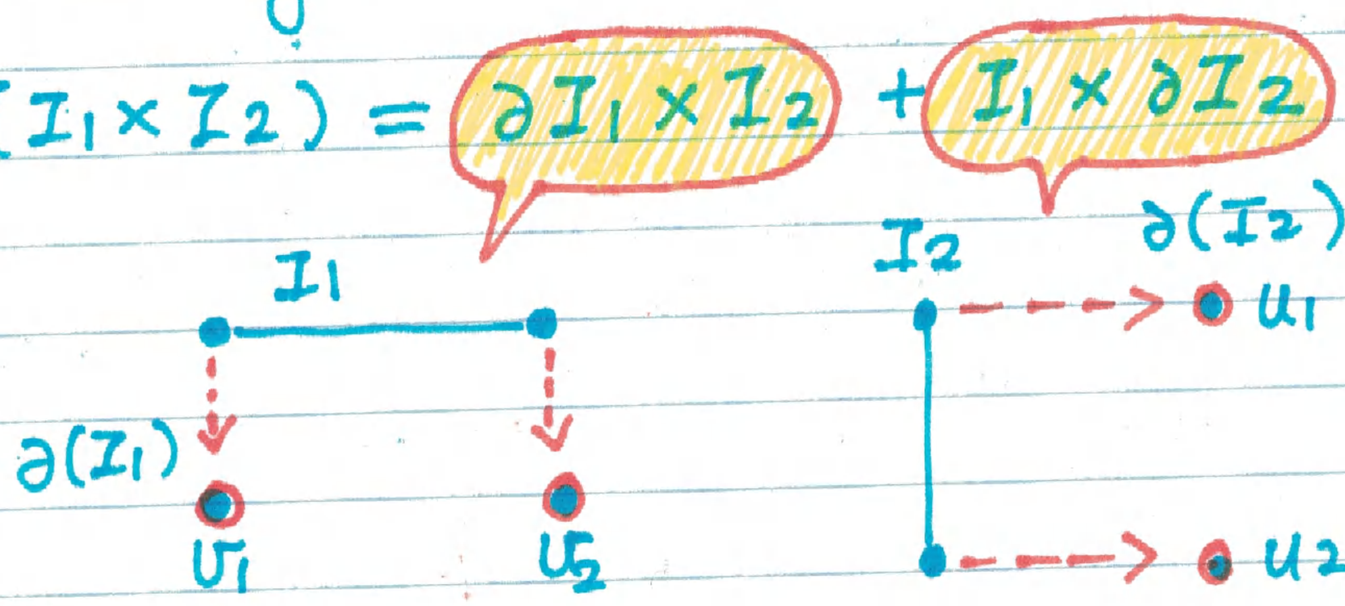
$\partial(I_2) = \{u_1, u_2\}$

$I_1 \times I_2$: the cartesian product

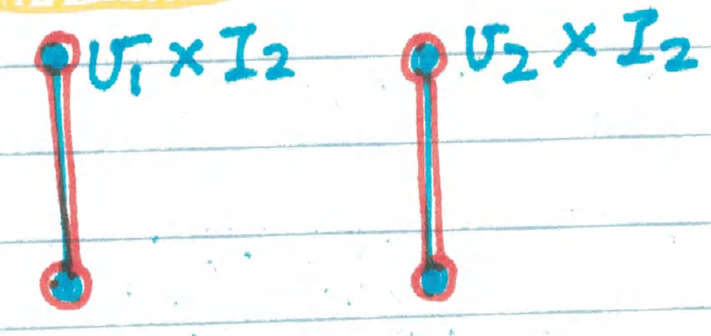


Consider the boundary of $I_1 \times I_2$:

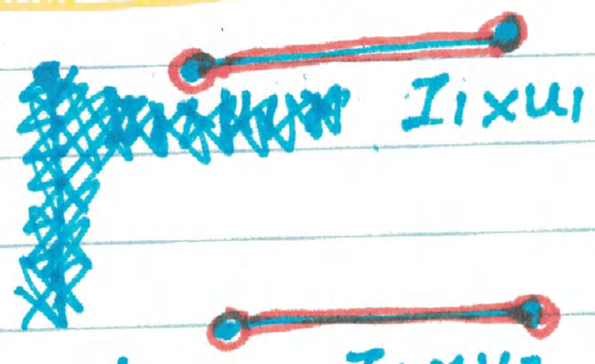
$$\partial(I_1 \times I_2) = \partial I_1 \times I_2 + I_1 \times \partial I_2$$



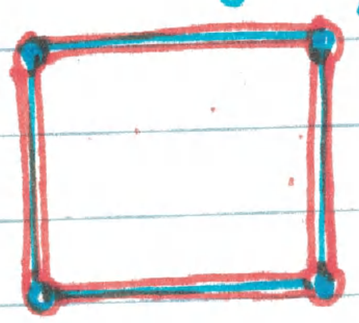
$\partial(I_1) \times I_2$



$I_1 \times \partial(I_2)$



add together



EILENBERG-ZILBER MAP

$$C(X) \otimes C(Y) \longrightarrow C(X \times Y)$$

There is a map in this direction, called the Eilenberg-Zilber Map

we name this map "nabla" ∇

$$\nabla : C(X) \otimes C(Y) \longrightarrow C(X \times Y)$$

$$\alpha \quad \beta \longmapsto \alpha \times \beta$$

$$\alpha : \Delta^m \to X$$

$$\beta : \Delta^n \to Y$$

$$\alpha \times \beta : \Delta^m \times \Delta^n \to X \times Y$$

The Eilenberg-Zilber map induces a map on simplicial chains:

$$C(\alpha \times \beta) : C(\Delta^m \times \Delta^n) \longrightarrow C(X \times Y)$$

* "σ" - THE FUNDAMENTAL CHAIN

In each degree

simplicial chain map

$$C_k(\alpha \times \beta) : C_k(\Delta^m \times \Delta^n) \longrightarrow C_k(X \times Y)$$

$$C_k(\alpha \times \beta) : C(\Delta^m_k \times \Delta^n_k) \longrightarrow C_k(X \times Y)$$

Recall, If $\Delta^m \times \Delta^n$ and $X \times Y$ are chain complexes the chain map is the family $\{f_k\}$

$$\cdots \leftarrow \Delta^m_{k-1} \times \Delta^n_{k-1} \xleftarrow{d} \Delta^m_k \times \Delta^n_k \cdots$$

$$\downarrow f_{k-1} \qquad \downarrow f_k$$

$$\cdots \leftarrow (X \times Y)_{k-1} \xleftarrow{d} (X \times Y)_k \cdots$$

Eilenberg-Zilber Map $\nabla : C(X) \otimes C(Y) \longrightarrow C(X \times Y)$

Induced simplicial Map $C(\alpha \times \beta) : C(\Delta^m \times \Delta^n) \longrightarrow C(X \times Y)$

$C_m(X)$ Free abelian group on nondegenerate $[m]$-simplices of X

$C_{m+n}(X \times Y)$ Free abelian group on nondegenerate $[m+n]$-simplices of $X \times Y$

Nondegenerate $[m+n]$-simplices

Pairs of maps (α, β)

$\alpha : \Delta^m \longrightarrow X$

$\beta : \Delta^n \longrightarrow Y$

(simplicial maps)

thus...

For any pair (α, β), we must construct an $(m+n)$-chain on $X \times Y$

This pair (α, β) yields a map:

$$\alpha \times \beta : \Delta^m \times \Delta^n \longrightarrow X \times Y$$

This map induces a map on simplicial chains:

$$C(\alpha \times \beta) : C(\Delta^m \times \Delta^n) \longrightarrow C(X \times Y)$$

DEFINE $\nabla(\alpha, \beta) = C_{m+n}(\alpha \times \beta)(\sigma) \in C_{m+n}(X \times Y)$

σ : THE FUNDAMENTAL CHAIN

→ The Fundamental Chain! "σ"

The Fundamental chain "σ" is an element of $C_{m+n}(\Delta^m \times \Delta^n)$
[Thus, it is an $m+n$-chain]
with the property that

$$* \quad \sigma = \sum_{\tau,\tau'} \text{sign}(\tau,\tau') \cdot (\tau,\tau')$$

where,

$\tau : \Delta^{m+n} \longrightarrow \Delta^m$ ($m+n$ simplex of Δ^m)
$\tau' : \Delta^{m+n} \longrightarrow \Delta^n$ ($m+n$ simplex of Δ^n)

such that (τ,τ') is NONdegenerate,
($m+n$ simplex of $\Delta^m \times \Delta^n$)

and $\text{sign}(\tau,\tau')$ is the sign of the "shuffle permutation" of (τ,τ')

↓

that is...

the permutation of $\{1,\dots,m+n\}$
whose first m terms enumerate positions i such that

$$\tau(i-1) < \tau(i)$$
and $$\tau'(i-1) = \tau'(i)$$

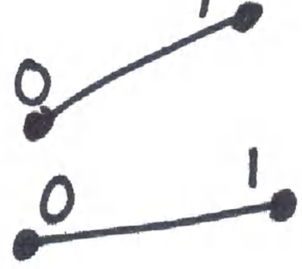
whereas, the last n terms enumerate positions i such that
$\tau(i-1) = \tau(i)$ and $\tau'(i-1) < \tau'(i)$.

I have no idea what just happened here!

What is this crazy "fundamental chain"?!

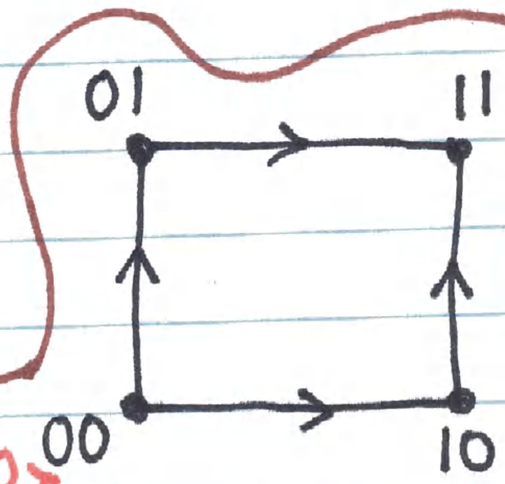
Resolution: The Fundamental Chain:

Ex: $X = \Delta^1$ $Y = \Delta^1$



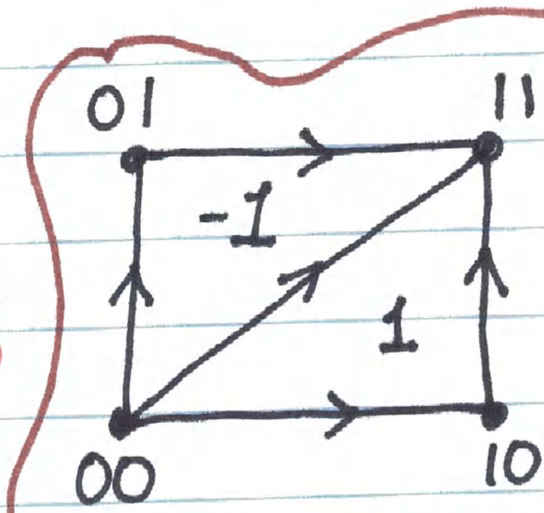
Construct the Eilenberg-Zilber map!

$$\nabla : C(\Delta^1) \otimes C(\Delta^1) \longrightarrow C(\Delta^1 \times \Delta^1)$$



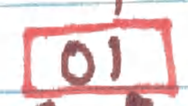
∇

EZ map



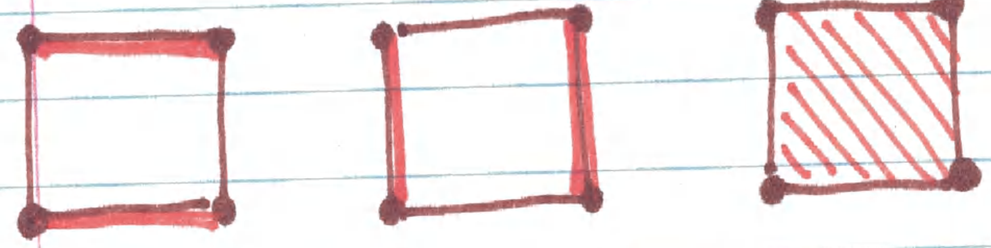
LEFT SIDE: Visualization of the tensor product of the simplicial chains of Δ^1 w/itself

NOT a simplicial set!



dimension of horizontal projection

dimension of vertical projection

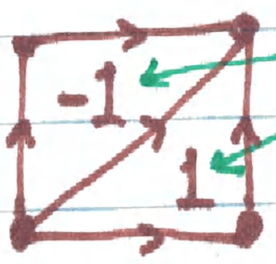


two generators in bidegree (1,0)

two generators in bidegree (0,1)

generator in bidegree (1,1)

RIGHT SIDE: Visualization of the simplicial chains of $\Delta^1 \times \Delta^1$



coefficients of the fundamental chains

OBSERVATION

When we compute the boundary of the fundamental chain, the diagonal will show up with opposite coefficients that will annihilate each other, such that the boundary is only the outer square.

LEMMA 22.7

The map ∇ satisfies $\nabla d = d \nabla$
(that is, it is a chain map)

Recall, a "chain map" $f: C \to D$ between chain complexes C and D is a sequence $f = \{f_n\}_{n \in \mathbb{Z}}$ where
$f_n: C_n \to D_n$
such that the square commutes for all values n.

$$\begin{array}{ccccccc} \cdots & \leftarrow & C_{n-1} & \leftarrow & C_n & \leftarrow & \cdots \\ & & \downarrow{\scriptstyle f_{n-1}} & \# & \downarrow{\scriptstyle f_n} & & \\ \cdots & \leftarrow & D_{n-1} & \leftarrow & D_n & \leftarrow & \cdots \end{array}$$

Proof:
[In each degree ...]

- $C(X) \otimes C(Y)$ is a free, graded abelian group on pairs of nondegenerate simplices of X, Y.

- Consider a pair (τ, τ') $\quad \tau \in X, \ \tau' \in Y$
  bidegree (m, n)
  τ has deg. m
  τ' has deg. n

- Now consider $\nabla d (\tau, \tau')$

$$\nabla d(\tau, \tau') = \nabla [\, d(\tau, \tau')]$$
$$= \nabla [\, d\tau \otimes \tau' + (-1)^m \tau \otimes d\tau' \,]$$

- Consider $d \nabla (\tau, \tau')$, and compare ↑

continued ...

$\nabla d(\tau,\tau') = d\nabla(\tau,\tau')$ ← we want to show this fact!

$\nabla d(\tau,\tau') = \nabla d(\tau\otimes\tau')$

$= \nabla[d\tau\otimes\tau' + (-1)^m \tau\otimes d\tau']$

$d\nabla(\tau,\tau') = d(\nabla(\tau,\tau')) = d(\nabla(\tau\otimes\tau'))$

↑

By definition, $\nabla(\tau,\tau')$

$\|$

$\nabla(\tau\otimes\tau') = C(\tau\otimes\tau')(\sigma_{m,n})$

where σ is the fundamental chain

$d(\nabla(\tau\otimes\tau')) = d(C(\tau\otimes\tau')(\sigma_{m,n}))$

$= C(\tau\otimes\tau')(\partial d(\sigma_{m,n}))$

$= C(\tau\otimes\tau')(d\Delta^m\times\Delta^n + \Delta^m\times d\Delta^n)$

$\sigma_{m-1,n}$ $\sigma_{m,n-1}$

$= \nabla[d\tau\otimes\tau'] \pm \nabla[\tau\otimes d\tau']$

$= C(d\tau\otimes\tau')(\sigma_{m-1,n}) \pm C(\tau\otimes d\tau')(\sigma_{m,n-1})$

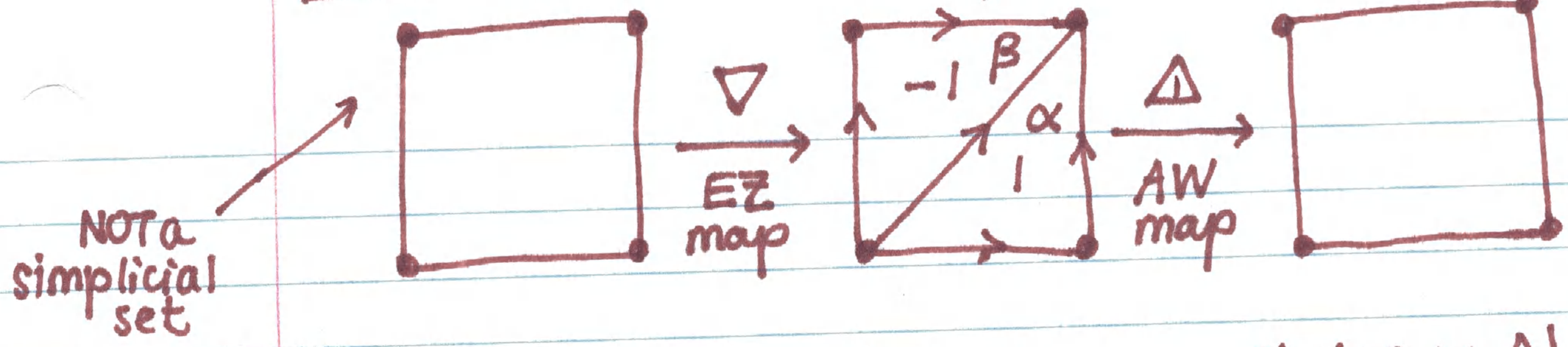
$= C(\tau\otimes\tau')(\partial\Delta^m\times\Delta^n \pm \Delta^m\times d\Delta^n)$

$= C(\tau\otimes\tau')(d\Delta^m\times\Delta^n) \pm C(\tau\otimes\tau')(\Delta^m\times d\Delta^n)$

$\sigma_{m-1,n}$ $\sigma_{m,n-1}$

Resolution of Lemma : $\nabla d = d\nabla$

Ex | simplicial chains



Chains on Δ^1 tensored with themselves

$C(\Delta^1) \otimes C(\Delta^1)$

Chains on $\Delta^1 \times \Delta^1$

$C(\Delta^1 \times \Delta^1)$

Chains on Δ^1 tensored with themselves

$C(\Delta^1) \otimes C(\Delta^1)$

$C_1(\Delta^1) = \mathbb{Z}$



Take $u \otimes u \in C(\Delta^1) \otimes C(\Delta^1)$

NOTE: Here is an "instructive example" of how to apply Δ and ∇ maps

$\Delta \circ \nabla = id$ [This composition is the identity]

$\Delta \circ \nabla (u \otimes u) = \Delta[\nabla(u \otimes u)] = \Delta[\alpha - \beta]$

$= \Delta\alpha - \Delta\beta = u \otimes u$

$\nabla : C(\Delta^1) \otimes C(\Delta^1) \longrightarrow C(\Delta^1 \times \Delta^1)$

$u \otimes u \longmapsto \alpha - \beta$

$u \in C(\Delta^1)$
u is a chain in Δ^1
u has degree
(a [1]-chain)

$\alpha - \beta \in \Delta^1 \times \Delta^1$
$\alpha - \beta$ is a chain in $\Delta^1 \times \Delta^1$
$\alpha - \beta$ has degree 2
(a [2]-chain)

([1]-chain) $\otimes$ ([1]-chain) $\longmapsto$ [2]-chain
(a very particular [2]-chain)

and $\Delta: C(\Delta^1 \times \Delta^1) \longrightarrow C(\Delta^1) \otimes C(\Delta^1)$

$$c_1\alpha + c_2\beta \longmapsto c_1 \Delta\alpha + c_2 \Delta\beta$$

where $\Delta\alpha = s(u \otimes u)$, some multiple
and $\Delta\beta = t(u \otimes u)$, some multiple

NOTE: $(C(\Delta^1) \otimes C(\Delta^1))_0 = \mathbb{Z}$

$(C(\Delta^1) \otimes C(\Delta^1))_1 = \mathbb{Z} \otimes \mathbb{Z}$

$(C(\Delta^1) \otimes C(\Delta^1))_2 = \mathbb{Z}$

* I am not sure what this means

To construct the Alexander-Whitney map,
substitute $a=1$, $b=1$
$s-t=1$
$s=t+1$
and choose parameters $t=0$, $s=1$

NOTE: There is a huge analogy between Calculus concepts and those of Algebraic Topology.

For example:

0-cochains "act like" (are analagous to) real-valued functions

0-chains "act like" measures, etc.

We can evaluate a cochain on a chain (like we can evaluate a function of a measure) and cochains are linear functionals on chains.

CUP PRODUCT

Let X be a simplicial set : $X \in sSet$

We define a "cup product" on X as follows :

The cup product

$$\cup : C^*(X) \otimes C^*(X) \xrightarrow{\Delta^*_{X,X}} C^*(X \times X) \xrightarrow{C^*(d)} C^*(X)$$

$\Delta^*_{X,X}$: Alexander-Whitney map for simplicial cochains

$C^*(d)$: The "diagonal" map

The cup product is a chain map given by the composition of the Alexander-Whitney map and the diagonal map.

→ *Generalized :*

Let A be a ring

[Recall, a "ring" is an abelian group equipped with a bilinear operation of multiplication

$mult. : A \otimes A \longrightarrow A$

and a unit element $1 \in A$

that are associative $x \cdot (y \cdot z) = (x \cdot y) \cdot z$

and unital $1 \cdot x = x \cdot 1 = x$

The generalized cup product

We define a "cup product" on A as follows :

$$\cup : C^*(X, A) \otimes C^*(X, A) \xrightarrow{\Delta^*_{X \times X}} C^*(X \times X, A \otimes A)$$

Alexander-Whitney map

$$\xrightarrow{C^*(d, A \otimes A)} C^*(X, A \otimes A) \xrightarrow{C^*(X, \mu)} C^*(X, A)$$

$C^*(d, A \otimes A)$: The "diagonal" map

$C^*(X, \mu)$: multiplication map

→ We can recover the map above as a special case of the generalized cup product [the case when $A = \mathbb{Z}$

The Alexander-Whitney Map for simplicial cochains

Recall, the Alexander-Whitney map

$$\Delta : C(X \times X) \longrightarrow C(X) \otimes C(X)$$

Apply the $\mathrm{Hom}(-, \mathbb{Z})$ functor to the map Δ ...

$$\Delta^* : \mathrm{Hom}(C(X \times X), \mathbb{Z}) \longleftarrow \mathrm{Hom}(C(X) \otimes C(X), \mathbb{Z})$$

~~$C^*(X \times X) \longleftarrow C^*(X) \otimes C^*(X)$~~

The AW-map for simplicial cochains goes in the reverse direction as the map for simplicial chains

$$\begin{array}{ccc} \mathrm{Hom}(C(X \times X), \mathbb{Z}) & \longleftarrow & \mathrm{Hom}(C(X) \otimes C(X), \mathbb{Z}) \\ \| & & \uparrow \text{map} \\ C^*(X \times X) & & C^*(X) \otimes C^*(X) \end{array}$$

We must specify the map (a bilinear map) from $C^*(X) \otimes C^*(X)$ to $\mathrm{Hom}(C(X) \otimes C(X), \mathbb{Z})$

$$C^*(X), C^*(X) \longrightarrow \mathrm{Hom}(C(X) \otimes C(X), \mathbb{Z})$$

$$f, g \longmapsto \begin{cases} C(X), C(X) \longrightarrow \mathbb{Z} \\ u, v \longmapsto f(u)g(v) \end{cases}$$

elements map to a map (an element of Hom) we specify this map to be the map that sends elements u and v to the product of their images under f and g, respectively.

LEMMA: Let X be a simplicial set — $X \in sSet$
A be a ring — $A \in Ring$
u be a cochain of degree m — $u \in C^m(X,A)$
v be a cochain of degree n — $v \in C^n(X,A)$

Then $u \cup v$ is a simplicial cochain on X of degree $m+n$

$u \cup v$ ← cup product

We compute the value of $u \cup v$ on an $(m+n)$-simplex $\alpha \in X$:

$$(u \cup v)(\alpha) = (C^*(X,\mu)(C^*(d, A \otimes A)(\Delta^*(u \otimes v))))(\alpha)$$

- $(u \cup v)(\alpha)$ ← cup product
- $C^*(X,\mu)$ ← ... is the same as ... ③ Apply multiplication map
- $C^*(d, A \otimes A)$ ← ② Apply diagonal map
- $\Delta^*(u \otimes v)$ ← ① Apply Alexander-Whitney map
- $u \in C^m(X,A)$ cochain degree m
- $v \in C^n(X,A)$ cochain degree n
- RESULT: $u \cup v \in C^{m+n}(X,A)$ cochain degree $m+n$

and this is the same result as if we evaluated the cochain on α and THEN applied the multiplication map to the interior:

$$= \mu[C^*(d, A \otimes A)(\Delta^*(u \otimes v))](\alpha)$$

$C^*(d, A \otimes A)$: apply d to α then evaluate by what's inside

$$= \mu[(\Delta^*(u \otimes v))(d(\alpha))]$$

$d(\alpha)$ ← "diagonal map" ⟶

$$u \cup v = \mu[(\Delta^*(u \otimes v))(d(\alpha))] = \mu[(\Delta^*(u \otimes v))(\alpha, \alpha)]$$

diagonal map

Alexander-Whitney map for simplicial cochains

(α, α)

component #1 — apply u

component #2 — apply v to degenerated simplex

$$= \mu[u(d_{m+1} \cdots d_{m+n}(\alpha)) \otimes v(d_0 \cdots d_{m-1}(\alpha))]$$

α m+n-simplex degenerate di (n) degree $n+m-n = m$ — apply u

α m+n simplex degenerate di (m) degree $n+m-m = n$ — apply v

Multiplication Map (multiplies elements $\mu(a \otimes b) = ab$)

$$= \boxed{u(d_{m+1} \cdots d_{m+n}(\alpha)) \cdot v(d_0 \cdots d_{m-1}(\alpha))}$$

This is how we derive the formula for cup product

$$u \cup v = u(d_{m+1} \cdots d_{m+n}(\alpha)) \cdot v(d_0 \cdots d_{m-1}(\alpha))$$

$u \cup v(\alpha) =$

- $u \cup v$ is a cochain. so, we must apply it to some simplex α.
- degenerate the last vertices (from m+1 to m+n) (apply u (deg m) to the now deg m result.)
- degenerate the first vertices (from 0 to m-1) (apply v (deg n) to the now deg n result.)

CUP PRODUCT

$$\cup\cup\upsilon(\alpha) = u(d_{m+1}\cdots d_{m+n}(\alpha))\cdot\upsilon(d_0\cdots d_{m-1}(\alpha))$$

$u \in C^m(X,A)$ cochain #1 of degree m
$\upsilon \in C^n(X,A)$ cochain #2 of degree n
$\alpha \in X_{m+n}$

The cup product is the "multiplication" operation defined on cochains. With this definition, we have introduced an algebraic ring structure on cochains

Previously: $A \in Ring$

$$sSet \xrightarrow{C^*(-,A)} DGA_A$$

sSet: simplicial set

DGA_A: differential graded algebra over A

Now (after defining cup product:)

$$sSet \xrightarrow{C^*(-,A)} DGA_A \xrightarrow{H^*} GA_A$$

the cohomology of differential graded algebra yields an ordinary graded algebra over A

NOTE: the cup product survives the passage to cohomology, and hence, the cup product is a topological invariant of a simplicial set.

(related to "weak equivalence")

THE COMPOSITION:

$$H^* \circ C^*(-,A) = H^*(X,A)$$

is called the

COHOMOLOGY RING

(due to Alexander (1935) and Kolmogoroff. (1936))

let p_i, q_i be cochains of degree i
and let $\star$ be the nondegenerate simplex in degree $m+n$

CUP PRODUCTS (generalized)

$\star \in X_0$ [deg 0 · deg 0 $\quad (p_0 \cup q_0)(\star) = p_0(\star) q_0(\star)$

$\star \in X_1$ [deg 1 · deg 0 $\quad (p_1 \cup q_0)(\star) = p_1(\star) q_0(d_0 \star)$
deg 0 · deg 1 $\quad (p_0 \cup q_1)(\star) = p_0(d_1 \star) q_1(\star)$
deg 1 · deg 1 $\quad (p_1 \cup q_1)(\star) = p_1(d_2 \star) q_1(d_0 \star)$

$\star \in X_2$ [deg 2 · deg 0 $\quad (p_2 \cup q_0)(\star) = p_2(\star) q_0(d_0 d_1 \star)$
deg 0 · deg 2 $\quad (p_0 \cup q_2)(\star) = p_0(d_1 d_2 \star) q_2(\star)$

$\star \in X_3$ [deg 1 · deg 2
deg 2 · deg 1 $<$ [deg 0 · deg 3
deg 3 · deg 0

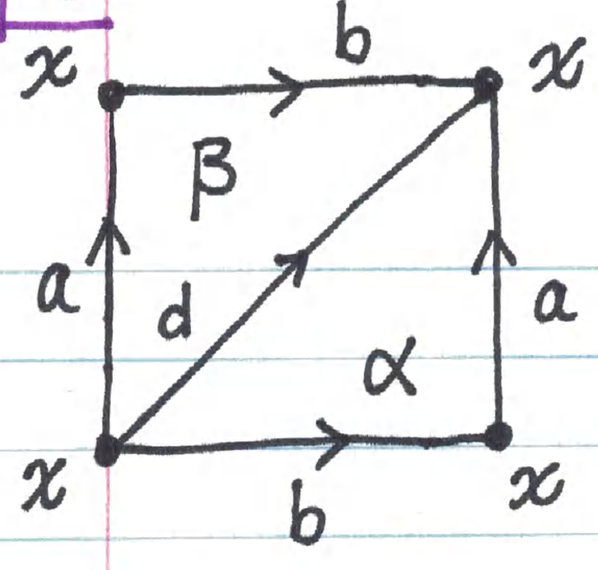
$\star \in X_4$ [deg 2 · deg 2
deg 1 · deg 3 $<$ [deg 0 · deg 4
deg 3 · deg 1 deg 4 · deg 0

$\star \in X_5$ [deg 2 · deg 3
deg 3 · deg 2
deg 3 · deg 3
⋮

IN GENERAL

Remove last vertices off first component until proper dimension, remove first vertices off last component until proper dimension.

Example



TORUS

[0]-simplices $\{x\} \longrightarrow \mathbb{Z}$

[1]-simplices $\{a,b,d\} \longrightarrow \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$

[2]-simplices $\{\alpha,\beta\} \longrightarrow \mathbb{Z}\oplus\mathbb{Z}$

Compute the cohomology ring of the torus!

Chain:

$$\cdots\ 0 \xleftarrow{d_0} \underset{x}{\mathbb{Z}} \xleftarrow{d_1} \underset{a\ b\ d}{\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}} \xleftarrow{d_2} \underset{\alpha\ \beta}{\mathbb{Z}\oplus\mathbb{Z}} \xleftarrow{d_3} 0\ \cdots$$

$d_0 = (0)$ $\quad d_1 = (0\ 0\ 0)$ $\quad d_2 = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ $\quad d_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Boundaries:

$d_0(x) = 0$

$d_1(a) = x - x = 0$

$d_1(b) = x - x = 0$

$d_1(d) = x - x = 0$

$d_2(\alpha) = a - d + b$

$d_2(\beta) = b - d + a$

boundary map (matrices)

$\text{range} \xleftarrow{d_n} \text{domain}$, $d_n = \begin{matrix} \\ \text{range} \end{matrix}\overset{\text{domain}}{\begin{bmatrix} & \end{bmatrix}}$

(generators of domain determine # columns, generators of range determine # rows)

(TRANSPOSE / SHIFT CHAIN MAPS TO DETERMINE COCHAIN MAPS)

Cochain:

$$\cdots\ 0 \xrightarrow{d^{-1}} \underset{x^*}{\mathbb{Z}} \xrightarrow{d^0} \underset{a^*\ b^*\ d^*}{\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}} \xrightarrow{d^1} \underset{\alpha^*\ \beta^*}{\mathbb{Z}\oplus\mathbb{Z}} \xrightarrow{d^2} 0\ \cdots$$

$d^{-1} = (0)$ $\quad d^0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad d^1 = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \end{pmatrix}$ $\quad d^2 = (0\ 0)$

Maps: $H^n = \ker d^n / \operatorname{Im} d^{n-1}$ Cohomology

H^2: $\ker d^2 = \mathbb{Z}\times\mathbb{Z} = \langle \alpha^*, \beta^* \rangle$, $\operatorname{Im} d^1 =$ ▭

H^1: $\ker d^1 =$ ▭, $\operatorname{Im} d^0 =$ [0] } To be determined →

H^0: $\ker d^0 = \mathbb{Z} = \langle x^* \rangle$, $\operatorname{Im} d^{-1} =$ [0]

Map d^0: $\mathbb{Z} \xrightarrow{\begin{pmatrix}0\\0\\0\end{pmatrix}} \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$ zero map $\operatorname{Im} d^0 = 0$

x^* ; $a^*\ b^*\ d^*$

Map d^1: $\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\begin{pmatrix}1&1&-1\\1&1&-1\end{pmatrix}} \mathbb{Z}\oplus\mathbb{Z}$

$a^*\ b^*\ d^*$; $\alpha^*\ \beta^*$

Take: $k_1 a^* + k_2 b^* + k_3 d^* \in \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$

Apply d^1:

$$d^1(k_1 a^* + k_2 b^* + k_3 d^*)$$
$$= k_1 d^1(a^*) + k_2 d^1(b^*) + k_3 d^1(d^*)$$
$$= k_1[\alpha^* + \beta^*] + k_2[\alpha^* + \beta^*] + k_3[-\alpha^* - \beta^*]$$

Apply to generators of $[2]$-chain $\{\alpha, \beta\}$

$$(k_1[\alpha^* + \beta^*] + k_2[\alpha^* + \beta^*] + k_3[-\alpha^* - \beta^*])(\alpha)$$

$$(k_1[\alpha^* + \beta^*] + k_2[\alpha^* + \beta^*] + k_3[-\alpha^* - \beta^*])(\beta)$$

$\alpha) \Rightarrow k_1(1) + k_2(1) + k_3(-1) = k_1 + k_2 - k_3$

$\beta) \Rightarrow k_1(1) + k_2(1) + k_3(-1) = k_1 + k_2 - k_3$

$$\Rightarrow (k_1 + k_2 - k_3)(\alpha^*) + (k_1 + k_2 - k_3)(\beta^*)$$

Image : set = arbitrary $[2]$-chain kernel : set = 0

$$c_1\alpha^* + c_2\beta^* = (k_1 + k_2 - k_3)\alpha^* + (k_1 + k_2 - k_3)\beta^*$$

$$0 = (k_1 + k_2 - k_3)\alpha^* + (k_1 + k_2 - k_3)\beta^*$$

$$\begin{cases} c_1 = k_1 + k_2 - k_3 \\ c_2 = k_1 + k_2 - k_3 \end{cases} \Rightarrow c_1 - c_2 = 0 \Rightarrow c_1 = c_2$$

$$\begin{cases} 0 = k_1 + k_2 - k_3 \\ 0 = k_1 + k_2 - k_3 \end{cases} \Rightarrow k_3 = k_1 + k_2$$

Arbitrary element: $c_1\alpha^* + c_1\beta^* = c_1(\alpha^* + \beta^*)$

set $c_1 = 1 \Rightarrow \langle \alpha^* + \beta^* \rangle$

→ arbitrary [1]-chain:

$$k_1 a^* + k_2 b^* + k_3 d^*$$
$$= k_1 a^* + k_2 b^* + (k_1 + k_2) d^*$$
$$= k_1 (a^* + d^*) + k_2 (b^* + d^*)$$

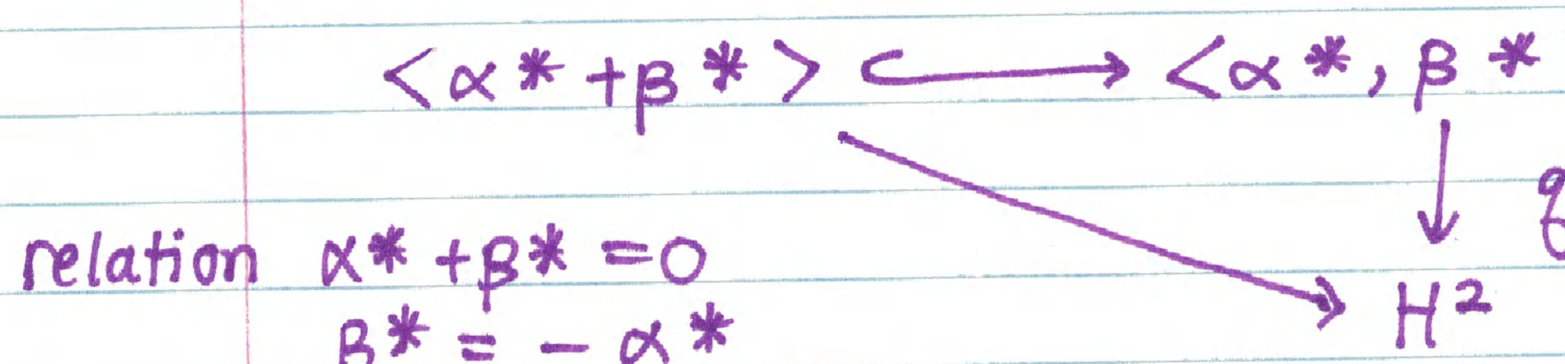
$$= \langle a^* + d^*,\ b^* + d^* \rangle$$

$$H^2 < \begin{matrix} \ker d^2 = \langle \alpha^*, \beta^* \rangle \\ \mathrm{Im}\, d^1 = \langle \alpha^* + \beta^* \rangle \end{matrix}$$

$$H^1 < \begin{matrix} \ker d^1 = \langle a^* + d^*,\ b^* + d^* \rangle \\ \mathrm{Im}\, d^0 = 0 \end{matrix}$$

$$H^0 < \begin{matrix} \ker d^0 = \langle x^* \rangle \\ \mathrm{Im}\, d^{-1} = 0 \end{matrix}$$

Quotient Maps: $H^2 = \dfrac{\langle \alpha^*, \beta^* \rangle}{\langle \alpha^* + \beta^* \rangle}$



relation $\alpha^* + \beta^* = 0$
$\beta^* = -\alpha^*$

$$q(\langle \alpha^*, \beta^* \rangle) = \langle \alpha^*, -\alpha^* \rangle \cong \langle \alpha^* \rangle$$

Cohomology of the Torus

so $H^2 = \langle \alpha^* \rangle \cong A$

$$H^1 = \frac{\langle a^* + d^*,\ b^* + d^* \rangle}{0}$$

$$H^1 = \langle a^* + d^*,\ b^* + d^* \rangle \cong A \oplus A$$

$$H^0 = \langle x^* \rangle \cong A$$

Cohomology Maps:

To compute cup products, we must choose cochain representatives.

To choose representatives, we construct sections of quotient maps

$\{ Z^n$ – n-cycles
$\{ H^n$ – n-homology

construct $\begin{cases} q^n : Z^n \longrightarrow H^n \\ s^n : U(H^n) \longrightarrow U(Z^n) \end{cases}$

such that:

$$U(q^n) \circ s^n = id_{H^n}$$

$H^0(X) \cong A = \langle x^* \rangle$
$Z^0(X) = \langle x^* \rangle$

quotient map $q^0 : Z^0 \longrightarrow H^0$ IDENTITY

so... section quotient map $s^0 : H^0 \longrightarrow Z^0$ is also IDENTITY

$s^0 : H^0 \longrightarrow Z^0$
$u \longmapsto u \cdot x^*$

$H^1(X) \cong A \oplus A = \langle a^* + d^*, b^* + d^* \rangle$
$Z^1(X) = \langle a^* + d^*, b^* + d^* \rangle$

quotient map $q^1 : Z^1 \longrightarrow H^1$
$k_1, k_2, k_3 \longmapsto k_1, k_2$
since $k_3 = k_1 + k_2$

so... section map $s^1 : H^1 \longrightarrow Z^1$
$k_1, k_2 \longmapsto k_1, k_2, (k_1 + k_2)$

$s^1 : H^1 \longrightarrow Z^1$
$v_1, v_2 \longmapsto v_1 a^* + v_2 b^* + (v_1 + v_2) d^*$

continued... ⟶

$H^2(X) \cong \langle \alpha^* \rangle$

$Z^2(X) \cong \langle \alpha^*, \beta^* \rangle$

quotient map $q^2 : H^2 \longrightarrow Z^2$

$k_1, k_2 \longmapsto k_1 - k_2$

$=$

w

so... section map $s^2 : Z^2 \longrightarrow H^2$

$w \longmapsto w + k_2,\ k_1 - w$

~~$w = k_1 + k_2$~~

~~$w = k_1$~~

$w = k_1 - k_2$

$k_1 - w = k_2$

$w + k_2 = k_1$

This is difficult to determine! But in involves making a choice in the representative

Take ~~$w + k_2 = 0$~~ $k_1 - w = 0$

Then $s^2(w) = (w + k_2)\alpha^* + (k_1 - w)\beta^*$

$\Rightarrow k_1 = w$

so $k_2 = 0$

$\Rightarrow w \longmapsto w \cdot \alpha^*$

Representatives:

$H^0(X) \cong A \longrightarrow Z^0 \qquad u \longmapsto u \cdot \alpha$

$H^1(X) \cong A \oplus A \longrightarrow Z^1 \qquad v_1 \oplus v_2 \longmapsto v_1 \cdot a + v_2 \cdot b + (v_1 + v_2) \cdot d$

$H^2(X) \cong A \longrightarrow Z^2 \qquad w \longmapsto w \cdot \alpha$

* Cup products give us a way to "multiply" elements.

Recall, the formula for cup product:

$u \in C^m$, $v \in C^n$, $\alpha \in X_{m+n}$ $\Rightarrow (u \cup v)(\alpha) = u(d_{m+1} \cdots d_{m+n}(\alpha))\, v(d_0 \cdots d_{m-1}(\alpha))$

that is, we evaluate u on the first m-vertices and we evaluate v on the last n-vertices.

Case 1: $m = 0$, $u \in C^0$; $n = 0$, $v \in C^0$; $m + n = 0$ ~~$\alpha \in X_0$~~ $(\eta \in X_0)$

η is a [0]-simplex (1 vertex)

$(u \cup v)(\eta) = u(\eta) \cdot v(\eta)$

so... $(u \cup v)(x) = u(x) \cdot v(x)$

Case 2: $m = 0$, $u \in C^0$; $n = 1$, $v \in C^1$; $m + n = 1$ $(\eta \in X_1)$

η is a [1]-simplex (2 vertices)

$(u \cup v)(\eta) = u(d_1(\eta)) \cdot v(\eta)$ $\longrightarrow$ so...

Case 2 continued...

$$(u \cup v)(a) = u(d_1(a)) \cdot v(a) = u(x) \cdot v(a)$$

$$(u \cup v)(b) = u(d_1(b)) \cdot v(b) = u(x) \cdot v(b)$$

$$(u \cup v)(d) = u(d_1(d)) \cdot v(d) = u(x) \cdot v(d)$$

Case 3: $m=1$, $u \in C^1$; $n=0$, $v \in C^0$; $m+n=1$ $(\eta \in X_1)$

η is a $[1]$-simplex

$(u \cup v)(\eta) = u(\eta) \cdot v(d_0(\eta))$

so... $(u \cup v)(a) = u(a) \cdot v(d_0(a)) = u(a) \cdot v(x)$

$(u \cup v)(b) = u(b) \cdot v(d_0(b)) = u(b) \cdot v(x)$

$(u \cup v)(d) = u(d) \cdot v(d_0(d)) = u(d) \cdot v(x)$

Case 4: $m=1$, $u \in C^1$; $n=1$, $v \in C^1$; $m+n=2$ $(\eta \in X_2)$

η is a $[2]$-simplex

$(u \cup v)(\eta) = u(d_2(\eta)) \cdot v(d_0(\eta))$

so... $(u \cup v)(\alpha) = u(d_2(\alpha)) \cdot v(d_0(\alpha)) = u(b) \cdot v(a)$

$(u \cup v)(\beta) = u(d_2(\beta)) \cdot v(d_0(\beta)) = u(a) \cdot v(b)$

Case 5: $m=0$, $u \in C^0$; $n=2$, $v \in C^2$; $m+n=2$ $(\eta \in X_2)$

η is a $[2]$-simplex

$(u \cup v)(\eta) = u(d_1 \cdot d_2(\eta)) \cdot v(\eta)$

so... $(u \cup v)(\alpha) = u(d_1 d_2(\alpha)) \cdot v(\alpha) = u(x) \cdot v(\alpha)$

$(u \cup v)(\beta) = u(d_1 d_2(\beta)) \cdot v(\beta) = u(x) \cdot v(\beta)$

→

Case 6: $m=2, u\in C^2$
$n=0, v\in C^0$
$m+n=2$ $(\eta \in X_2)$

η is a [2]-simplex

$(u \cup v)(\eta) = u(\eta)\cdot v(d_0 d_1(\eta))$

So... $(u\cup v)(\alpha) = u(\alpha)\cdot v(d_0 d_1(\alpha))$
$= u(\alpha)\cdot v(x)$

$(u\cup v)(\beta) = u(\beta)\cdot v(d_0 d_1(\beta))$
$= u(\beta)\cdot v(x)$

Now, we allow these products to act on our representatives

use case!

degree 0: $(u\cup v)(x) = u(x)\cdot v(x)$ $\qquad u \longmapsto u\cdot x$

so... $(u\cdot x^*) \cup (u'\cdot x^*) = u\cdot u'\cdot x^*$ [coeff of x · coeff of x]

degree 1: $(u\cup v)(a) = u(x)\cdot v(a)$ [coeff of x · coeff of a]
$(u\cup v)(b) = u(x)\cdot v(b)$ [coeff of x · coeff of b]
$(u\cup v)(d) = u(x)\cdot v(d)$ [coeff of x · coeff of d]

(when $m=0$, $n=1$)

Use case 2

so... $(u\cdot x^*) \cup (v_1 a^* + v_2 b^* + (v_1+v_2) d^*)$

$= u\cdot v_1\cdot a^* + u\cdot v_2\cdot b^* + u(v_1+v_2)\cdot d^*$

degree 1: $(u\cup v)(a) = u(a)\cdot v(x)$ [coeff of a · coeff of x]
$(u\cup v)(b) = u(b)\cdot v(x)$ [coeff of b · coeff of x]
$(u\cup v)(d) = u(d)\cdot v(x)$ [coeff of d · coeff of x]

Use case 3

(when $m=1$, $n=0$)

so... $(v_1 a^* + v_2 b^* + (v_1+v_2) d^*) \cup (u\cdot x^*)$

$= v_1\cdot u\cdot a^* + v_2\cdot u\cdot b^* + (v_1+v_2)\cdot u\cdot d^*$

degree 2: so, you see, the cases we discussed previously [case 1 – case 6] give us "multiplication instructions" for how to extract coefficients of products of cochains!

keep going!

use case 4: degree 2 (when $m=1$, $n=1$)

$(u \cup v)(\alpha) = u(b) \cdot v(a)$

$(u \cup v)(\beta) = u(a) \cdot v(b)$

so... $(v_1 a^* + v_2 b^* + (v_1+v_2) d^*) \cup (v_1' a^* + v_2' b^* + (v_1'+v_2') d^*)$ (deg 1, deg 1)

$= v_2 v_1' \cdot \alpha^* + v_1 v_2' \cdot \beta^*$

use cases: degree 2 (when $m=0$, $n=2$)

$(u \cup v)(x) = u(x) v(x)$

$(u \cup v)(\beta) = u(x) v(\beta)$

so... $(u \cdot x^*) \cup (w \cdot \alpha^*)$ (deg 0, deg 2)

$= uw \cdot \alpha^* + u \cdot (0) \cdot \beta^* = uw \cdot \alpha^*$ ($=0$)

use case 6: degree 2 (when $m=2$, $n=0$)

$(u \cup v)(\alpha) = u(\alpha) \cdot v(x)$

$(u \cup v)(\beta) = u(\beta) \cdot v(x)$

so... $(w \cdot \alpha^*) \cup (u \cdot x^*)$ (deg 2, deg 0)

$= w \cdot u \cdot a^* + (0) \cdot u \cdot \beta^* = wu \cdot \alpha^*$ ($=0$)

Now, we know how to multiply in every pair of degrees.
[NOTE: When $m+n > 2$, the product vanishes]

Why does the product vanish? *

Resolve

continue. →

degree 0 generator $[x^*] \Rightarrow u=1$, $w=1$

$$(ux^*) \cup (u'x^*) = uu' \cdot x^*$$

$$(x^*) \cup (x^*) = x^*$$

degree $0+1$ generator $[a^*+d^*]$, $[b^*+d^*]$

$$(u \cdot x^*) \cup (v_1 a^* + v_2 b^* + (v_1+v_2) d^*) = uv_1 a^* + uv_2 b^* + u(v_1+v_2) d^*$$

$$(x^*) \cup (a^*+d^*) = a^*+d^*$$
$$(x^*) \cup (b^*+d^*) = b^*+d^*$$

degree $1+0$ generator $[a^*+d^*]$, $[b^*+d^*]$

$$(v_1 a^* + v_2 b^* + (v_1+v_2) d^*) \cup (ux^*) = v_1 u a^* + v_2 u b^* + (v_1+v_2) u d^*$$

$$(a^*+d^*) \cup (x^*) = a^*+d^*$$
$$(b^*+d^*) \cup (x^*) = b^*+d^*$$

degree $1+1$ generator $[\alpha^*]$

$$(v_1 a^* + v_2 b^* + (v_1+v_2) d^*) \cup (v_1' a^* + v_2' b^* + (v_1'+v_2') d^*) = v_2 v_1' \alpha^* + v_1 v_2' \beta^*$$

$$(a^*+d^*) \cup (a^*+d^*) = 0$$
$$(a^*+d^*) \cup (b^*+d^*) = \beta^* = -\alpha^*$$ by relations
$$(b^*+d^*) \cup (a^*+d^*) = \alpha^*$$
$$(b^*+d^*) \cup (b^*+d^*) = 0$$

$0+2$ ... degree ... $2+0$

$$(ux^*) \cup (w\alpha^*) = uw\alpha^*$$

$$(x^*) \cup (\alpha^*) = \alpha^*$$

$$(w\alpha^*) \cup (ux^*) = wu\alpha^*$$

$$(\alpha^*) \cup (x^*) = \alpha^*$$

→ FINALLY! The cohomology ring can be computed:

generators: x^*, a^*+bl^*, b^*+d^*, α^*

relations: x^* is like a "unit"
(since it can be multiplied by anything and yield the other)

α^* can be generated by $(a^*+d^*)(b^*+d^*)$

so... generators are only a^*+d^* and b^*+d^*

Also NOTICE: $(a^*+d^*)^2=0$

$(b^*+d^*)^2=0$

$(a^*+d^*)(b^*+d^*) = -[(b^*+d^*)(a^*+d^*)]$

So.... the **COHOMOLOGY RING** can be expressed:...

$$A[\underbrace{a^*+d^*, b^*+d^*}_{\text{generators!}}] \Big/ \underbrace{((a^*+d^*)^2=0,\ (b^*+d^*)=0,\ (a^*+d^*)(b^*+d^*) = -((b^*+d^*)(a^*+d^*))}_{\text{relations}}$$

(modulo!)

let $s = a^*+d^*$ and $t = b^*+d^*$

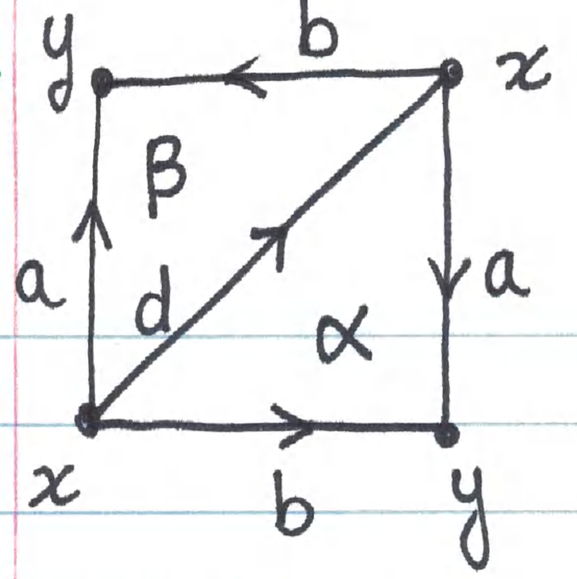
then,

$$A[s,t] \Big/ (s^2=0,\ t^2=0,\ st=-ts)$$

is the cohomology ring. ▨

REAL PROJECTIVE PLANE

Example



[0]-simplices $\{x, y\} \to \mathbb{Z} \oplus \mathbb{Z}$
[1]-simplices $\{a, b, d\} \to \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$
[2]-simplices $\{\alpha, \beta\} \to \mathbb{Z} \oplus \mathbb{Z}$

Compute the cohomology ring of the real projective plane!

Chain:

$$\cdots 0 \xleftarrow{\cdot\begin{pmatrix}0 & 0\end{pmatrix}} \underset{x\ y}{\mathbb{Z}\oplus\mathbb{Z}} \xleftarrow{\begin{pmatrix}-1 & -1 & 0\\ 1 & 1 & 0\end{pmatrix}} \underset{a\ b\ d}{\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}} \xleftarrow{\begin{pmatrix}1 & -1\\ -1 & 1\\ 1 & 1\end{pmatrix}} \underset{\alpha\ \beta}{\mathbb{Z}\oplus\mathbb{Z}} \xleftarrow{\begin{pmatrix}0\\ 0\end{pmatrix}} 0 \cdots$$
$$d_0 \qquad\qquad d_1 \qquad\qquad d_2 \qquad\qquad d_3$$

Boundaries:
$d_0(x) = 0$
$d_0(y) = 0$
$d_1(a) = y - x$
$d_1(b) = y - x$
$d_1(d) = x - x = 0$
$d_2(\alpha) = a - b + d$
$d_2(\beta) = b - a + d$

Cochain:

$$\cdots 0 \xrightarrow[d^{-1}]{\begin{pmatrix}0\\ 0\end{pmatrix}} \underset{x^*\ y^*}{\mathbb{Z}\times\mathbb{Z}} \xrightarrow[d^0]{\begin{pmatrix}-1 & 1\\ -1 & 1\\ 0 & 0\end{pmatrix}} \underset{a^*\ b^*\ d^*}{\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}} \xrightarrow[d^1]{\begin{pmatrix}1 & -1 & 1\\ -1 & 1 & 1\end{pmatrix}} \underset{\alpha^*\ \beta^*}{\mathbb{Z}\times\mathbb{Z}} \xrightarrow[d^2]{(0\ 0)} 0 \cdots$$

Maps: $H^n = \ker d^n / \operatorname{Im} d^{n-1}$ cohomology

$$H^2 \begin{cases} \ker d^2 = \mathbb{Z}\times\mathbb{Z} = \langle \alpha^*, \beta^* \rangle \\ \operatorname{Im} d^1 = \square \end{cases}$$

$$H^1 \begin{cases} \ker d^1 = \square \\ \operatorname{Im} d^0 = \square \end{cases}$$

$$H^0 \begin{cases} \ker d^0 = \square \\ \operatorname{Im} d^{-1} = 0 \end{cases}$$

To be determined...

continued... ⟶

Map d^0: $\mathbb{Z}\times\mathbb{Z} \xrightarrow{\begin{pmatrix}-1 & 1\\ -1 & 1\\ 0 & 0\end{pmatrix}} \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$

$x^*\ y^*$ $\qquad$ $a^*\ b^*\ d^*$

Take $k_1 x^* + k_2 y^* \in \mathbb{Z}\times\mathbb{Z}$

Apply d^0:

$$d^0(k_1 x^* + k_2 y^*)$$
$$= k_1 d^0(x^*) + k_2 d^0(y^*)$$
$$= k_1(-a^* - b^*) + k_2(a^* + b^*)$$

Apply to generators of $[1]$-chain: $\{a, b, d\}$

$$[k_1(-a^* - b^*) + k_2(a^* + b^*)](a)$$
$$[k_1(-a^* - b^*) + k_2(a^* + b^*)](b)$$
$$[k_1(-a^* - b^*) + k_2(a^* + b^*)](d)$$

a) $\Rightarrow k_1(-1) + k_2(1) = -k_1 + k_2$
b) $\Rightarrow k_1(-1) + k_2(1) = -k_1 + k_2$
d) $\Rightarrow k_1(0) + k_2(0) = 0$

$$\Rightarrow (-k_1 + k_2)a^* + (-k_1 + k_2)b^* + (0)d^*$$

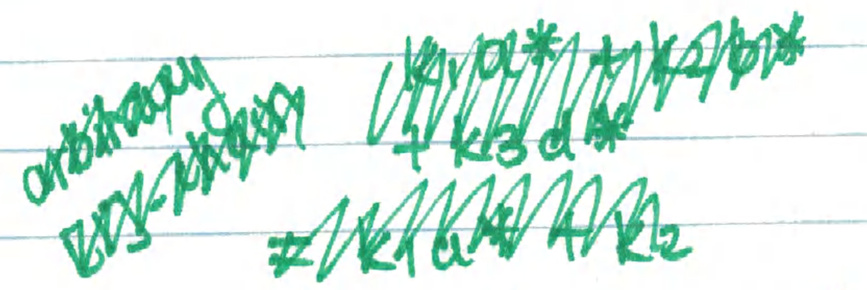
Image (set $\in$ arbitrary $[1]$-chain) $\qquad$ kernel (set $= 0$)

$c_1 a^* + c_2 b^* + c_3 d^* = (-k_1 + k_2)(a^* + b^*)$ $\qquad$ $0 = (-k_1 + k_2)(a^* + b^*)$

$\begin{cases} c_1 = -k_1 + k_2 \\ c_2 = -k_1 + k_2 \\ c_3 = 0 \end{cases} \Rightarrow c_1 - c_2 = 0$, $c_1 = c_2$ $\qquad$ $\begin{cases} 0 = -k_1 + k_2 \\ 0 = -k_1 + k_2 \end{cases} \Rightarrow k_1 = k_2$

so $c_1 a^* + c_2 b^*$
$= c_1 a^* + c_1 b^*$
$= c_1(a^* + b^*)$ set $c_1 = 1$
$\Rightarrow \langle a^* + b^* \rangle$

arbitrary elements



continued ...

→ arbitrary [0]-chain $k_1 x^* + k_2 y^*$

$= k_1 x^* + k_1 y^*$

$= k_1 (x^* + y^*)$

set $k_1 = 1 \Rightarrow \langle x^* + y^* \rangle$

Map d':

$$\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \xrightarrow{\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix}} \mathbb{Z} \times \mathbb{Z}$$

$a^* \quad b^* \quad d^* \qquad\qquad \alpha^* \quad \beta^*$

Take $k_1 a^* + k_2 b^* + k_3 d^* \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$

Apply d':

$d'(k_1 a^* + k_2 b^* + k_3 d^*)$

$= k_1 d'(a^*) + k_2 d'(b^*) + k_3 d'(d^*)$

$= k_1 (\alpha^* - \beta^*) + k_2 (-\alpha^* + \beta^*) + k_3 (\alpha^* + \beta^*)$

Apply to generators of [2]-chain: $\{\alpha, \beta\}$:

$[k_1 (\alpha^* - \beta^*) + k_2 (-\alpha^* + \beta^*) + k_3 (\alpha^* + \beta^*)] (\alpha)$

$[k_1 (\alpha^* - \beta^*) + k_2 (-\alpha^* + \beta^*) + k_3 (\alpha^* + \beta^*)] (\beta)$

$\alpha) \Rightarrow k_1 (1) + k_2 (-1) + k_3 (1) = k_1 - k_2 + k_3$

$\beta) \Rightarrow k_1 (-1) + k_2 (1) + k_3 (1) = -k_1 + k_2 + k_3$

$\Rightarrow (k_1 - k_2 + k_3) \alpha^* + (-k_1 + k_2 + k_3) \beta^*$

Image (set = arbitrary [2]-chain)

Kernel (set = 0)

image:

$\longrightarrow c_1\alpha^* + c_2\beta^* = (k_1 - k_2 + k_3)\alpha^* + (-k_1 + k_2 + k_3)\beta^*$

$$\begin{cases} c_1 = k_1 - k_2 + k_3 \\ c_2 = -k_1 + k_2 + k_3 \end{cases} \Rightarrow c_1 + c_2 = 2k_3$$

$$c_1\alpha^* + c_2\beta^* = c_1\alpha^* + (2k_3 - c_1)\beta^*$$

arbitary C_2-chain

$c_1 = 1,\ k_3 = 0 \rightarrow \alpha^* - \beta^*$

$c_1 = 0,\ k_3 = 1 \rightarrow 2\beta^*$

$$\Rightarrow \langle \alpha^* - \beta^*, 2\beta^* \rangle$$

kernel: $0 = (k_1 - k_2 + k_3)\alpha^* + (-k_1 + k_2 + k_3)\beta^*$

$$\begin{cases} 0 = k_1 - k_2 + k_3 \\ 0 = -k_1 + k_2 + k_3 \end{cases} \Rightarrow \boxed{0 = 2k_3} \qquad \begin{aligned} k_3 &= k_2 - k_1 \\ k_1 &= k_2 - k_3 \end{aligned}$$

arbitrary C_1-chain $k_1 a^* + k_2 b^* + k_3 d^*$

$k_1 a^* + k_1 b^* + k_3 d^*$

$k_1 (a^* + b^*)$

choose $k_1 = 1,\ k_3 = 0 \Rightarrow k_2 = 1 \Rightarrow a^* + b^*$

choose $k_1 = 0,\ k_3 = u \Rightarrow k_2 = u \Rightarrow ub^* + ud^*$

must pick $u \in \mathrm{Tor}(\mathbb{Z}/2, A)$

$$\Rightarrow \langle a^* + b^*, ub^* + ud^* \mid u \in \mathrm{Tor}(\mathbb{Z}/2, A) \rangle$$

$$H^2 = \frac{\ker d^2}{\mathrm{Im} d^1} = \frac{\langle \alpha^*, \beta^* \rangle}{\langle \alpha^* - \beta^*, 2\beta^* \rangle}$$

$$H^1 = \frac{\ker d^1}{\mathrm{Im} d^0} = \frac{\langle a^* + b^*, ub^* + ud^* \mid u \in \mathrm{Tor}(\mathbb{Z}/2, A) \rangle}{\langle a^* + b^* \rangle}$$

$$H^0 = \frac{\ker d^0}{\mathrm{Im} d^{-1}} = \frac{\langle x^* + y^* \rangle}{0}$$

Quotient Maps:

$$H^2 = \frac{\langle \alpha^*, \beta^* \rangle}{\langle \alpha^* - \beta^*, 2\beta^* \rangle}$$

$$\begin{array}{ccc} \langle \alpha^* - \beta^*, 2\beta^* \rangle & \hookrightarrow & \langle \alpha^*, \beta^* \rangle \\ & \searrow & \downarrow q \\ & & H^2 \end{array}$$

Matrix correspondance

$$\begin{array}{l} \\ \langle \alpha^* \rangle \\ \langle \beta^* \rangle \\ \langle \alpha^* - \beta^* \rangle \\ \langle 2\beta^* \rangle \end{array} \begin{array}{c} \begin{array}{cc} \alpha^* & \beta^* \end{array} \\ \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline 1 & -1 \\ 0 & 2 \end{array}\right] \end{array}$$

$$\sim \left[\begin{array}{cc} 0 & 1 \\ 0 & 1 \\ \hline 1 & -1 \\ 0 & 2 \end{array}\right]$$

add $-1R_3$ to R_1

$$\sim \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \\ \hline 1 & -1 \\ 0 & 2 \end{array}\right] \rightarrow \langle \beta^* \rangle$$

add $-1R_2$ to R_1

generator of quotient group where $\beta^* \in A/2A$

$$\langle \beta^* \mid \beta^* \in A/2A \rangle$$

$$H^1 = \frac{\langle a^* + b^*, ub^* + ud^* \mid u \in \mathrm{Tor}(\mathbb{Z}/2, A) \rangle}{\langle a^* + b^* \rangle} \rightarrow$$

Matrix correspondance

$$\begin{array}{c} \\ a+b \\ ub+ud \\ a+b \end{array}\begin{array}{c} \begin{array}{ccc} a^* & b^* & d^* \end{array} \\ \left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & u & u \\ \hline 1 & 1 & 0 \end{array}\right] \end{array} \sim \text{add } (-1)R_3 \text{ to } R_1 \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & u & u \\ \hline 1 & 1 & 0 \end{array}\right] >$$

generator $\langle \widetilde{ub} \mid u \in Tor(\mathbb{Z}/2, A) \rangle$

$\widetilde{u(b^*+d^*)}$

Cohomology Maps:

$H^0(X) \cong A = \langle x^* + y^* \rangle$ (A)

$Z^0(X) = \langle x^* + y^* \rangle$

$u \in A$

$u \longmapsto u \cdot (x^* + y^*)$

$H^1(X) \cong Tor(\mathbb{Z}/2, A) = \langle \widetilde{ub^*+d^*} \mid u \in Tor(\mathbb{Z}/2, A) \rangle$ (Tor(Z/2, A))

$Z^1(X) = \langle a^*+b^*, ub^*+ud^* \mid u \in Tor(\mathbb{Z}/2, A) \}$

$v \in Tor(\mathbb{Z}/2, A)$

$w \longmapsto w(ub^*+ud^*)$

$H^2(X) \cong \langle \beta^* \mid \beta^* \in A/2A \rangle$ (A/2A)

$Z^2(X) = \langle \alpha^*, \beta^* \rangle$

$w \in A/2A$

$w \longmapsto w' \beta^*$

$q^1 : Z^1 \longrightarrow H^1$

CAP PRODUCT

The cap product is the topological analog of integration,
pairing a chain and a cochain
like Integration pairs a function and measure.

PROPOSITION:

For each $Y \in Ch$

There is a UNIQUE FUNCTOR $Hom : Ch^{op} \times Ch \longrightarrow Ch$

with a NATURAL MAP $Hom(Y,Z) \otimes Y \longrightarrow Z$

This map called the evaluation map has a special "universal property":

chain maps $X \longrightarrow Hom(Y,Z)$ (are in bijection with) $\longleftrightarrow$ chain maps $X \otimes Y \longrightarrow Z$

and how do we recognize this bijection?
Construct the map as follows:

START WITH: $X \longrightarrow Hom(Y,Z)$

TENSOR WITH Y: $X \otimes Y \longrightarrow Hom(Y,Z) \otimes Y$

COMPOSE WITH EVALUATION MAP: $X \otimes Y \longrightarrow Hom(Y,Z) \otimes Y \longrightarrow Z$

END RESULT: $X \otimes Y \longrightarrow Z$

* I am not sure why we need this yet — cool result, though! (?)

CAP PRODUCTS continued...

CAP PRODUCTS ARE THE ANALOG OF INTEGRATION; PAIRING CHAINS AND COCHAINS IN TOPOLOGY IS LIKE PAIRING FUNCTIONS AND MEASURES IN ANALYSIS.

$A \in Ring$ (A is a ring)
$X \in sSet$ (X is a simplicial set)

THERE IS A CANONICAL PAIRING CALLED THE "KRONECKER PAIRING"

$$\langle _, _ \rangle : C^*(X,A) \otimes C(X,A) \longrightarrow A$$

If same degree... $f, x \longmapsto f(x)$
otherwise... $f, x \longmapsto 0$

In other words (Gary!), the kronecker pairing evaluates the cochain on the chain. This map $\langle _, _ \rangle$ is a bichain map, for

$$\langle d(f \otimes x) \rangle = \langle \partial^*(f) \otimes x + (-1)^f f \otimes \partial x \rangle$$

The action of the Kronecker paining

$$= \langle \partial^*(f) \otimes x \rangle + (-1)^f \langle f \otimes \partial x \rangle$$

$$= (\partial^*(f))(x) + (-1)^f f(\partial x)$$

$$= -(-1)^f f(\partial x) + (-1)^f f(\partial x) = 0$$

The pairing only works on cochains and chains of the same degree!

$\pi(C^*(X,A))$
$\pi : coCh \longrightarrow Ch$

$X_n \xrightarrow{d^n} X_{n+1} \longmapsto X_{n+1} \xleftarrow{(-1)^{n+1} - (-1)^n d^n} X_n$

deg n deg n+1 deg -n-1 deg -n

A little lemma

THE PAIRING $\pi(C^*(X,A)) \otimes C(X,A) \longrightarrow A$
IS A CHAIN MAP.

① Recall, the map that sends abelian groups to chains

$$Ab \longrightarrow Ch$$
$$A \longmapsto \cdots 0 \longleftarrow A \longleftarrow 0 \cdots$$

sends an abelian group to a chain with A in deg 0 and 0's elsewhere.

② To show the pairing $\langle -, - \rangle$ is a chain map, we need only show the case $n=1$
(since A as a chain only has a nonzero group for $n=0$)

$$\begin{array}{ccc} C_{n-1} & \longleftarrow & C_n \\ \downarrow & & \downarrow \\ D_{n-1} & \longleftarrow & D_n \end{array} \quad \Big\} \text{ show that the diagram commutes}$$

We show ② commutes:

$$\begin{array}{ccc} \bigoplus_{n-1} C^{n-1}(X,A) \otimes C_{n-1}(X,A) & \xleftarrow{d} & \bigoplus_{n} C^n(X,A) \otimes C_n(X,A) \\ \downarrow & & \downarrow \\ A & \longleftarrow & 0 \end{array}$$

composition #1 $\quad f \otimes x$

composition #2

Take $(f, x) \in \bigoplus_n C^n(X,A) \otimes C_n(X,A)$

$f \in C^n(X,A)$, so f has chain degree $-n$

$$\begin{aligned} d\langle f, x \rangle &= \langle d(f \otimes x) \rangle \\ &\overset{\text{Apply definition}}{=} \langle df \otimes x + (-1)^{-n} f \otimes dx \rangle \\ &= \langle df \otimes x \rangle + (-1)^{+n} \langle f \otimes dx \rangle \\ &= \langle df, x \rangle + (-1)^{+n} \langle f, dx \rangle \\ &= (-1)^{n+1} \langle f, dx \rangle + (-1)^n \langle f, dx \rangle = 0 \end{aligned}$$

$X_n \xrightarrow{d^n} X_{n+1}$, $\quad X_{n+1} \longleftarrow X_n$, $\quad (-1)^{n+1} d^n$

(f has cochain deg n, f has chain deg $-n$)

$= $ composition #2

(the zero map) ▨

NOW, consider the following statement:

THE PAIRING BETWEEN COCHAINS AND CHAINS DESCENDS TO THE LEVEL OF COHOMOLOGY AND HOMOLOGY.

This produces a pairing:

$$H^*(X,A) \otimes H(X,A) \longrightarrow A$$

which is actually just a collection of pairings:

$$H^m(X,A) \otimes H_n(X,A) \longrightarrow A$$

which vanish if $m \neq n$.

What we learned previously:

- Kronecker pairing pairs a cochain and a chain.
- Only cochains/chains of the same degree are paired.
- Kronecker pairing is a chain map

and now...

- kronecker pairing induces cohomology/homology pairing.

PROOF: $f \in Z^m(X,A)$ (f is a cochain of degree m)
$x \in Z_n(X,A)$ (f is a chain of degree n)

let $y \in C_{n+1}(X,A)$ [a chain of degree n+1]
$g \in C^{m-1}(X,A)$ [a cochain of degree m-1]

We need to show $f(x) = f(x+dy) = (f + d^*g)(x)$

$f(x+dy) = f(x) + f(dy) = f(x) + (d^*f)(y) = f(x) + 0 = f(x)$

and

$(f + d^*g)(x) = f(x) + (d^*g)(x) = f(x) + g(dx)$
$= f(x) + g(0) = f(x).$ ∎

THE SLANT PRODUCT

The slant product for simplicial cochains is defined as the composition of the Alexander-Whitney map and the Kronecker pairing.

"MEASURE"

$$\backslash\!: C^*(X,A) \otimes C(Y \times X, A) \longrightarrow C^*(X,A) \otimes C(Y,A) \otimes C(X,A)$$

(ALEXANDER-WHITNEY MAP)

KRONECKER PAIRING $\langle -,- \rangle$

$$C^*(X,A) \otimes C(Y,A) \otimes C(X,A) \longrightarrow C(Y,A)$$

The slant product for simplicial chains is the cochain map:

THE COMPOSITION

$$/\!: C^*(X \times Y, A) \otimes C(X,A) \longrightarrow C^*(Y,A)$$

defined by the composition:

"THE COMPOSITION OF FIBERWISE INTEGRATION"

$$C^*(X \times Y, A) \otimes C(X,A) = \mathrm{Hom}(C(X \times Y), A) \otimes C(X) \otimes A$$

EILENBERG-ZILBER MAP

$$\mathrm{Hom}(C(X) \otimes C(Y), A) \otimes C(X) \otimes A$$

ADJOINT TO EVALUATION MAP

$$\mathrm{Hom}(C(Y), A) \otimes A$$

MULTIPLICATION ON A

$$C^*(Y,A)$$

THE SLANT PRODUCT IS A CHAIN MAP

and... THE SLANT PRODUCT INDUCES A MAP ON HOMOLOGY GROUPS!

$$\backslash\!: H^m(X,A) \otimes H_n(Y \times X, A) \longrightarrow H_{n-m}(Y,A)$$

CAP PRODUCT

$A \in \text{Ring}$ $X \in s\text{Set}$ (w/ coeffs in A)

We define the cap product:

$$\cap : C^m(X,A) \otimes C_n(X,A) \longrightarrow C_{n-m}(X,A)$$

defined as the composition:

$$C^m(X,A) \otimes C_n(X,A) \longrightarrow C^m(X,A) \otimes C_n(X \times X, A) \longrightarrow C_{n-m}(X,A)$$

↑ map #1 of composition

$C^m(X,A) \otimes C_n(d,A)$

↑ diagonal map

↑ map #2 of composition

Slant-product for simplicial cochains

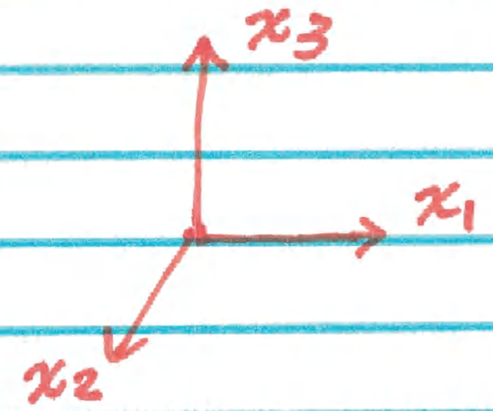
AND THIS MAP INDUCES A CAP PRODUCT ON HOMOLOGY / COHOMOLOGY GROUPS:

$$\cap : H^m(X,A) \otimes H_n(X,A) \longrightarrow H_{n-m}(X,A)$$

EXAMPLE: SPHERE!

$$S^n = \{ x \in \mathbb{R}^{n+1} \mid \|x\| = 1 \}$$

Let $n=2$; $S^2 = \{ x \in \mathbb{R}^3 \mid \|x\| = 1 \}$



① OPEN COVER

For each $x \in \mathbb{R}^3$, $x = (x_1, x_2, x_3)$
We define an open cover as follows:

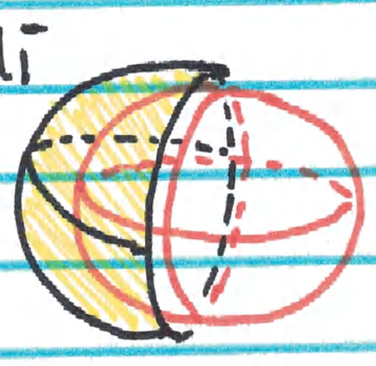
$\{U_i^+, U_i^-\} = U$ (open cover)

$U_i^+ = \{ x \in S^n \mid x_i > 0 \}$
$U_i^- = \{ x \in S^n \mid x_i < 0 \}$



So, when $n=2$, we have:

$U_1^+ = \{ x \in S^2 \mid x_1 > 0 \}$



$U_1^- = \{ x \in S^2 \mid x_1 < 0 \}$

$U_2^+ = \{ x \in S^2 \mid x_2 > 0 \}$



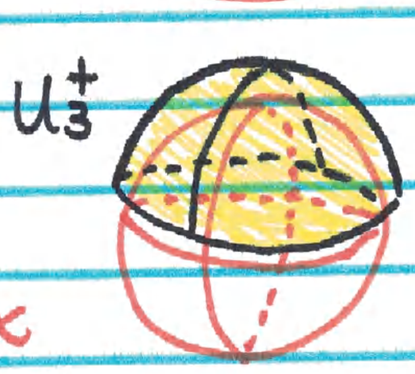
$U_2^- = \{ x \in S^2 \mid x_2 < 0 \}$

$U_3^+ = \{ x \in S^2 \mid x_3 > 0 \}$



$U_3^- = \{ x \in S^2 \mid x_3 < 0 \}$

NOTE: $U_i^+ \cap U_i^- = \emptyset$, $\forall i$



② k-SIMPLEX

A nondegenerate k-simplex of the nerve of this open cover $N(S^2, U)$ is an injective map $k \longrightarrow n$

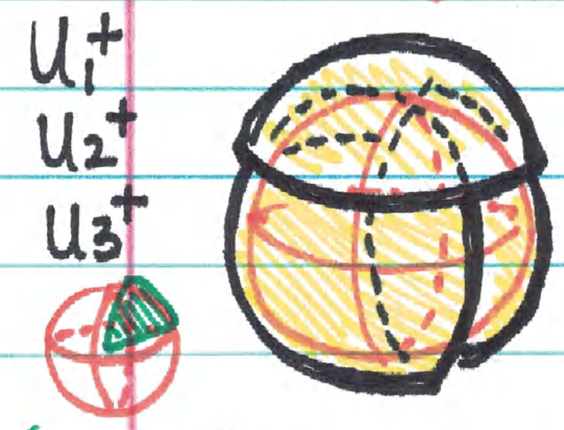


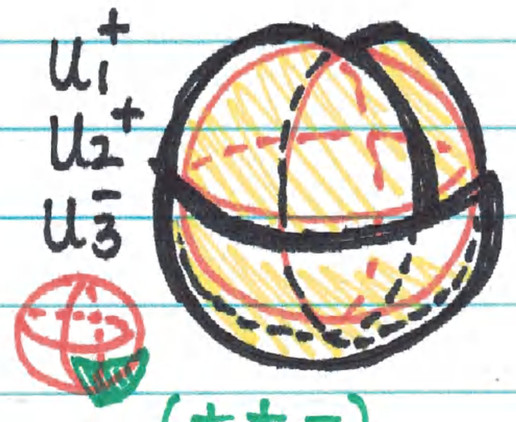
⟶

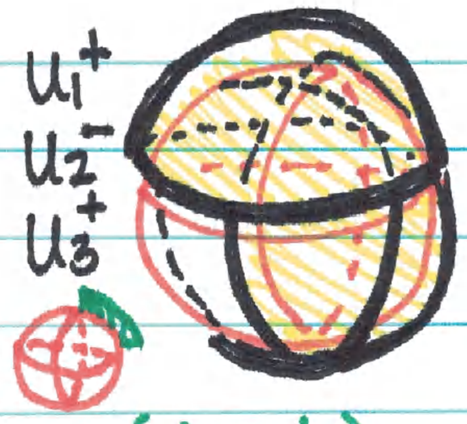
→ map of simplices $[k] \to [n]$
together with a map of sets $[k] \to \{+,-\}$
to indicate whether we choose U_i^+ or U_i^-.

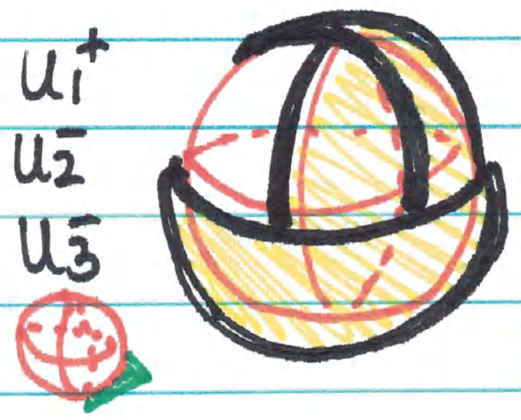
NOTE: There are exactly 2^{n+1} nondegenerate $[n]$-simplices corresponding to the 2^{n+1} maps $[n] \to \{+,-\}$

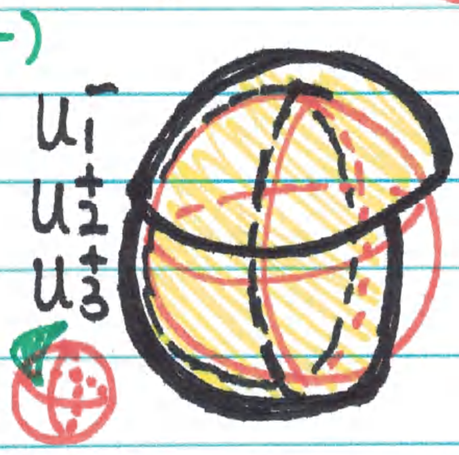
Here, $n=2$; there are $2^{2+1} = 8$ nondegenerate $[2]$-simplices corresponding to the 8 intersections of $U_i \pm$'s:

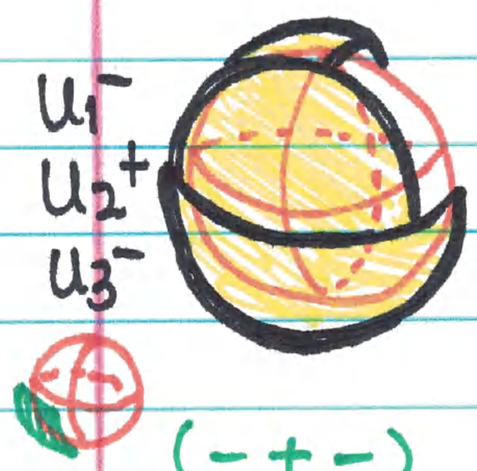


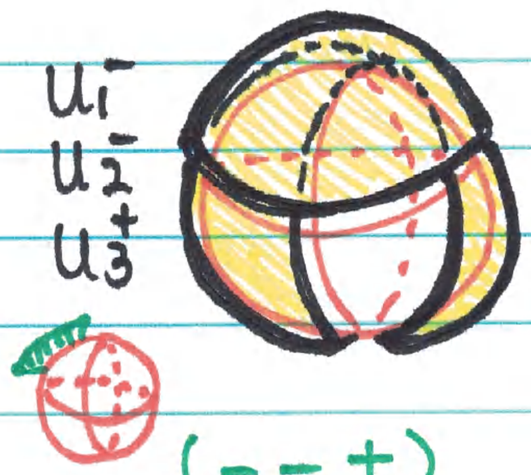


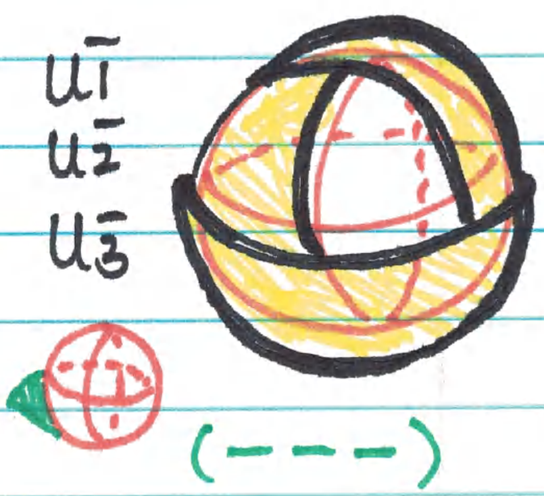










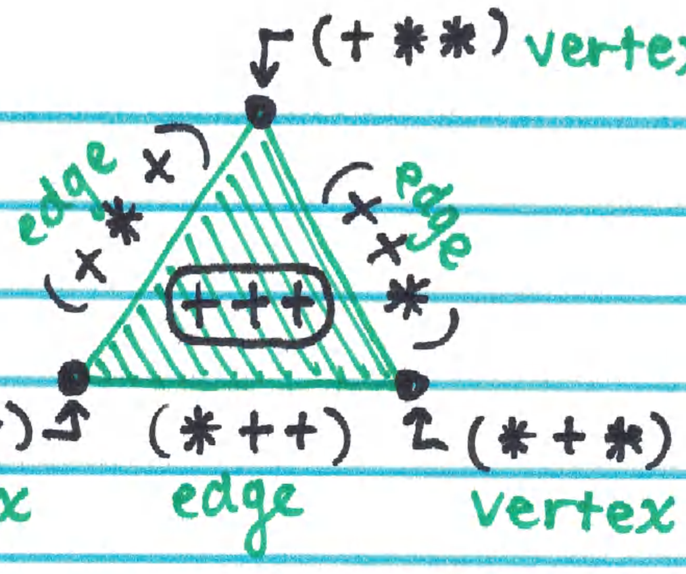


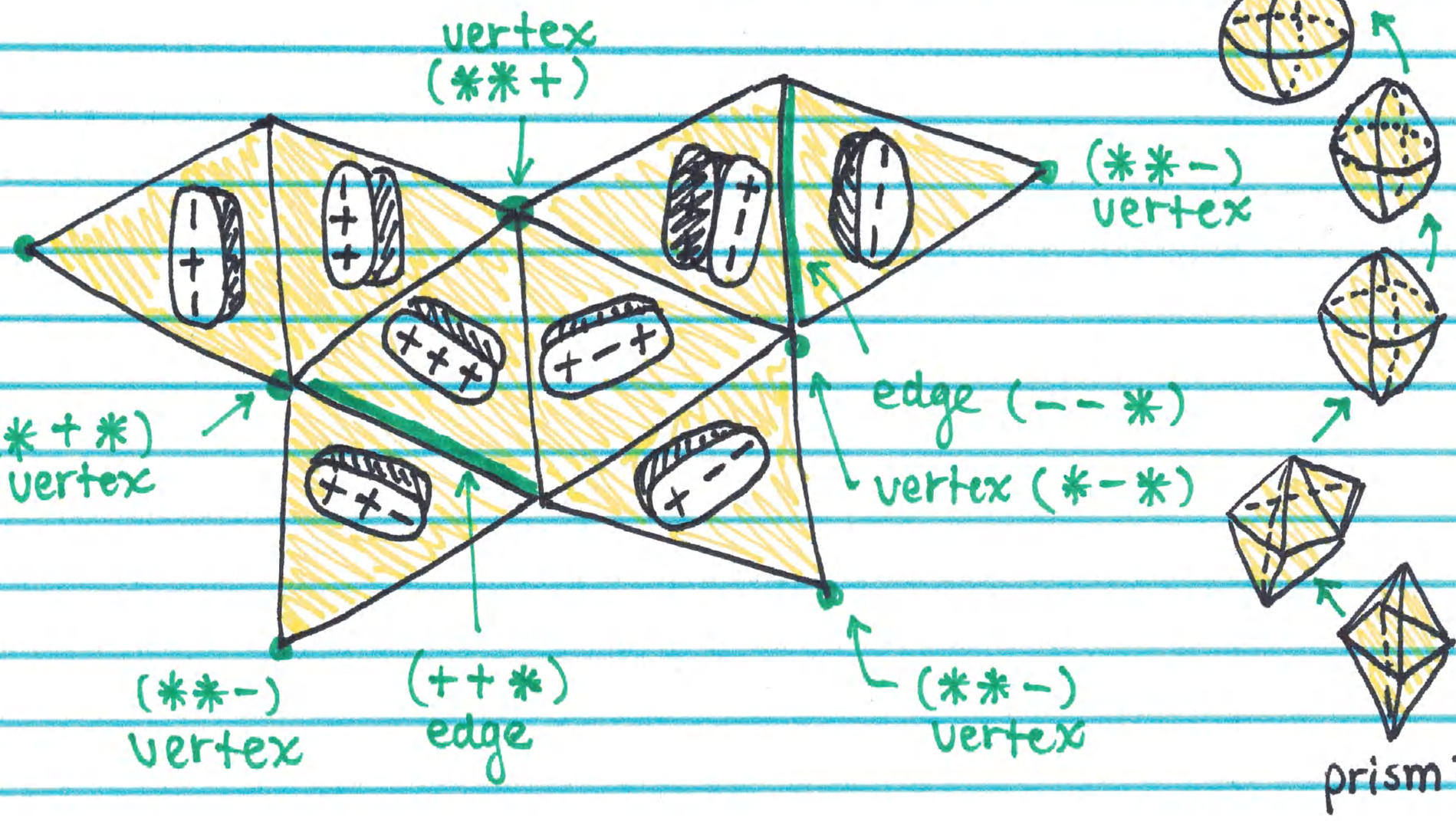
There are $(n+1)2^n$ nondegenerate $[n+1]$-simplices,
so here, $n=2$,
there are $(3)2^2 = 12$ nondegenerate $[1]$-simplices

$(*++)$ $(*-+)$ $(+*+)$ $(-*+)$ $(++*)$ $(-+*)$
$(*+-)$ $(*--)$ $(+*-)$ $(-*-)$ $(+-*)$ $(--*)$

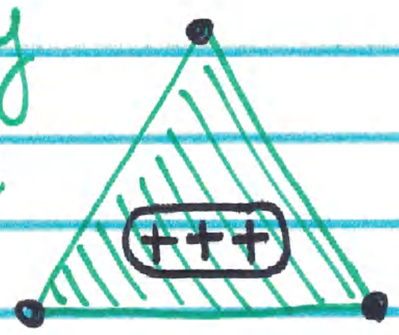
$U_1^+ \cap U_2^+ \cap U_3^+ = (+++)$ a [2]-simplex





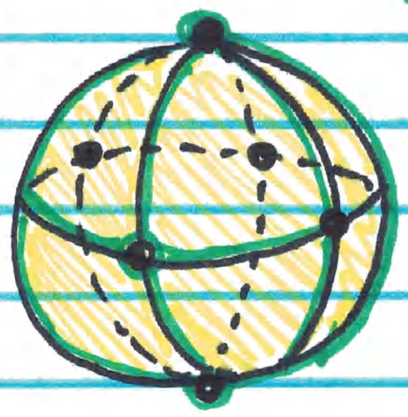


Boundary of the [2]-simplex



$\partial(+++) = (*++) - (+*+) + (++*)$

NONDEGENERATE SIMPLICES



[0]-simplices	{(+**), ... (**-)}	$\mathbb{Z}^6$
[1]-simplices	{(++*), ... (*--)}	$\mathbb{Z}^{12}$
[2]-simplices	{(+++), ... (---)}	$\mathbb{Z}^8$

COMPUTE HOMOLOGY FROM HERE
LIKE NORMAL (use trick in notes)

TOPOLOGY^2

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notes posted online on website

§13 COPRODUCTS AND COEQUALIZERS

coproduct notion: We assemble two pictures, side by side, without intersections, like a disjoint union of sets.

DEFINITION

The "coproduct" of objects X and Y in a category $\mathcal{C}$

is a triple $\begin{cases} X \sqcup Y & \text{(an object in } \mathcal{C}\text{)} \\ \iota_X : X \longrightarrow X \sqcup Y & \text{(morphisms in } \mathcal{C}\text{)} \\ \iota_Y : Y \longrightarrow X \sqcup Y & \text{(morphisms in } \mathcal{C}\text{)} \end{cases}$

such that
the following universal property is satisfied:

$\forall Z \in \mathcal{C}$

the map $(\hom(\iota_X, Z), \hom(\iota_Y, Z))$:

$\hom(X \sqcup Y, Z) \longrightarrow \hom(X, Z) \times \hom(Y, Z)$

$f \in \hom(X \sqcup Y, Z)$

$f : X \sqcup Y \longrightarrow Z$

$(\hom(\iota_X, Z), \hom(\iota_Y, Z))(f) = (f \circ \iota_X, f \circ \iota_Y)$

$f \circ \iota_X \in \hom(X, Z)$

$f \circ \iota_Y \in \hom(Y, Z)$

is a bijection.

NOTE: $f : X \longrightarrow Z$, $g : Y \longrightarrow Z$ > the inverse image of (f, g) under the map $(\hom(\iota_X, Z), \hom(\iota_Y, Z))$ is called the "copairing"

The "copairing" $[f, g] : X \sqcup Y \longrightarrow Z$

Lemma:

$X \in \mathcal{C}$ (an object in a category)
$Y \in \mathcal{C}$ (an object in a category)

GIVEN TWO COPRODUCTS OF X & Y

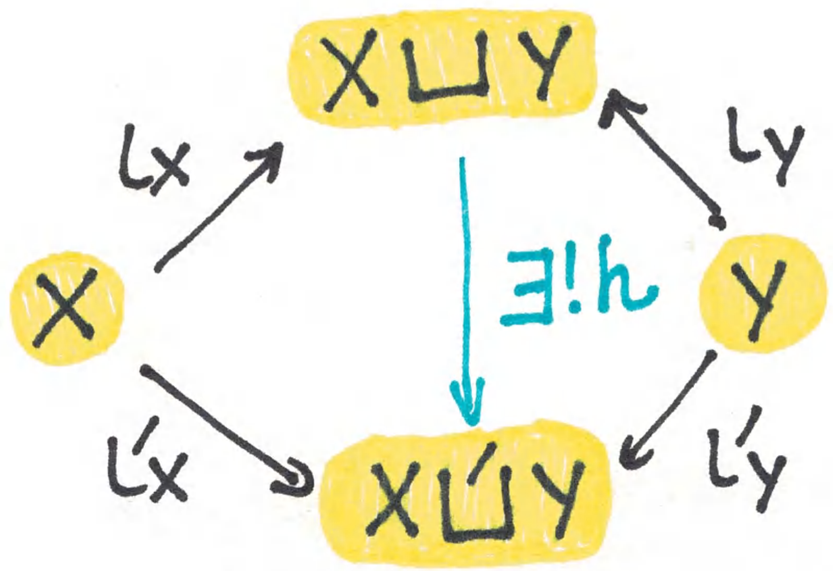
coproduct #1

$(X \sqcup Y,\ \iota_X : X \longrightarrow X \sqcup Y,\ \iota_Y : Y \longrightarrow X \sqcup Y)$

coproduct #2

$(X \acute{\sqcup} Y,\ \acute{\iota}_X : X \longrightarrow X \acute{\sqcup} Y,\ \acute{\iota}_Y : Y \longrightarrow X \acute{\sqcup} Y)$

* THERE IS A UNIQUE MAP $h : X \sqcup Y \longrightarrow X \acute{\sqcup} Y$ *
* THAT MAKES THE FOLLOWING DIAGRAM COMMUTE. *



Furthermore, $h : X \sqcup Y \longrightarrow X \acute{\sqcup} Y$ is an isomorphism.

When $\mathcal{C}$ = Set, the coproduct of X and Y is the disjoint union of sets, with ι_X and ι_Y as injection maps

When $\mathcal{C}$ = sSet, the coproduct of X and Y exists, and is defined by:

$$\begin{cases} (X \sqcup Y)_m = X_m \sqcup Y_m \\ (X \sqcup Y) f \doteq X_n \sqcup Y_n \longrightarrow X_m \sqcup Y_m \\ \quad \text{for a function } f : m \longrightarrow n \end{cases} \qquad \begin{cases} \iota_{X_m} : X_m \to X_m \sqcup Y_m \\ \iota_{Y_m} : Y_m \longrightarrow X_m \sqcup Y_m \\ \text{injection maps} \end{cases}$$

When $\mathcal{C} = \{X_i\}_{i \in I}$ (objects in a category)

Then the coproduct can be defined

$$\bigsqcup_{i \in I} X_i \quad \text{w/morphisms} \quad L_i : X_i \longrightarrow \bigsqcup_{i \in I} X_i$$

*BUT NOTE: The coproduct does not always exist!

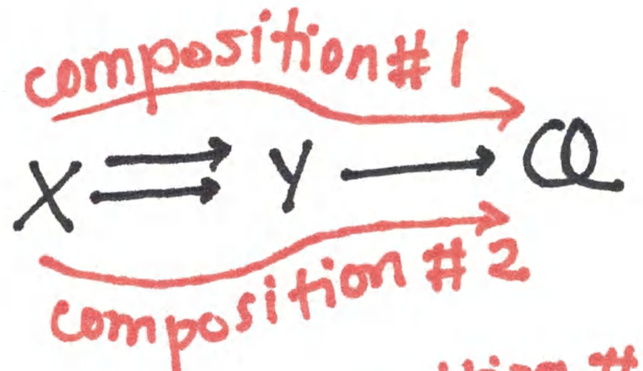
DEFINITION

A "coequalizer fork" of morphisms
(made up of two morphisms $f : X \longrightarrow Y$ and $g : X \longrightarrow Y$)
is a morphism:

$$q : Y \longrightarrow Q$$

such that $q \circ f = q \circ g$

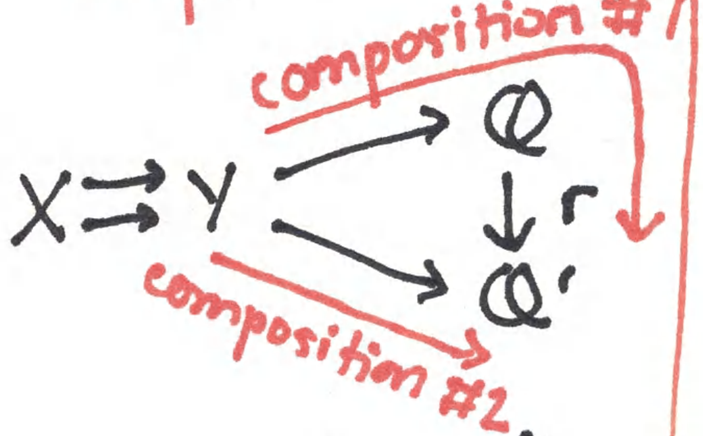
$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \longrightarrow Q$$



A "morphism of coequalizer forks";
is a morphism

$$r : Q \longrightarrow Q'$$

such that the following diagram commutes:



The "coequalizer" of morphisms $f : X \longrightarrow Y$ and $g : X \longrightarrow Y$

is a coequalizer fork

$$q : Y \longrightarrow Q$$

such that

for any coequalizer fork $q' : Y \longrightarrow Q'$

there is a UNIQUE morphism of coequalizer forks.

COPRODUCT

$X \sqcup Y$

$\iota_X : X \longrightarrow X \sqcup Y$

$\iota_Y : Y \longrightarrow X \sqcup Y$

A triple of an object and two injection morphisms

This new object is much like assembling a new picture where both objects X and Y are present but non-intersecting

The injection maps tell how each original object is embedded in this new object.

COEQUALIZER

a coequalizer fork morphism

$q : Y \longrightarrow Q$

that has the property

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{q} Q$$

$$q \circ f = q \circ g$$

such that the universal property of coequalizers holds:

$$X \rightrightarrows Y \xrightarrow{q} Q \qquad Y \xrightarrow{q'} Q' \quad \exists! r : Q \to Q'$$

This morphism q called the coequalizer is defined based on the morphisms f and g.

*NOTE: coproducts and coequalizers do not always exist!

§26

LIMITS & COLIMITS

"diagram"

A functor $\mathcal{D} : I \longrightarrow \mathcal{C}$

I: indexing category

$\mathcal{C}$: the category in question

"small diagram"

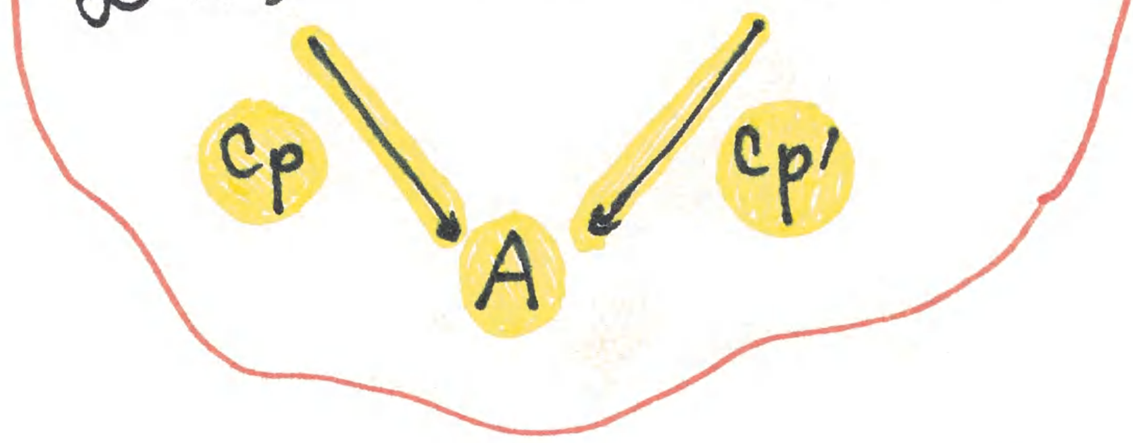
A functor $\mathcal{D} : I \longrightarrow \mathcal{C}$

"small" indexing category

the category in question

"cocone" under a diagram $\mathcal{D}$

An object $A \in \mathcal{C}$ together with a family of injection morphisms $cp : \mathcal{D}(P) \longrightarrow A$ for any $P \in \mathcal{C}$ such that for any arbitrary morphism $f : P \longrightarrow P'$ in I:



$I = \{P_1, P_2, P_3, \ldots\}$

$\mathcal{D}(P_1) \in \mathcal{C}$
$\mathcal{D}(P_2) \in \mathcal{C}$
$\mathcal{D}(P_3) \in \mathcal{C}$

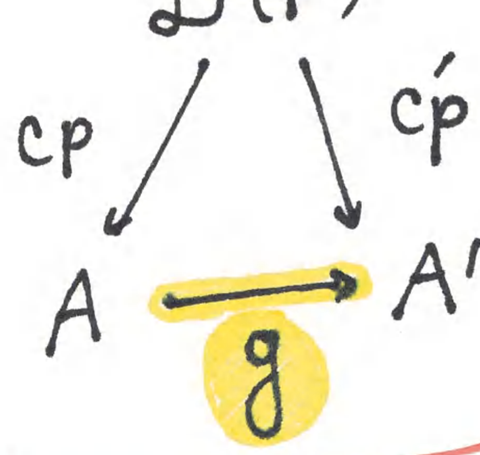
$f : P_1 \to P_2$
$g : P_1 \to P_3$
$h : P_2 \to P_3$

$\mathcal{D}(f) : \mathcal{D}(P_1) \to \mathcal{D}(P_2)$
$\mathcal{D}(g) : \mathcal{D}(P_1) \to \mathcal{D}(P_3)$
$\mathcal{D}(h) : \mathcal{D}(P_2) \to \mathcal{D}(P_3)$

So a cocone is another object A in the category $\mathcal{C}$ with injective morphisms from the image of P under $\mathcal{D}$ to this object A.

"morphism of cocones" under a diagram $\mathcal{D}$

A morphism $g : A \longrightarrow A'$ such that the following triangle commutes for an arbitrary object $p \in I$



The "category of cocones" under a diagram $\mathcal{D}$

objects: cocones under $\mathcal{D}$

morphisms: morphisms of cocones under $\mathcal{D}$

"Initial object" in a category

For $A \in \mathcal{C}$, A is an initial object in the category $\mathcal{C}$ if

$\forall B \in \mathcal{C}$ (for all other objects)

$\exists!$ morphism $A \longrightarrow B$

Examples of initial objects in your favorite categories:

$\mathcal{C} = \mathbf{set}$, $A = \emptyset$ (the empty set)

$\mathcal{C} = \mathbf{Ab}$, $A = \{0\}$ (the trivial group)

$\mathcal{C} = \mathbf{Ring}$, $A = \mathbb{Z}$ (the integers)

$\mathcal{C} = \mathbf{Field}$, does NOT have an initial object (there are no morphisms of fields w/ different characteristics, etc.)

$\mathcal{C} = \Delta$, does NOT have an initial object

$\mathcal{C} = \mathbf{Cat}$, $A = \emptyset$ (the empty category)

$\mathcal{C} = \mathbf{sSet}$, $A = \emptyset$ (the empty simplicial set)

$\mathcal{C} = \mathbf{Top}$, $A = \emptyset$ (the empty topological space)

*NOTE: Initial objects are unique up to a unique isomorphism.

"colimit" of a diagram

The colimit of a diagram $\mathcal{D}: I \longrightarrow \mathcal{C}$ is the **initial object** in the category of cocones under $\mathcal{D}$.

* The colimit is a cocone. It is the special cocone such that there is a morphism of cocones from the colimit to any other cocone

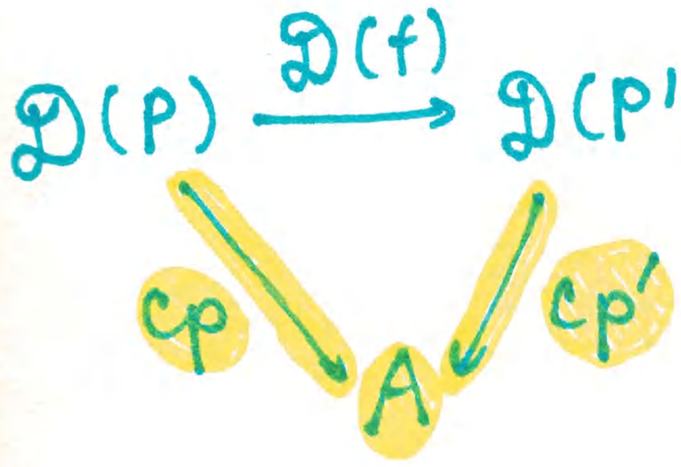
* Because initial objects are unique up to a unique isomorphism, the colimit (an initial object) is unique up to a unique isomorphism.

cocomplete category

A category that **admits** **colimits** of all small diagrams

*NOTE: The colimit of $\mathcal{D}$ is a cocone (which is also the initial object of the category of cocones)

The colimit is (i) an object A
(ii) a family of injection morphisms $\{cp\}$



BUT by abuse of language, we often refer to the **object A** (ignoring maps $\{cp\}$) as the colimit of $\mathcal{D}$.

When including maps $\{cp\}$, we specifically say "colimit cocone."

The "Universal property of colimits"

There is a natural bijection between maps

$$\operatorname{colim} \mathcal{D} \longrightarrow A$$

and families of maps

$$c_p : \mathcal{D}(p) \longrightarrow A$$

Example #1

indexing category $\mathcal{I} = \{0, 1\}$
(with NO non-identity morphisms)
that is, this category is just two objects
0 and 1, and one morphism
(the identity morphism)

A **diagram** (a functor) $\mathcal{D} : \mathcal{I} \longrightarrow \mathcal{C}$ is a pair of objects

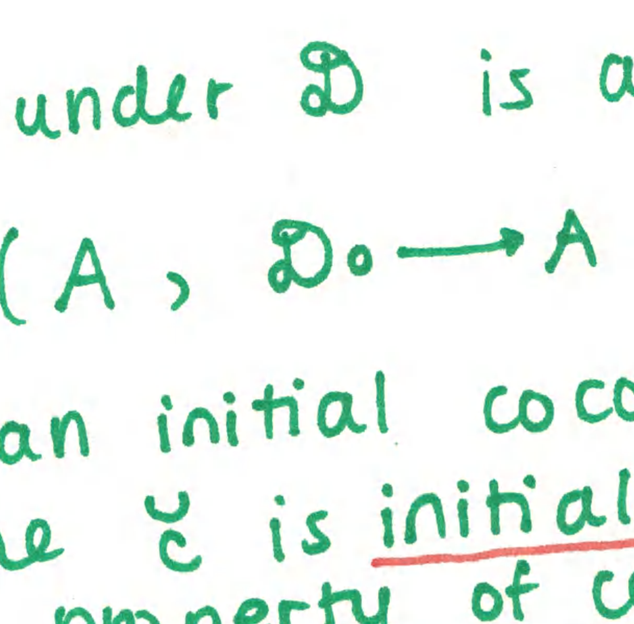
$\mathcal{D}_0$ and $\mathcal{D}_1 \in \mathcal{C}$

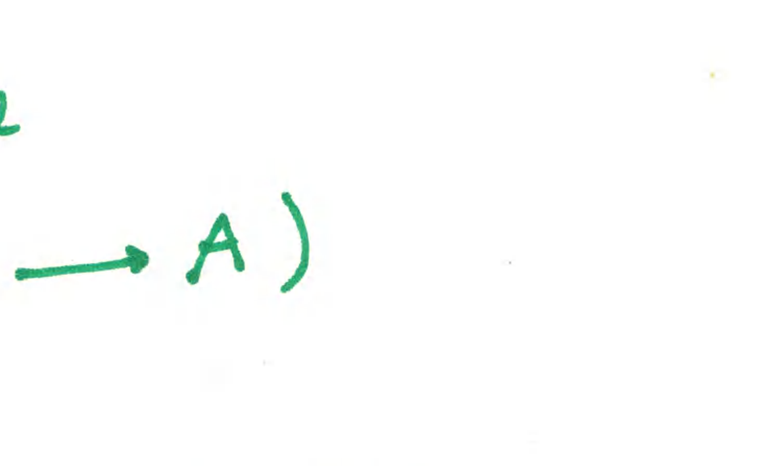
A **cocone** $\overset{\vee}{c}$ under $\mathcal{D}$ is a triple

$$(A, \mathcal{D}_0 \longrightarrow A, \mathcal{D}_1 \longrightarrow A)$$

A **colimit** is an initial cocone,
and the cocone $\overset{\vee}{c}$ is initial if it satisfies the universal property of coproducts.

WHY?





* Under what conditions (if any) does there exist a unique morphism of cocones given any other cocone

Example #2

indexing category $I = \{0 \rightrightarrows 1\}$

(that is, a category with objects 0 and 1 and exactly 2 non-identity morphisms)

A diagram $\mathcal{D}: I \longrightarrow \mathcal{C}$ is

a pair of objects $\mathcal{D}_0, \mathcal{D}_1 \in \mathcal{C}$

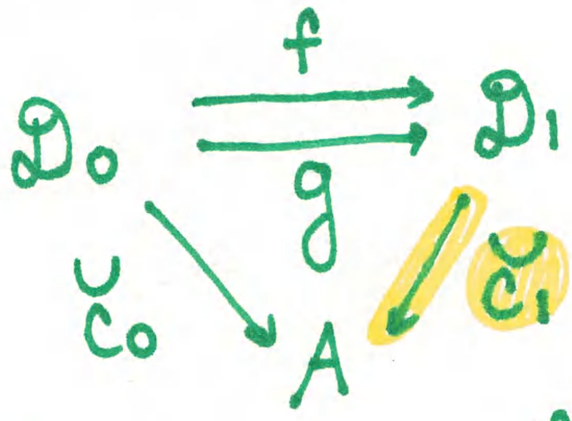
and two morphisms $f, g: \mathcal{D}_0 \longrightarrow \mathcal{D}_1$

A cocone $\check{C}$ under $\mathcal{D}$ is a morphism

$$\check{c}: \mathcal{D}_1 \longrightarrow A$$

such that $c \circ f = c \circ g$

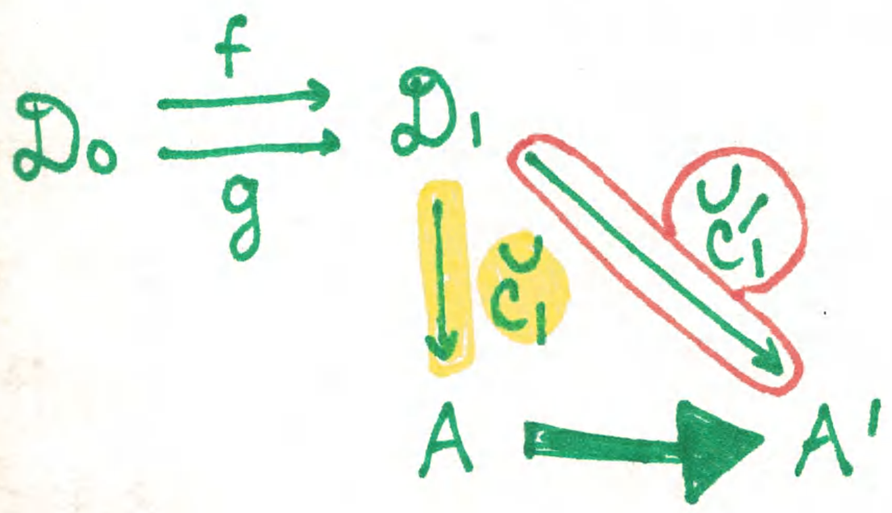
[The triple $(A, \check{c}_0: \mathcal{D}_0 \longrightarrow A, \check{c}_1: \mathcal{D}_1 \longrightarrow A)$]



A colimit is an initial cocone and the cocone $\check{c}$ is initial (here) it satisfies the property of coequalizers.

WHY

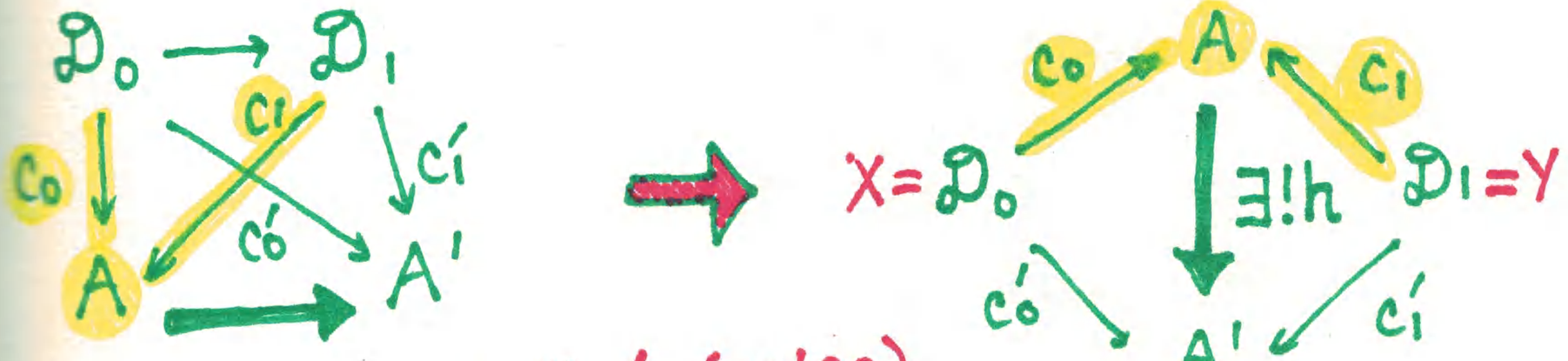
*we only need $\check{c}_1$ because the condition $c \circ f = c \circ g$ forces the diagram to commute. A is implied. $\check{c}_0$ is defined by $\check{c}_1$.



*Under what conditions (if any) does there exist a unique morphism of cocones.

Example 1

Why is this the coproduct?



Recall, Lemma 13.4 (notes)

If X and Y are objects in a category $\mathcal{C}$ and

$(X \amalg Y, \iota_X : X \to X \amalg Y, \iota_Y : Y \to X \amalg Y)$

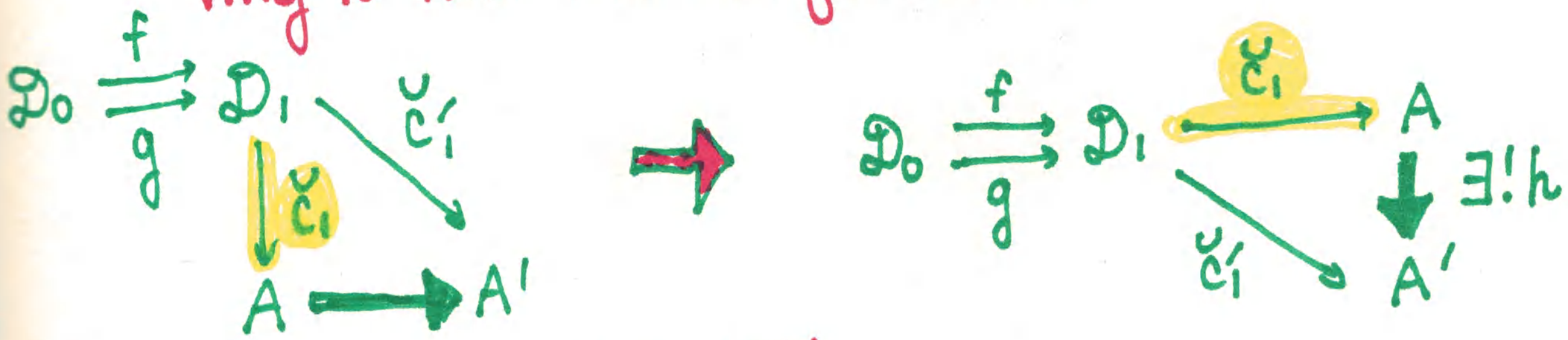
$(X \acute{\amalg} Y, \iota_X' : X \to X \acute{\amalg} Y, \iota_Y : Y \to X \acute{\amalg} Y)$

are coproducts of X and Y, then... ... there exists a unique map $h : X \amalg Y \to X \acute{\amalg} Y$ that makes the diagram commute.

So, the universal property of coproducts satisfies the conditions for the existence of a unique homomorphism from A to $A' \in \mathcal{C}$, making the cocone (A, c_0, c_1) an initial object.

Example 2

Why is this the coequalizer?



Recall, Definition 13.14 (notes)

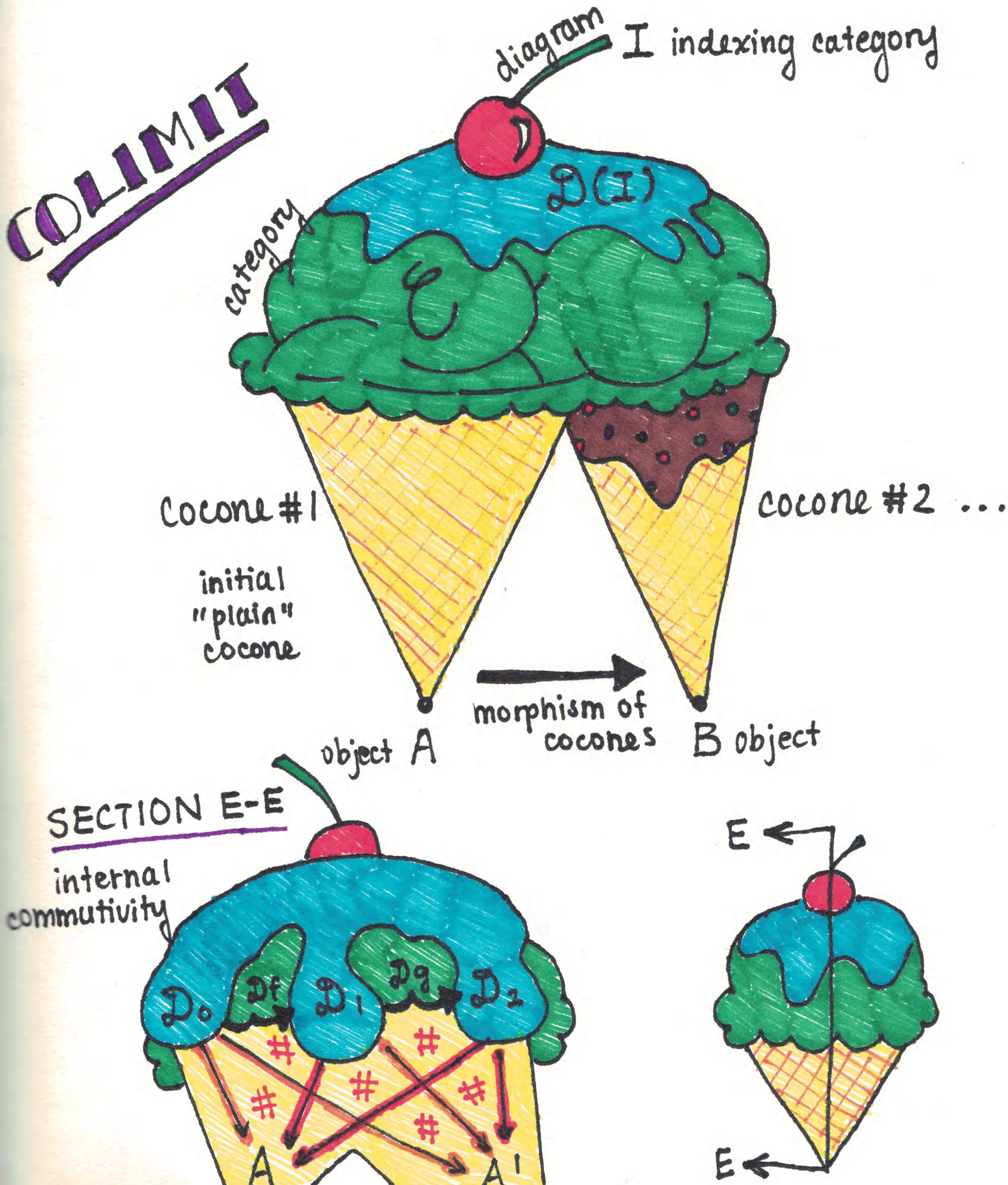
If $f, g : X \to Y$ are morphisms in a category $\mathcal{C}$ and $q : Y \to Q$ is a coequalizer fork (that is, $qof = qog$), then the universal property of coequalizers holds:

for any other coequalizer fork $q' : Y \to Q'$, ... there exists a unique morphism of coequalizer forks: $h : Q \to Q'$ such that the diagram commutes.

so... the universal property of coequalizers satisfies the conditions for the existence of a unique homomorphism from $A \in \mathcal{C}$ to $A' \in \mathcal{C}$, making the cocone $(A, \check{c}_0, \check{c}_1)$ initial.

→ Conclusions from Examples:

- When the indexing category has two objects and No non-identity morphisms, the colimit is the coproduct of the ~~category~~ objects.
- When the indexing category has two objects and two non-identity morphisms, the colimit is the coequalizer of the maps.



Example #3

indexing category $\mathcal{I} = \emptyset$.

For any category $\mathcal{C}$, there is a unique empty diagram:

THE EMPTY DIAGRAM $\quad \mathcal{D}: \emptyset \longrightarrow \mathcal{C}$

The **colimit** of such a diagram is just the initial object of $\mathcal{C}$ (if it exists).

Example #4

indexing category $\mathcal{I} = \{1\}$

(consisting of a single object 1 and a single identity morphism.)

A **DIAGRAM** $\quad \mathcal{D}: \mathcal{I} \longrightarrow \mathcal{C}$

is a single object, $\mathcal{D}_1 \in \mathcal{C}$

The **COLIMIT** of $\mathcal{D}$ is $\mathcal{D}_1$.

Example #5

indexing category $\mathcal{I} = \{0 \to 1 \to 2 \to \dots\}$

(a colimit indexed by such a category is called a "sequential colimit")

$0 \xrightarrow{f_0} 1$, $1 \xrightarrow{f_1} 2$ > $0 \xrightarrow{f_1 \circ f_0} 2$ composition of maps

A **diagram** $\quad \mathcal{D}: \mathcal{I} \longrightarrow \mathcal{C}$ is a collection

$\mathcal{D}(i)$, $\forall i \geq 0$ together with

morphisms $\mathcal{D}(i) \xrightarrow{\mathcal{D}f_i} \mathcal{D}(i+1)$

so... $\quad \mathcal{I} = \{0 \xrightarrow{f_0} 1 \xrightarrow{f_1} 2 \xrightarrow{f_2} \dots\}$

$$\mathcal{D}_0 \xrightarrow{\mathcal{D}f_0} \mathcal{D}_1 \xrightarrow{\mathcal{D}f_1} \mathcal{D}_2 \xrightarrow{\mathcal{D}f_2} \dots$$

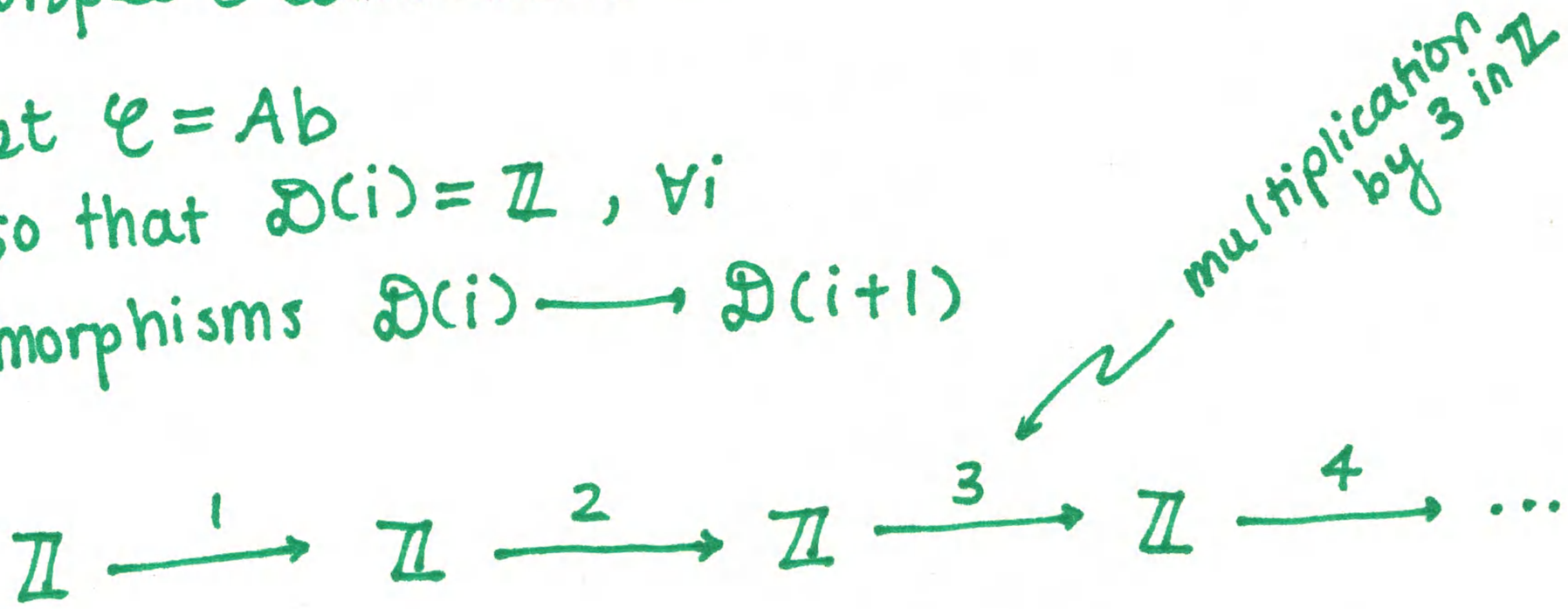
Continued... ⟶

Example 5 continued...

Let $\mathcal{C} = Ab$

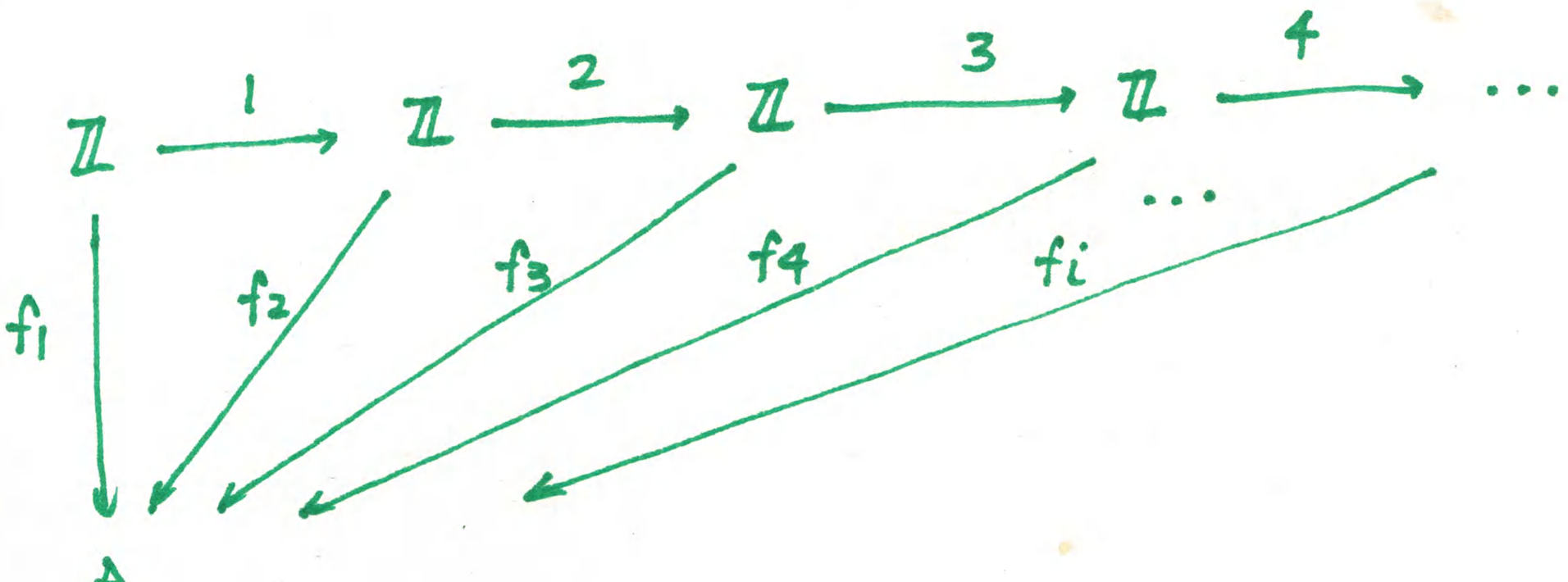
so that $\mathcal{D}(i) = \mathbb{Z}$, $\forall i$

w/morphisms $\mathcal{D}(i) \longrightarrow \mathcal{D}(i+1)$



We wish to determine the colimit of this diagram.

Given $A \in \mathcal{C}$



These triangles must commute:

$n \in \mathbb{Z}$,

$f_1(n) = f_2(n)$

$f_2(n) = f_3(2n) = 2 \cdot f_3(n)$, $\quad \frac{1}{2} f_2(n) = f_3(n)$

$f_3(n) = f_4(3n) = 3 \cdot f_4(n)$, $\quad \frac{1}{3} f_3(n) = f_4(n)$

thinking... development...

$f_4(n) = \frac{1}{3}\left(\frac{1}{2}\right)(f_1(n))$ and this relationship continues...

$f_4(n) = \frac{1}{3!} f_1(n)$

$f_k(n) = \frac{1}{(k-1)!} f_{k-1}(n)$

$\vdots$

$f_1(k) = n!\, f_n(k)$

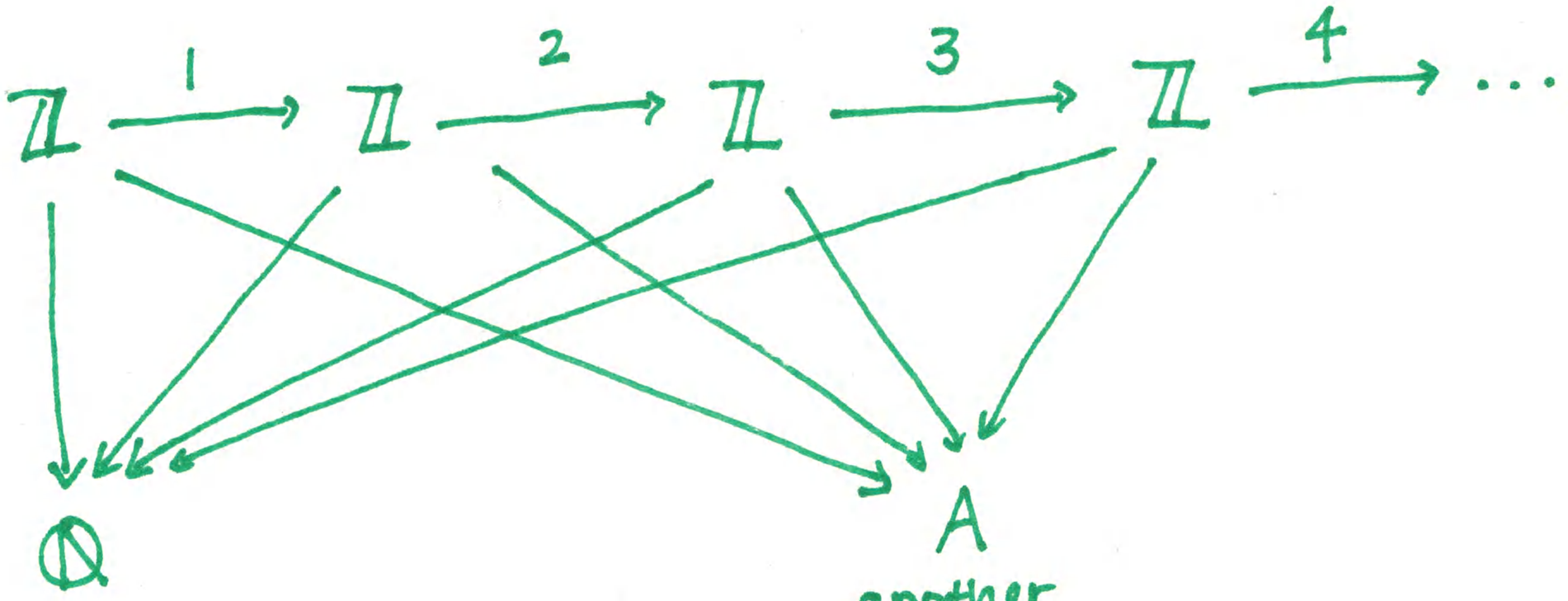
$\frac{f_1(k)}{n!} = f_n(k)$ ⟲

When $f_1(k) = m$ we have $f_n(k) = \frac{m}{n!}$

What $A \in \mathcal{C} = Ab$ satisfies such a relation?

$A = \mathbb{Q}$. (We verify that $\mathbb{Q}$ is the initial object...)

continued... ⟶

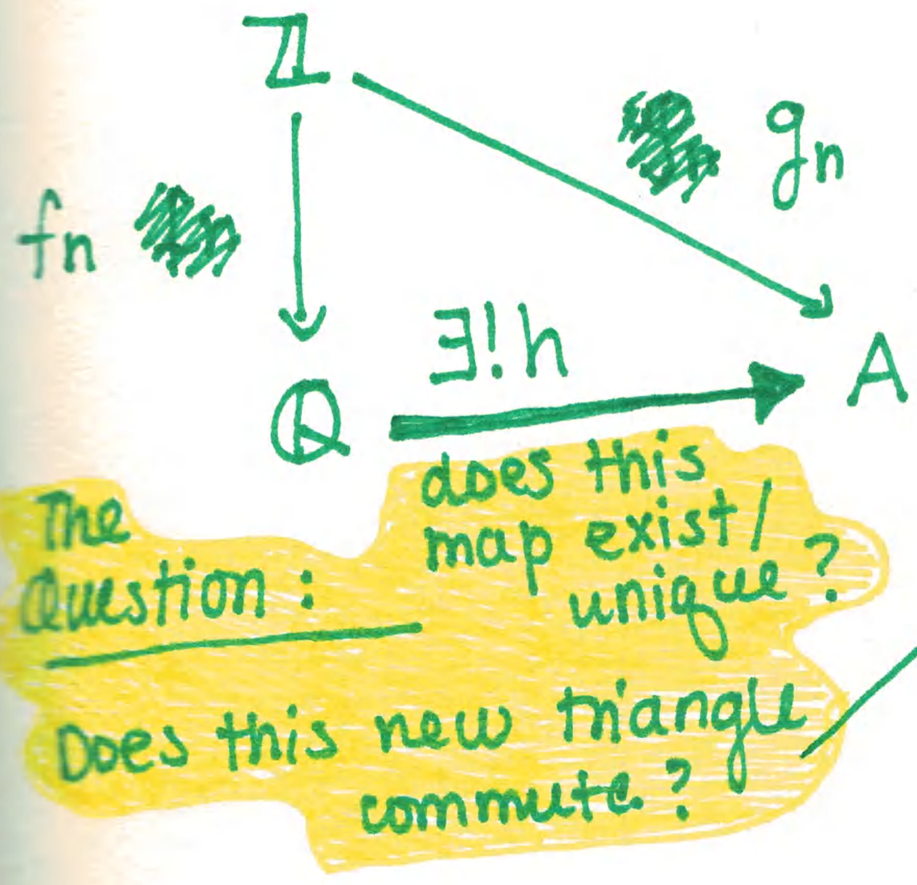


the proposed initial object (initial cocone?)

another object (another cocone)

We already determined The maps Ln ...

$$f_n(m) = \frac{m}{n!}$$



The Question: does this map exist/ unique? Does this new triangle commute?

Assume h is a morphism of cocones (so the triangle commutes)

Answer #1: If such an h exists, it must be unique

UNIQUENESS

$$g_n(m) = (h \circ f_n)(m) = h(f_n(m))$$

$$g_n(m) = h(m/n!)$$

$$g_n(m(n-1)!) = (h \circ f_n)(m(n-1)!) = h(f_n(m(n-1)!))$$

$$g_n(m(n-1)!) = h\left(\frac{m(n-1)!}{n!}\right) = h\left(\frac{m}{n}\right)$$

For n=1 $g_1(m) = h(m/1!) = h(m)$ $\quad g_1(m(1-1)!) = h\left(\frac{m}{1}\right) = h(m)$

For n=2 $g_2(m) = h(m/2!) = h(m/2)$ $\quad g_2(m(2-1)!) = h(m/2)$

For n=3 $g_3(m) = h(m/3!)$ $\quad g_3(m(2)!) = h\left(\frac{m}{3}\right)$

? HOW DOES THIS PROVE UNIQUE?

This proves uniqueness because
→ we can map $\forall r \in \mathbb{Q}$ (representable as $r = \frac{m}{n}$, $m,n \in \mathbb{Z}$)
uniquely under $h: \mathbb{Q} \longrightarrow A$

Answer #2 **EXISTENCE**

$\forall r \in \mathbb{Q}$, define $h(r) = g_n(rn!)$
where $n \in \mathbb{Z}$ is chosen such that $rn! \in \mathbb{Z}$.

$g_n(rn!)$: idependent of choice of n
since $g_n(nk) = g_{n-1}(k)$, $\forall n > 0$, $\forall k \in \mathbb{Z}$.

To show h is a homomorphism of abelian groups:

$$h(r+r') = g_m((r+r')m!)$$

$$* \begin{cases} = g_m(rm!) + g_m(r'm!) \\ = g_n(rn!) + g_{n'}(r'n'!) \end{cases}$$

where
$m \geq n$
$m \geq n'$

$$= h(r) + h(r')$$

Example #6

Indexing category $\mathcal{I} = \{ 0 \to 1 \;;\; 0 \to 2 \}$

which can be written as $\longrightarrow$ $\mathcal{I} = \{ 1 \leftarrow 0 \rightarrow 2 \}$

(3 objects, 2 nonidentity morphisms)

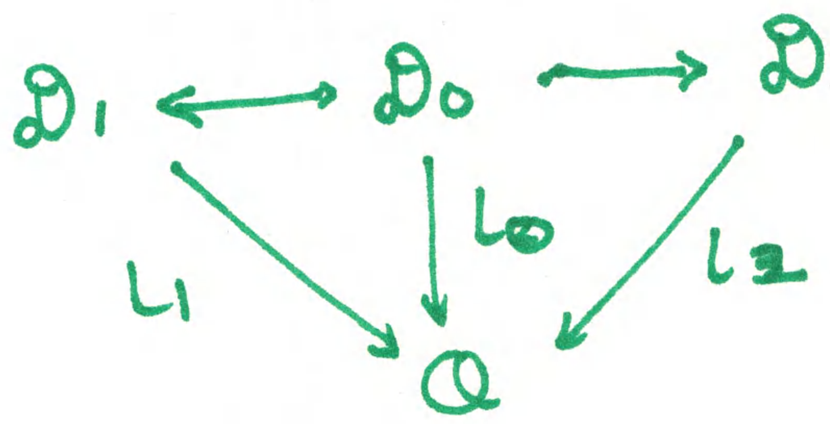
A **diagram** $\mathcal{D} : \mathcal{I} \longrightarrow \mathcal{C}$ is

$$\mathcal{D}_1 \longleftarrow \mathcal{D}_0 \longrightarrow \mathcal{D}_2$$

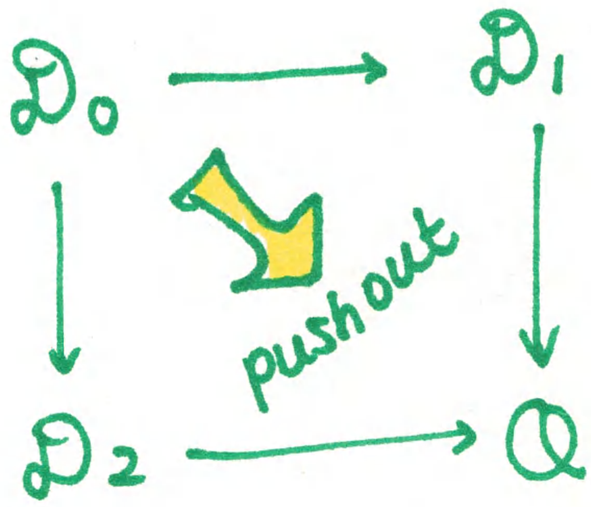
morphism #1 morphism #2

A pair of morphisms in $\mathcal{C}$.

The "pushout" of $\mathcal{D}$ is defined as the colimit



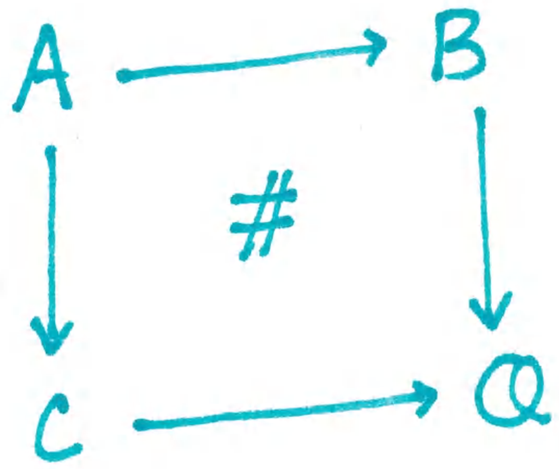
and this diagram can be depicted (with less information) as:



This square is called **"cocartesian"** if it arises as a pushout of $\mathcal{D}_0 \longrightarrow \mathcal{D}_1$ and $\mathcal{D}_0 \longrightarrow \mathcal{D}_2$

The morphism $\mathcal{D}_1 \longrightarrow \mathcal{Q}$ is called the **"cobase change"** of $\mathcal{D}_0 \longrightarrow \mathcal{D}_2$ along $\mathcal{D}_0 \longrightarrow \mathcal{D}_1$

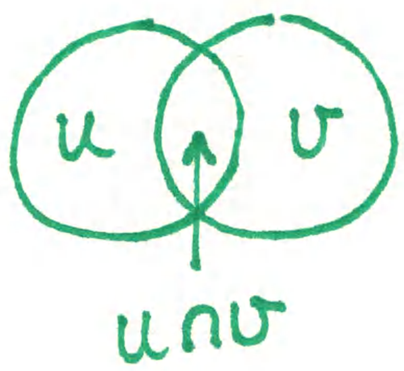
THE UNIVERSAL PROPERTY OF PUSHOUTS

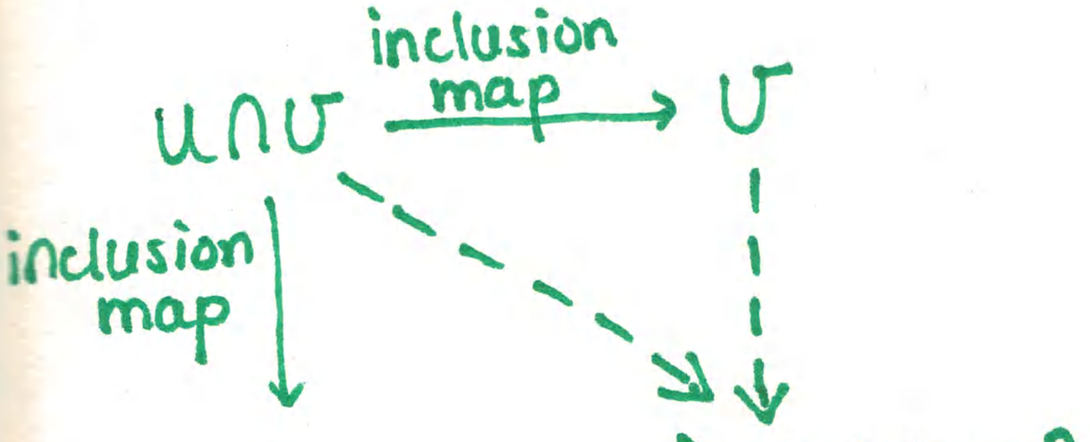


Morphisms $B \sqcup_A C \longrightarrow Q$ are in natural bijection with pairs of morphisms $b: B \longrightarrow Q$, $c: C \longrightarrow Q$ such that the above square commutes.

Example of pushouts

$U, V \in \text{set}$



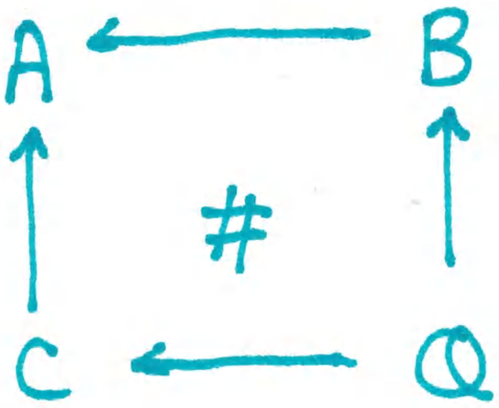


Recall, a pushout is the colimit of a diagram consisting of two morphisms with a common domain.
The pushout is an object and two morphisms that "complete the commutative square."

The "Universal Property of Pushouts" yields the most "general" way to complete the commutative square.

PULLBACKS

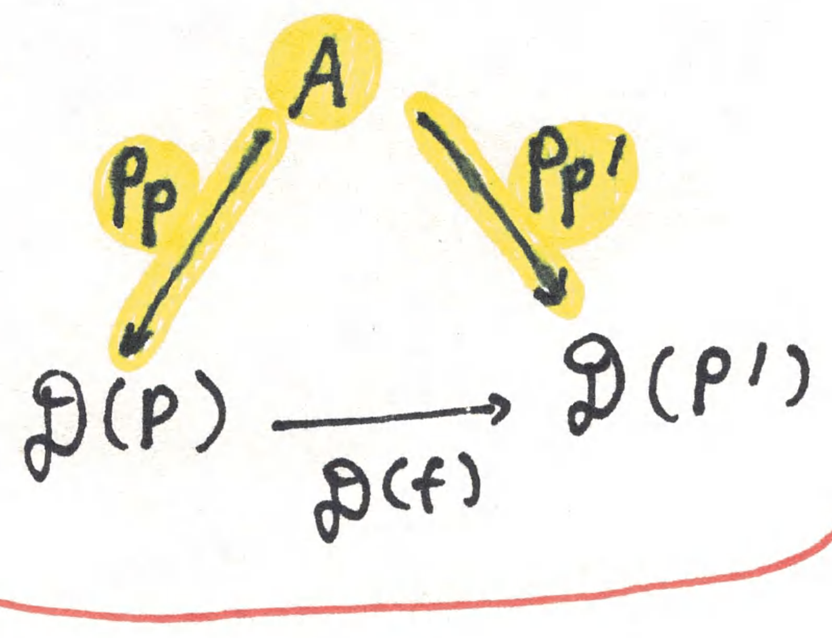
are dual to pushouts



A pullback is the limit of a diagram consisting of two morphisms with a common codomain

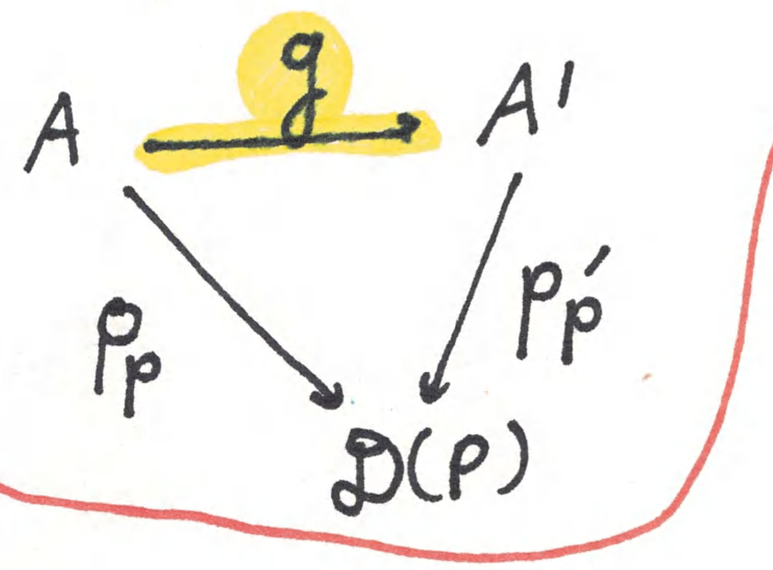
"cone" over a diagram $\mathcal{D}$.

An object $A \in \mathcal{C}$ together with a family of projection morphisms $p_p : A \longrightarrow \mathcal{D}(P)$ for any $P \in \mathcal{C}$ such that for any arbitrary morphism $f : P \longrightarrow P'$ in I:



"morphism of cones" over a diagram $\mathcal{D}$

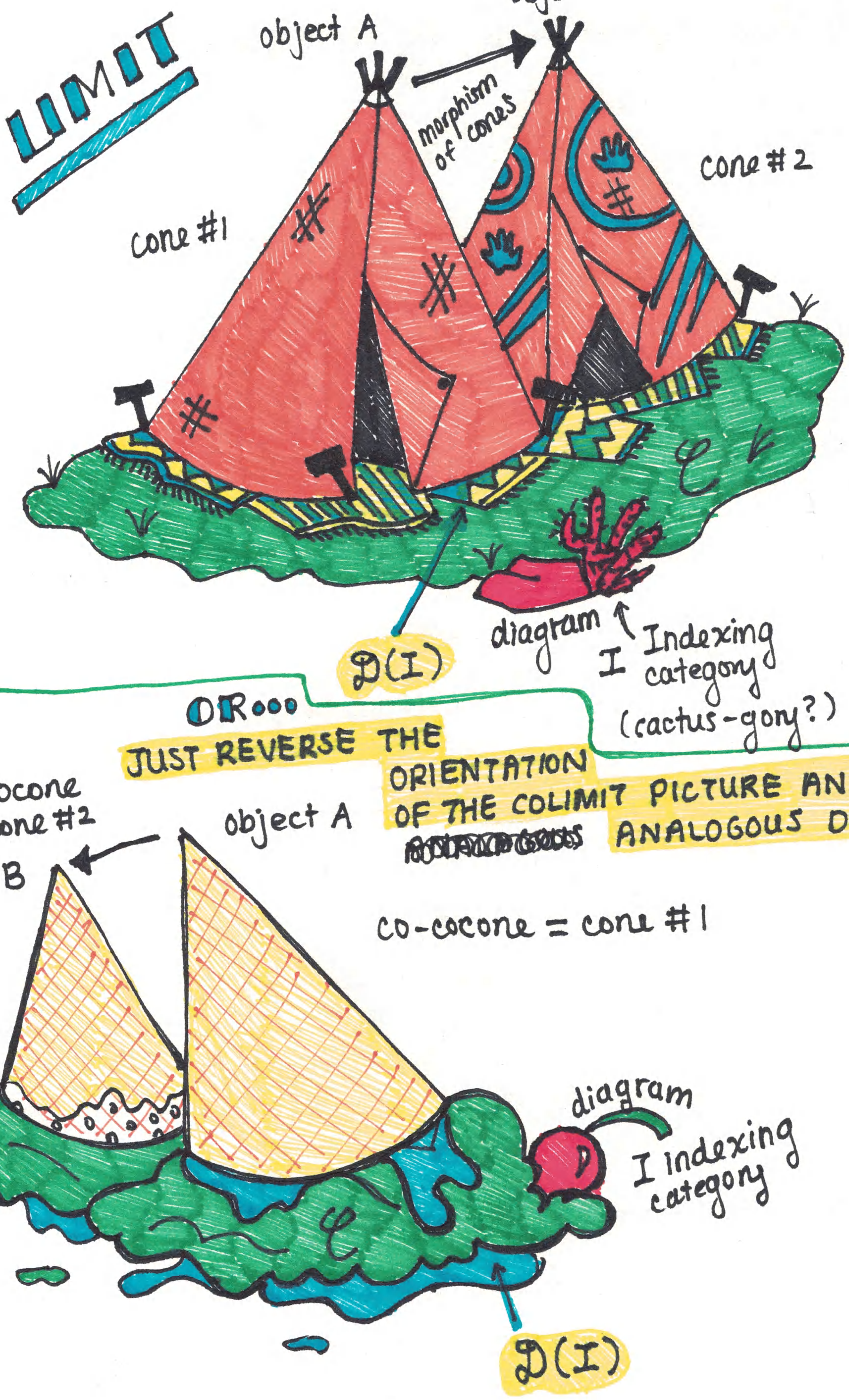
A morphism $g : A \longrightarrow A'$ such that the following triangle commutes for an arbitrary object $P \in I$:



The "category of cones" over a diagram $\mathcal{D}$

objects: cones over $\mathcal{D}$

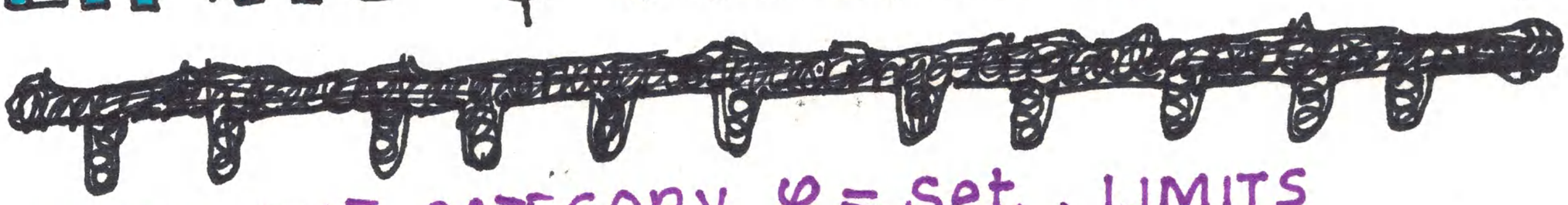
morphisms: morphisms of cones over $\mathcal{D}$



OR...

JUST REVERSE THE ORIENTATION OF THE COLIMIT PICTURE AND TAKE ANALOGOUS DEFINITIONS

LIMITS & COLIMITS



WHEN THE CATEGORY $\mathcal{C} = \mathbf{Set}$, LIMITS AND COLIMITS ARE GIVEN BY THE FOLLOWING:

$$\rightarrow \operatorname{colim} \mathcal{D} = \left\{ (P, x) \;\middle|\; P \in I,\ x \in \mathcal{D}(P) \right\}$$

* modulo *

The equivalence relation generated by

$(P, x) \sim (P', \mathcal{D}(f)(x))$

where $f: P \to P'$

*NOTE:

$$\bigsqcup_{P \in I} \mathcal{D}(P) = \left\{ *(P, x) \;\middle|\; \begin{matrix} P \in I \\ x \in \mathcal{D}(P) \end{matrix} \right\}$$

and the morphisms $f: P \to P'$ in the equivalence relation tell you what elements to identify and how to identify them

---

$$\rightarrow \lim \mathcal{D} = \left\{ (x_P)_{P \in I} \;\middle|\; x_P \in \mathcal{D}(P) \text{ and } \mathcal{D}(f)(x_P) = x_{P'} \ \forall f: P \to P' \right\}$$

THESE RESTRICTED DEFINITIONS OF COLIMIT AND LIMIT IN THE CATEGORY OF SETS CAN BE USED TO SOLVE EXERCISES 26.25 AND 26.34 IN THE NOTES, WHERE THE INDEXING CATEGORY $I = BG$.

RECALL, BG AS A CATEGORY IS DEFINED

OBJECTS: $\mathrm{Ob}(BG) = \{*\}$

MORPHISMS: $\mathrm{Mor}(BG) = \mathcal{U}(G)$

PROPOSITION Suppose $\mathcal{C}$ is a cocomplete category and I is an indexing category.

Then there is a "colimit functor"

$$\text{colim}: \mathcal{C}^I \longrightarrow \mathcal{C}$$

whose value on objects of $\mathcal{C}^I$ is given by the colimit.*

$\mathcal{C}^I = \text{Fun}(I, \mathcal{C})$ functions from index category to category

I-indexed diagrams

DEF 14.10 IN DR. PAVLOU'S NOTES

*To the object of the colimit, not to the morphism!

Recall, "cocomplete" category ⇒ category admits colimits of all small diagrams

We need to consider cocomplete categories so we can guarentee that colimits exist, so that we can construct this colimit functor

NOTE: If $\mathcal{C}$ admits coequalizers and small coproducts, then it is cocomplete.

We define colimits for morphisms of diagrams (natural transformations of diagrams)

Recall, "natural transformations"

If $\mathcal{C}$ and $\mathcal{D}$ are categories, $F, G: \mathcal{C} \to \mathcal{D}$ are functors, then a "natural transformation" $t: F \to G$ is a family $\{t_x\}_{x \in \mathcal{C}}$ of morphisms in $\mathcal{C}$ such that the "naturality property" is satisfied for all morphisms $f: X \to Y$ in $\mathcal{C}$.

"naturality property":

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle t_x}\downarrow & \# & \downarrow{\scriptstyle t_y} \\ G(X) & \xrightarrow{G(f)} & G(Y) \end{array}$$

(the square commutes)

Continued... (define colimit functor...)

We need to define this colim fuctor on morphisms of diagrams ("natural transformations")

diagram #1 $\mathcal{D} : I \longrightarrow \mathcal{C}$

diagram #2 $\mathcal{D}' : I \longrightarrow \mathcal{C}$

(Developing idea of how we need to construct the colimit functor)

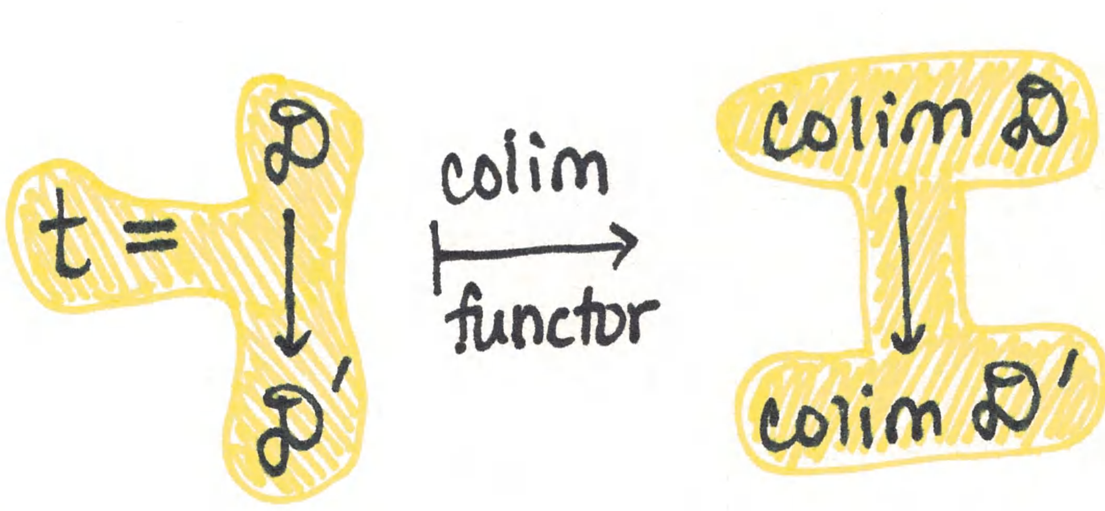
$t : \mathcal{D} \longrightarrow \mathcal{D}'$

natural transformation

The functor colim: $\mathcal{C}^I \longrightarrow \mathcal{C}$ will send t to a map of colimits:

colim(t) is a map $\operatorname{colim} \mathcal{D} \longrightarrow \operatorname{colim} \mathcal{D}'$

(colimit of $\mathcal{D}$) (colimit of $\mathcal{D}'$)



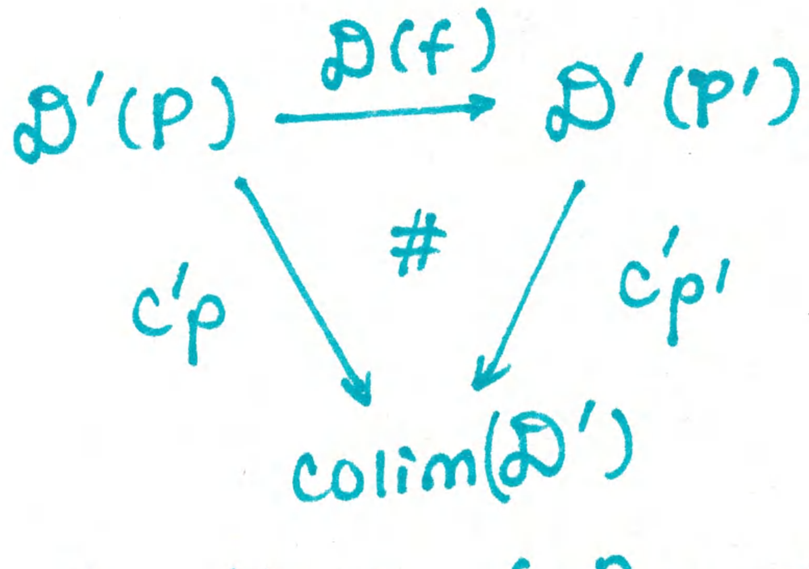
and this mapping must satisfy the naturality property of natural transformations:

$$\begin{array}{ccc} \mathcal{D}(P) & \xrightarrow{\mathcal{D}(f)} & \mathcal{D}(P') \\ {\scriptstyle tp}\downarrow & \# & \downarrow{\scriptstyle tp'} \\ \mathcal{D}'(P) & \xrightarrow[\mathcal{D}'(f)]{} & \mathcal{D}'(P') \end{array}$$

for all functions $f : P \longrightarrow P'$.

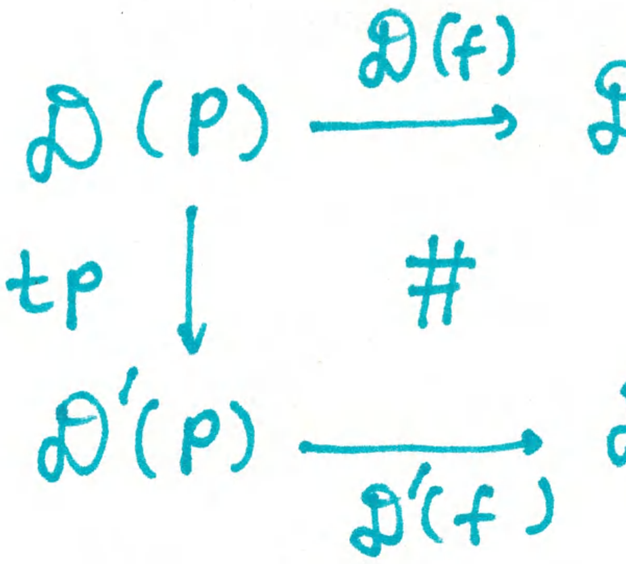
(Proof following ... continued on next page ...) ⟶

By definition of the colimit of $\mathcal{D}'$,
we have the cocone:

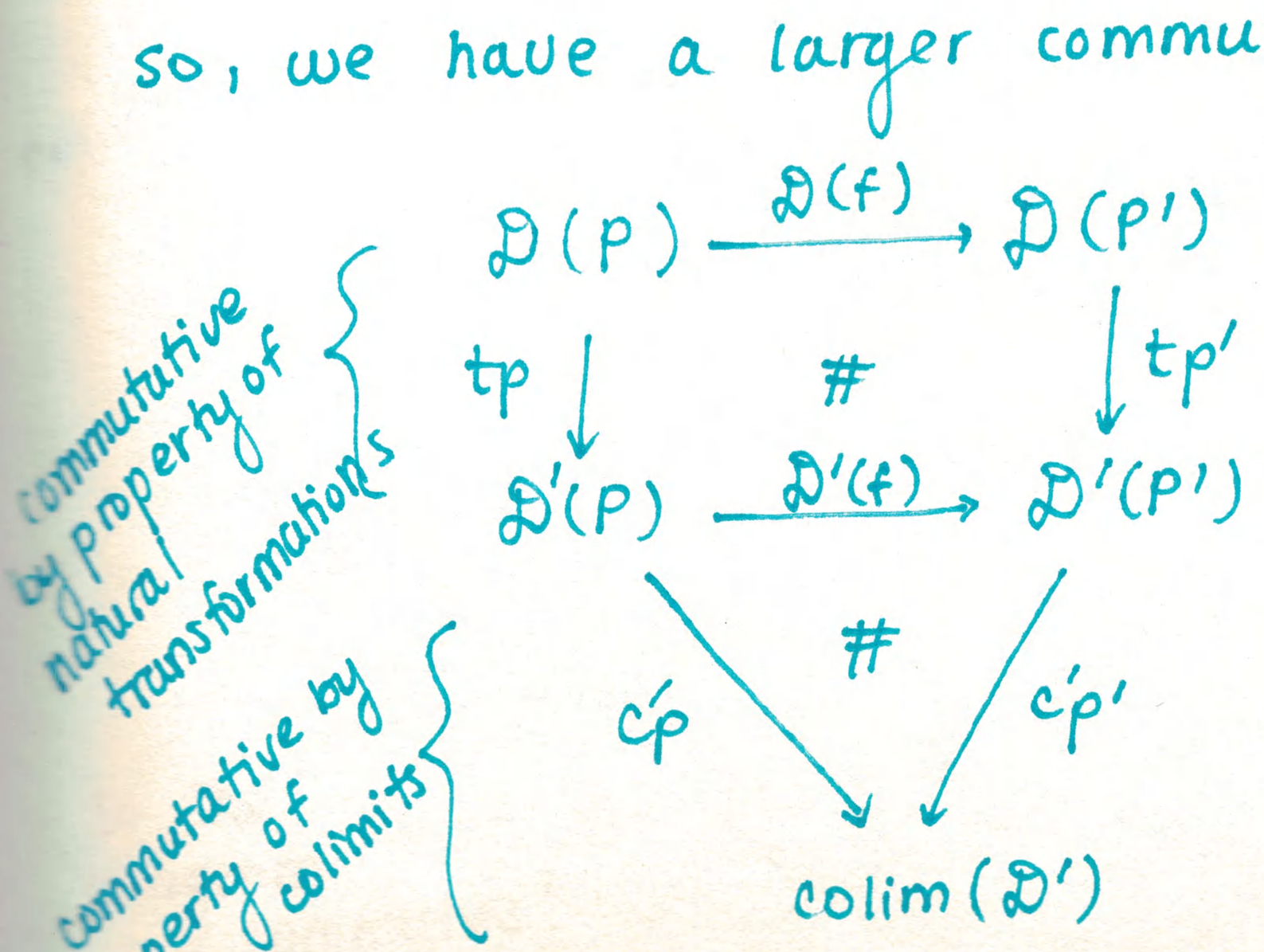


for all functions $f: P \longrightarrow P'$.

and by the naturality property of natural transformations we have the commutative diagram



so, we have a larger commutative diagram:



Now, it remains to show that this larger diagram commutes

natural transformation #1: $t' : \mathcal{D} \longrightarrow \mathcal{D}'$

natural transformation #2: $t'' : \mathcal{D}' \longrightarrow \mathcal{D}''$

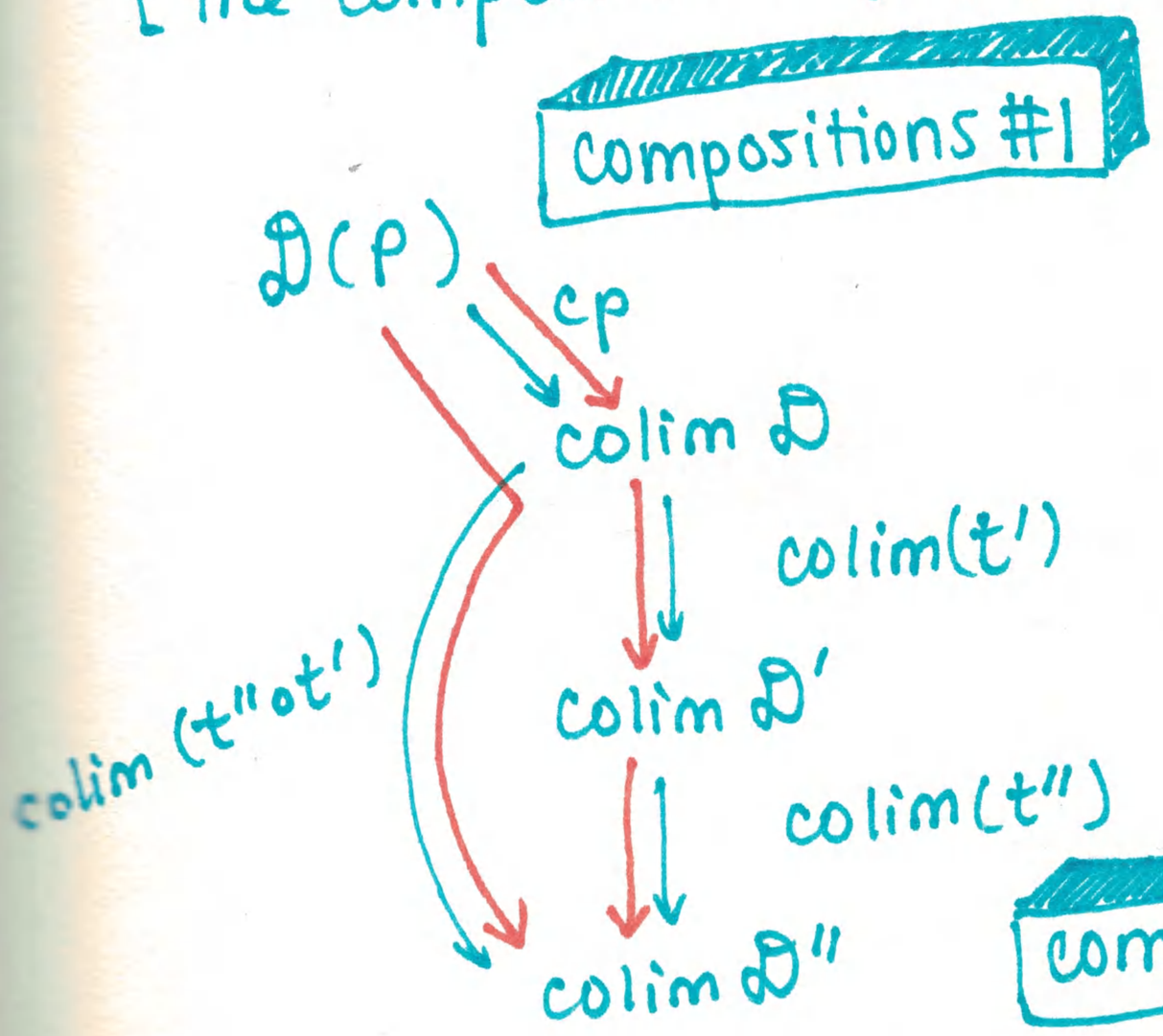
Then we have $\mathcal{D} \xrightarrow{t'} \mathcal{D}' \xrightarrow{t''} \mathcal{D}''$, with $t'' \circ t' : \mathcal{D} \longrightarrow \mathcal{D}''$

We will define $\operatorname{colim}(t) : \operatorname{colim} \mathcal{D} \longrightarrow \operatorname{colim} \mathcal{D}'$ by the universal property of colimits as the collection of morphisms of the form:

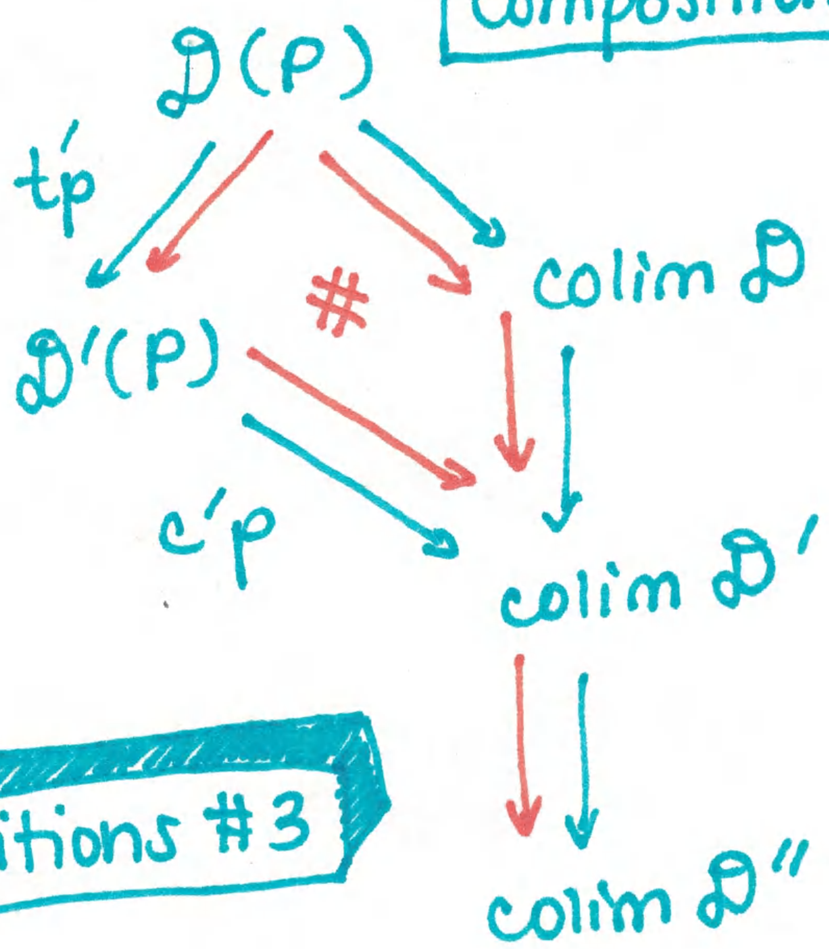
$$\mathcal{D}(P) \longrightarrow \operatorname{colim}(\mathcal{D}')$$

[the composition: $\mathcal{D}(P) \xrightarrow{t_P} \mathcal{D}'(P) \xrightarrow{c'_P} \operatorname{colim}(\mathcal{D}')$]

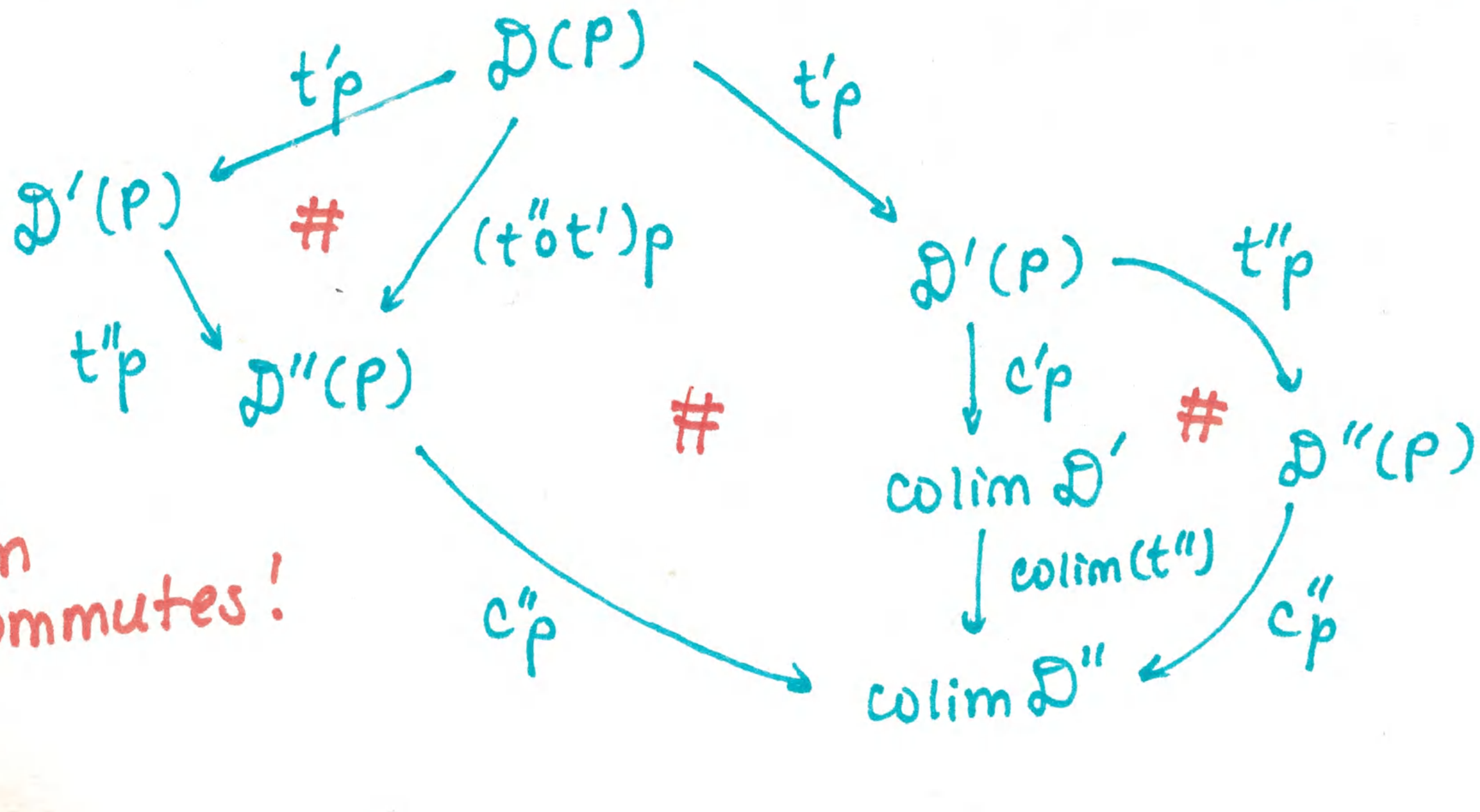
Compositions #1



Compositions #2



Compositions #3



So the diagram commutes!

§25

BROUWER FIXED POINT Theorem

contractible metric space (or topological space) X

$\exists b \in X$ such that

[redacted] there is a continuous map

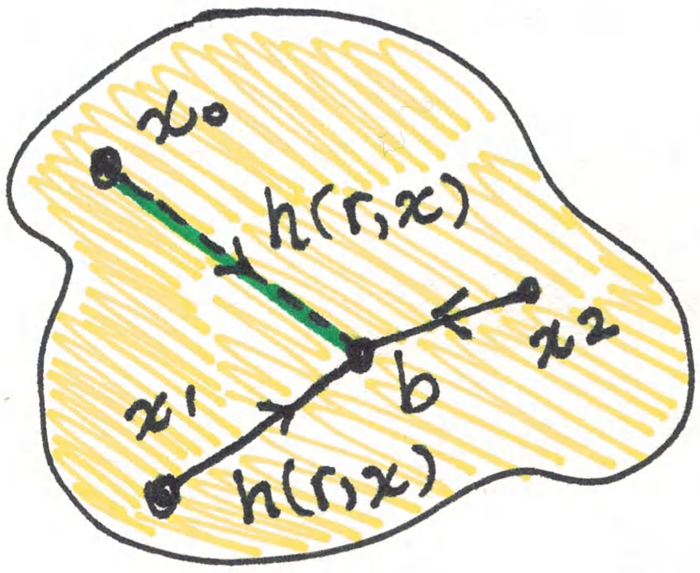
$h : [\mathbb{R} \times X \longrightarrow X$

(or $h : [0,1] \times X \longrightarrow X$)

with

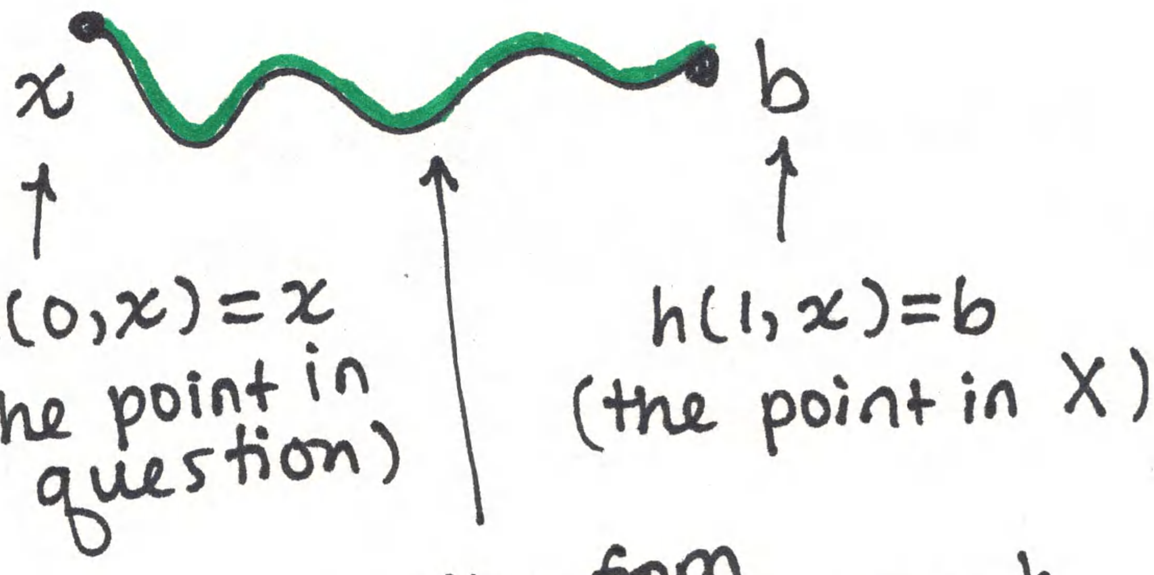
$h(0,x) = x \quad h(1,x) = b$

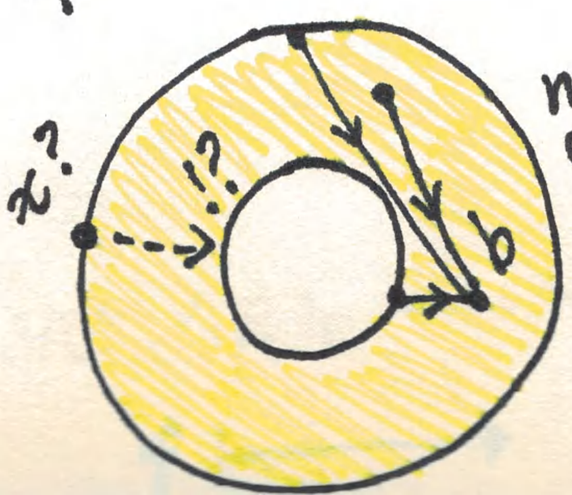
for all points $x \in X$





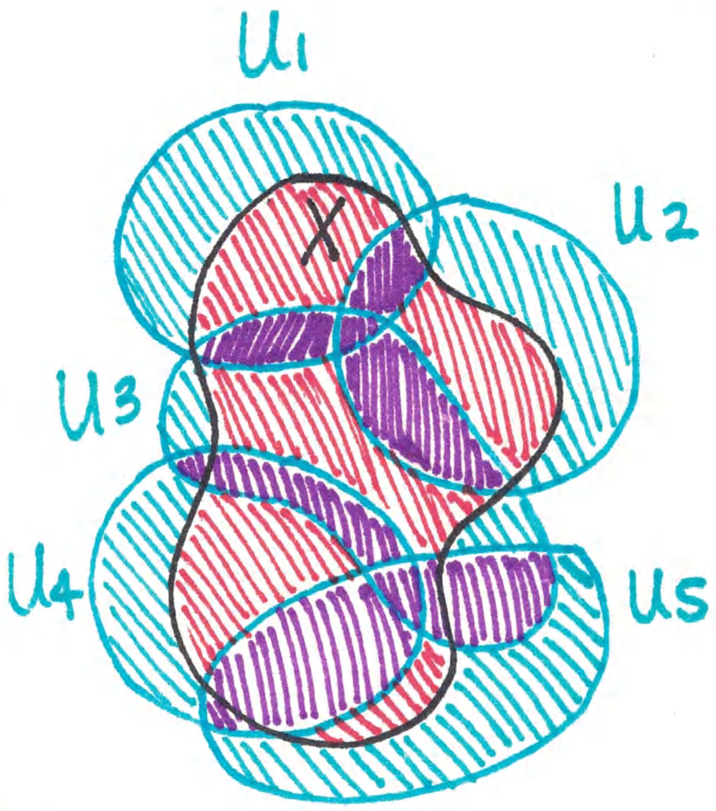
contractible space



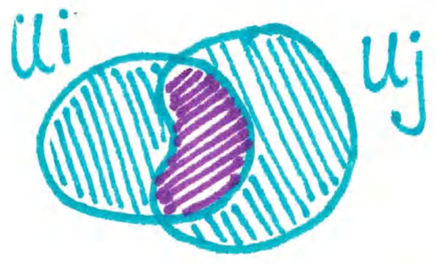


not contractible space

Nerve Theorem



Suppose $\{U_i\}$ is an open cover of a metric or topological space X, such that



every finite intersection

$U_{i_0} \cap U_{i_1} \cap \ldots \cap U_{i_k}$

is either empty or a contractible space

Then, there is a simplicial map from the nerve of the open cover $\mathcal{U}$ to the singular simplicial set of X:

$$N(X, \mathcal{U}) \longrightarrow \mathrm{Sing}(X)$$

that is a simplicial "weak equivalence".

In particular, applying the functors

H (homology) or H* (cohomology)

produces isomorphisms.

(The homology/cohomology of the nerve of the open cover is isomorphic to the homology/cohomology of the space X.)

Recall, (Def §17) "Nerve"
[Paul Alexandroff, 1927]

The nerve of (X, Y, R) is the simplicial set $N(X, Y, R)$ defined by: $[N(X,Y,R)]_m$ is the set of maps

$$\left\{ f : \mathcal{U}([m]) \longrightarrow Y \;\middle|\; \begin{array}{l} \exists x \in X \\ (x,y) \in R \\ \forall y \in \mathrm{Im} f \end{array} \right\}$$

w/ structure map

$$N(X,Y,R)f : N(X,Y,R)_n \longrightarrow N(X,Y,R)_m$$

$$f \longmapsto f \circ \mathcal{U}(f)$$

EXAMPLE 25.3

Consider the sphere $S^n = \{ x \in \mathbb{R}^{n+1} \mid \|x\| = 1 \}$

Ex S^1: 

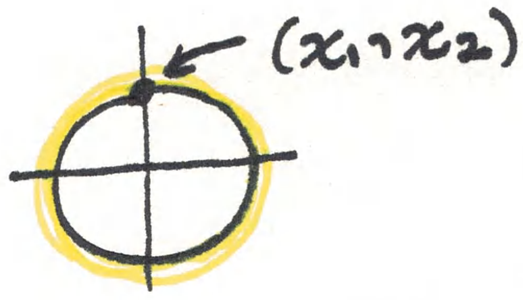
$x \in \mathbb{R}^2$
$\|x\| = 1$

Ex S^2: 

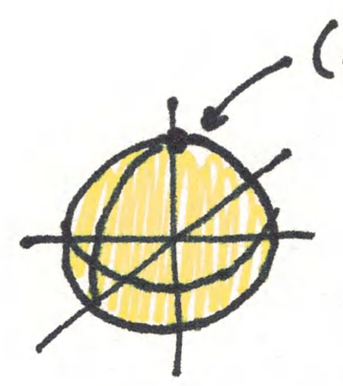
$x \in \mathbb{R}^3$
$\|x\| = 1$

We define an open cover of S^n: $\{ U_i^+, U_i^- \}$

$$U_i^+ = \{ x \in S^n \mid x_i > 0 \}$$
$$U_i^- = \{ x \in S^n \mid x_i < 0 \}$$

Ex S^1: 

$$U_1^+ = \{ x \in S^1 \mid x_1 > 0 \}$$
$$U_1^- = \{ x \in S^1 \mid x_1 < 0 \}$$
$$U_2^+ = \{ x \in S^1 \mid x_2 > 0 \}$$
$$U_2^- = \{ x \in S^1 \mid x_2 \leq 0 \}$$

Ex S^2: 

$$U_1^+ = \{ x \in S^2 \mid x_1 > 0 \}$$
$$U_1^- = \{ x \in S^2 \mid x_1 < 0 \}$$
$$U_2^+ = \{ x \in S^2 \mid x^2 > 0 \}$$
$$U_2^- = \{ x \in S^2 \mid x_2 < 0 \}$$
$$U_3^+ = \{ x \in S^2 \mid x_3 > 0 \}$$
$$U_3^- = \{ x \in S^2 \mid x_3 < 0 \}$$

* NOTE: since $U_i^+ \cap U_i^- = \emptyset$, we never have to consider intersections of $+, -$ for the same value of i.

Consider the nerve $N(S^n, \mathcal{U})$

Nondegenerate $[k]$-simplex of $N(S^n, \mathcal{U})$:

an injective map of simplices $[k] \longrightarrow [n]$
and a map of sets $[k] \longrightarrow \{+, -\}$

Nondegenerate $[n]$-simplex of $N(S^n, \mathcal{U})$

an injective map of simplices $[n] \longrightarrow [n]$
(hence, an isomorphism)

and a map of sets $[k] \longrightarrow \{+, -\}$

→ There are 2^{n+1} maps $[n] \longrightarrow \{+, -\}$

Likewise...

→ There are $(n+1)2^n$ maps $[n-1] \longrightarrow [n]$ $[n-1] \longrightarrow \{+, -\}$

Nondegenerate $[n]$-simplices

+ + + ⋯ + + +
+ + + ⋯ + + −
+ + + ⋯ + − +
+ + + ⋯ + − −
+ + + ⋯ − + +
+ + + ⋯ − + −
+ + + ⋯ − − +
+ + + ⋯ − − −
⋮
etc...

Nondegenerate $[n-1]$-simplices

+ + + ⋯ * + +
+ + + ⋯ * + −
+ + + ⋯ * − +
+ + + ⋯ * − −
+ + + ⋯ + * +
+ + + ⋯ + * −
+ + + ⋯ − * +
+ + + ⋯ − * −
⋮
etc...

where * indicates a missing entry

Simplicial Chain:

$$\cdots \leftarrow A^{(n+1)2^n} \xleftarrow{\partial} A^{2^{n+1}} \leftarrow \cdots$$

$A^{(n+1)2^n}$: (deg n-1)

$A^{2^{n+1}}$: (deg n)

describe the differential (boundary map) of the chain ↑

DOMAIN:

direct sum of copies of A indexed by a sequence of n+1 signs

The ith face map replaces the ith sign by *

First face map ⟶ replaces first sign w/ *

Second face map ⟶ replaces second sign w/ *

⋮

ith face map $\xrightarrow{\text{etc...}}$ replaces ith sign w/ *

For instance... take

$(++-+\cdots+-+) \in A^{2^{n+1}}$

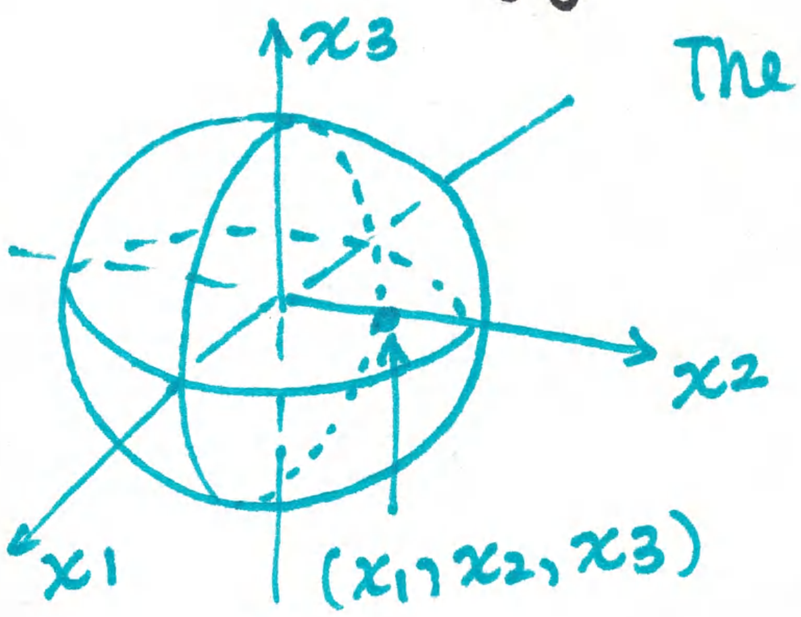
$$\partial(++-+\cdots+-+) = (*+-+\cdots+-+)$$
$$-(+*-+\cdots+-+)$$
$$+(++*+\cdots+-+)$$
$$-(++-*\cdots+-+)$$
$$\vdots$$
$$\pm(++-+\cdots*-+)$$
$$\mp(++-+\cdots+*+)$$
$$\pm(++-+\cdots+-*)$$

alternating signs, removing ith sign

Exercise 25.10

Consider the sphere $S^2 = \{x \in \mathbb{R}^3 \mid \|x\| = 1\}$
We identify x with $-x$ for any $x \in S^{n=2}$.

(a) Use the nerve theorem to compute the singular homology of the resulting space $\mathcal{Q}$.



When we identify x with $-x$ we are identifying (x_1, x_2, x_3) with $(-x_1, -x_2, -x_3)$

The previous open cover of S^2 was:

$$U_1^+ = \{x \in \mathbb{R}^3 \mid x_1 > 0\}$$
$$U_1^- = \{x \in \mathbb{R}^3 \mid x_1 < 0\}$$
$$U_2^+ = \{x \in \mathbb{R}^3 \mid x_2 > 0\}$$
$$U_2^- = \{x \in \mathbb{R}^3 \mid x_2 < 0\}$$
$$U_3^+ = \{x \in \mathbb{R}^3 \mid x_3 > 0\}$$
$$U_3^- = \{x \in \mathbb{R}^3 \mid x_3 < 0\}$$

So, we can reduce our open cover:

PREVIOUSLY

+ + +
+ + −
+ − +
+ − −
− + +
− + −
− − +
− − −

4 nondegenerate [2]-simplices

+ + +	+ + −	+ − +	+ − −
‖	‖	‖	‖
− − −	− − +	− + −	− + +

and we have a chain:

$$\cdots \longleftarrow A^6 \longleftarrow A^4 \longleftarrow 0$$

A^6 (deg n−1 = 1) A^4 (deg n = 2)

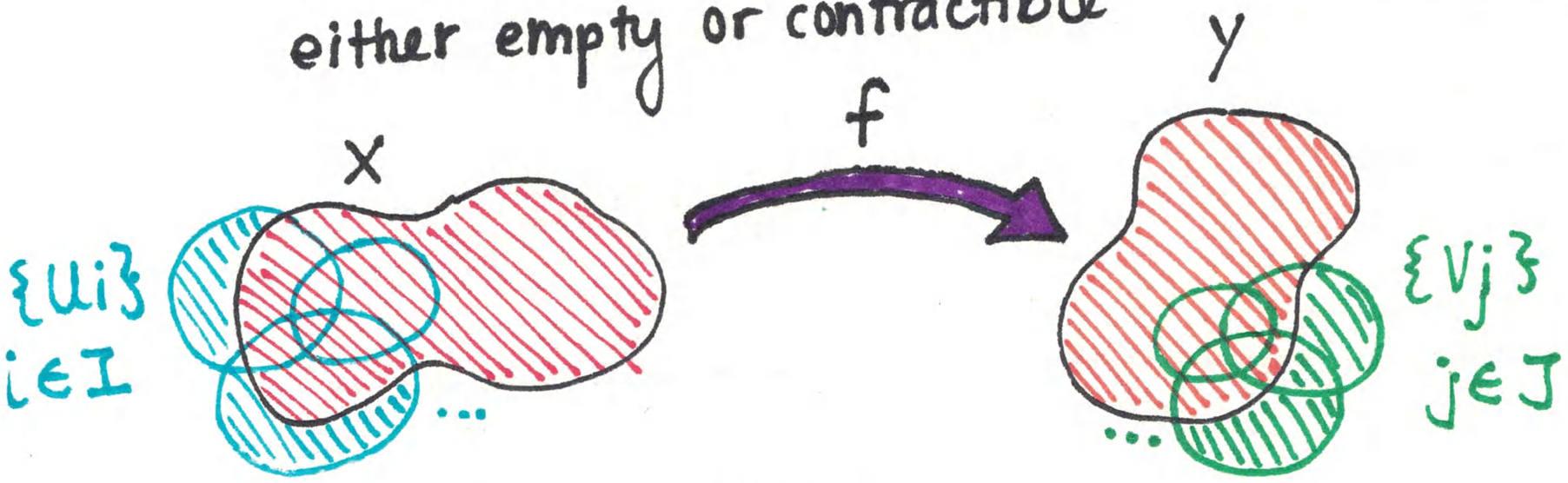
(continued...)

Relative Nerve Theorem



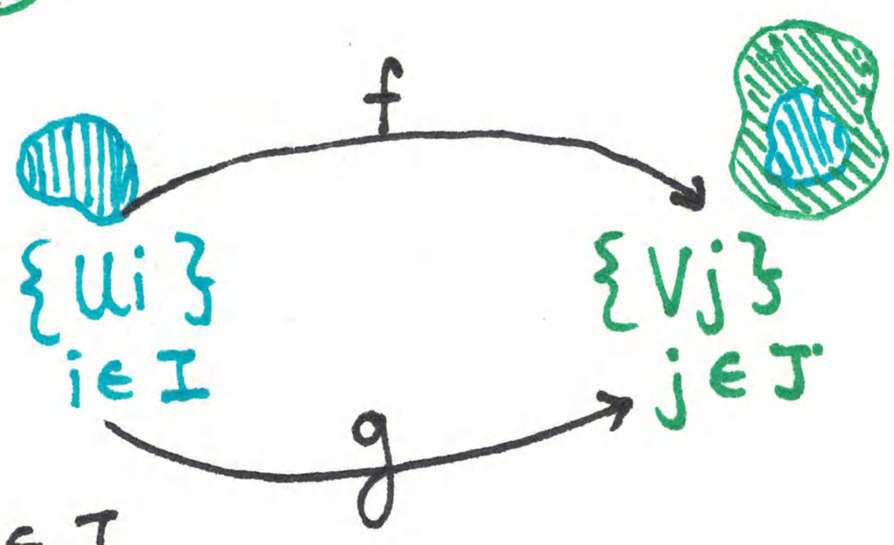
Suppose $(X, \mathcal{U})$ is a space with an open cover as in the previous "nerve theorem", and $(Y, \mathcal{V})$ is another pair with the same property.

every finite intersection $U_{i_0} \cap U_{i_1} \cap \dots \cap U_{i_k}$ is either empty or contractible



suppose $f: X \longrightarrow Y$ is a continuous map

and $g: I \longrightarrow J$ is a map of indexing sets such that $f(U_i) \subseteq V_{g(i)}$, $\forall i \in I$



Then g induces a simplicial map

$$N(f,g): N(X,\mathcal{U}) \longrightarrow N(Y,\mathcal{V})$$

and the simplicial maps

$$N(X,\mathcal{U}) \longrightarrow sing(X)$$
$$N(Y,\mathcal{V}) \longrightarrow sing(Y)$$

can be chosen in such a way as to make the diagram commute:

$$\begin{array}{ccc} N(X,\mathcal{U}) & \xrightarrow{N(f,g)} & N(Y,\mathcal{V}) \\ \# & & \# \\ \downarrow & & \downarrow \\ sing(X) & \xrightarrow{sing(f)} & sing(Y) \end{array}$$

Using the Relative Nerve Theorem

space #1 $(X, \mathcal{U})$

$X = S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$

$\mathcal{U} = \{U_i^{\pm}\}$

open cover #1

$U_i^+ = \{x \in S^n \mid x_i > 0\}$

$U_i^- = \{x \in S^n \mid x_i < 0\}$

space #2 $(Y, \mathcal{V})$

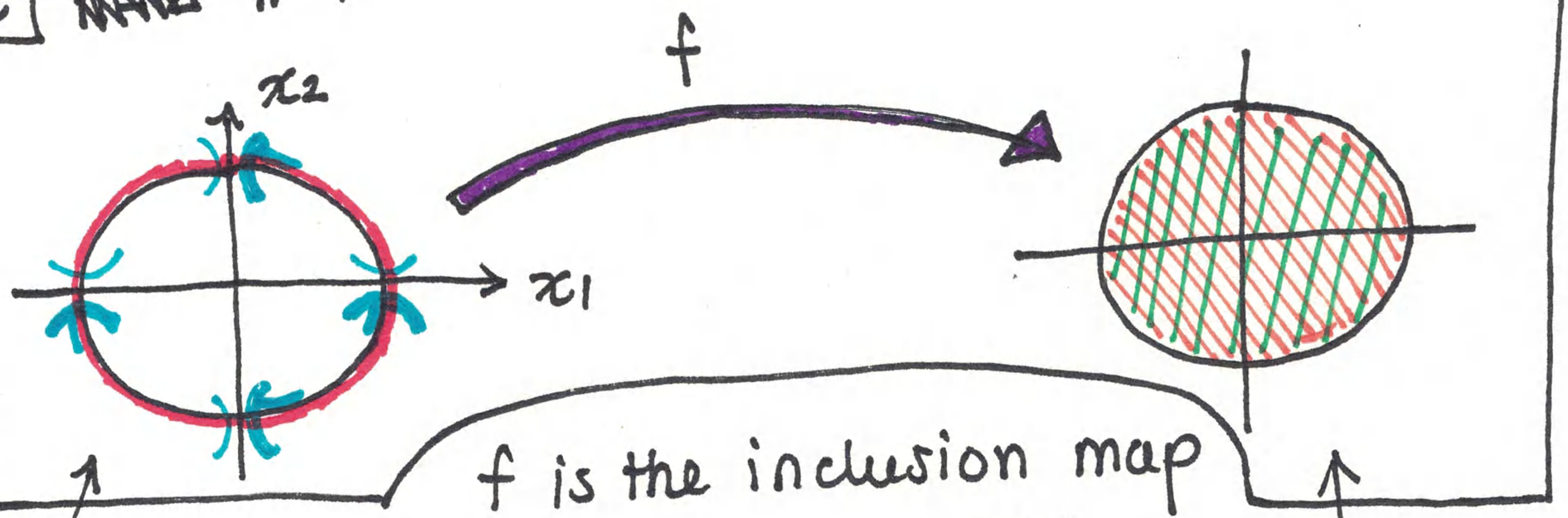
$Y = D^{n+1} = \{x \in \mathbb{R}^{n+1} \mid \|x\| \leq 1\}$

$\mathcal{V} = D^{n+1}$

open cover #2

(the disk is a covering of itself, and hence is contractible)

Ex $n = 1$



f is the inclusion map

$i: S^n \longrightarrow D^{n+1}$

a continuous map

The nerve of $(S^n, \{U_i^{\pm}\})$ is the set of nondegenerate $[n]$-simplices iindexed by $[n] \times \{+,-\}$

$N(S^n, \{U_i^{\pm}\}) \cong [n] \times \{+,-\}$

The nerve of $(D^{n+1}, \{D^{n+1}\})$ is a single vertex

$N(D^{n+1}, \{D^{n+1}\}) \cong \Delta^0$

so we have a map of indexing sets:

$g: [n] \times \{+,-\} \longrightarrow \{*\}$

More on this topic to follow in later discussion! Now... moving on!!

PART II

§28 THE FUNDAMENTAL GROUPOID

28.1 : Definitions of Groupoids and Introductory concepts

28.2 : Classification of Groupoids

28.3 : Construction of the Functor $\Delta \to$ Grpd

A **GROUPOID** is a category in which all morphisms are isomorphisms

Grpd is the category of groupoids, which is a subcategory of the category of categories!

* NOTE: We can construct a groupoid from a category in several ways, 1 one, by forcing all morphisms to be invertible in the most obvious way

category

$x \xrightarrow{f} y$

f^{-1} does NOT necessarily exist

Groupoid

$x \underset{f^{-1}}{\overset{f}{\rightleftarrows}} y$

we define the inverse as part of the groupoid structure

* Is this defined in the notes

Ex #1 The discrete category (on a set of objects) is a groupoid

$S \in$ set

All morphisms are invertible because there are no nonidentity morphisms.

$Ob(\mathcal{C}) = U(S)$ $\quad \{ s_1 \quad s_2 \quad s_3 \}$

$Mor(\mathcal{C}) = id$

↑ no morphisms other than identity

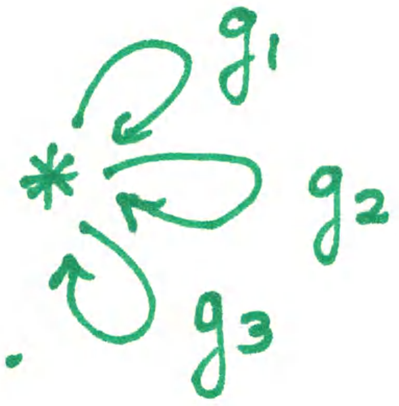
Ex The delooping category BG of a monoid G is a groupoid iff G is a group.

Recall, the delooping category:

Given G, we form category BG,

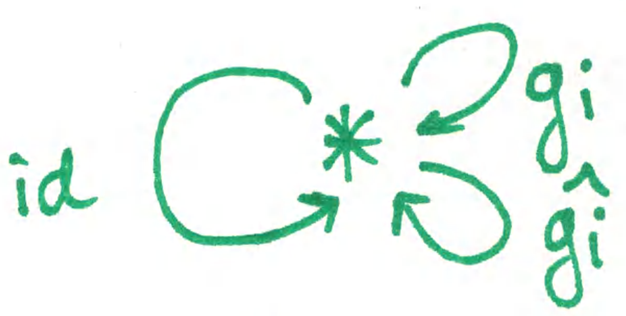
where $ob(BG) = \{*\}$

and $Mor(BG) = U(G)$



The composition of functions in BG is the binary operation on G (G is a monoid, so it has an associative binary operation)

These morphisms g_i are invertible if $\exists \hat{g}_i \in G$ such that $\hat{g}_i \circ g_i = id$, that is



Then $\hat{g}_i = g_i^{-1}$

That is, each element of monoid G has an inverse

$\Rightarrow$ G is a group!

GENERATORS & RELATIONS OF GROUPOIDS

a system of generators and relations for a groupoid is:

A set of objects $\mathcal{O}$

... and ...

A set of generating isomorphisms $\{ f_i : x_i \longrightarrow y_i \}$ $x_i, y_i \in \mathcal{O}$

... and ...

A set of relations $f_{ik}^{\pm 1} \circ f_{ik-1}^{\pm 1} \circ \cdots \circ f_{i0}^{\pm 1} = f_{il}^{\pm 1} \circ f_{il-1}^{\pm 1} \circ \cdots \circ f_{i0}^{\pm 1}$

This relation is specified

A groupoid generated by such a set (of generators and relations) is specified by a universal property!

THE UNIVERSAL PROPERTY OF GROUPOIDS

- Groupoid #1 (G)

  objects of G: $\mathcal{O}$

  generating isomorphisms of G: $\{f_i : x_i \longrightarrow y_i\}$

  relations: $f_{ik}^{\pm 1} \circ f_{ik-1}^{\pm 1} \circ \cdots \circ f_{i0}^{\pm 1} = f_{i'k'}^{\pm 1} \circ f_{i'k'-1}^{\pm 1} \circ \cdots \circ f_{i'0}^{\pm 1}$

- Groupoid #2 (H)

  objects of H: $\mathcal{E}$

  generating isomorphisms $\{g_j : r_j \longrightarrow s_j\}$

  relations: $g_{jk}^{\pm 1} \circ g_{jk-1}^{\pm 1} \circ \cdots \circ g_{j0}^{\pm 1} = g_{j'k'}^{\pm 1} \circ g_{j'k'-1}^{\pm 1} \circ \cdots \circ g_{j'0}^{\pm 1}$

- Morphism of Groupoids:

$$\{G \longrightarrow H\} \cong \begin{cases} \mathcal{O} \xrightarrow{h} \mathcal{E} & \begin{cases} x_j, y_j \in G \\ h(x_j), h(y_j) \in H \end{cases} \\ g_j : h(x_j) \longrightarrow h(y_j) \text{ in } H \\ g_j \text{ satisfy the relation in } H \end{cases}$$

Universal Property of Groupoids

GROUPOID GENERATED by a system of GENERATORS AND RELATIONS

Recall,
generators / relations

set of objects & → The given set is the set of objects in the groupoid

generating isomorphisms {fi} → The morphisms form equivalence classes of chains of composable generators

relations (fik ∘ ... ∘ fi0 = fil ∘ ... ∘ fi0)

The compositions can have length 0 or more

The equivalence relation allows for zero or more substitutions from the given set of relations.

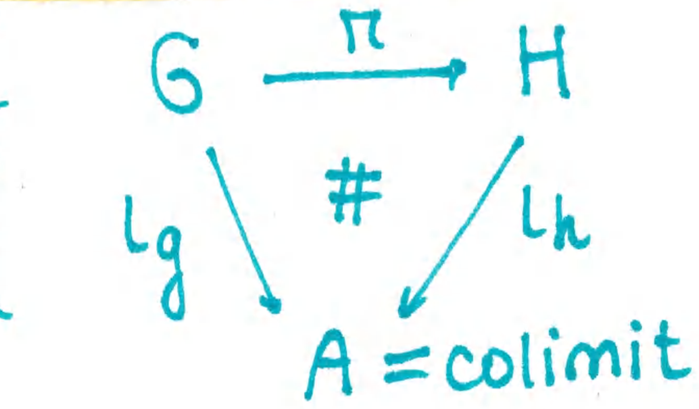
GRPD is COCOMPLETE

To say Grpd is cocomplete is to say that colimits in Grpd always exist.

recall, ▶ Grpd (a category)

- Ob(Grpd) = groupoids
- Mor(Grpd) = morphisms of groupoids

looks like this for example



PROOF! We will construct the colimit by specifying its generators and relations.

Let $\mathcal{D}: I \longrightarrow$ Grpd be a diagram (where I is a small indexing category)

i

$\mathcal{D}(i) \in$ Grpd

so $\mathcal{D}(i)$ is a groupoid (hence, a category)

$Ob(\mathcal{D}(i))$ [the objects of the groupoid $\mathcal{D}(i)$]
is a set. Set is cocomplete. so, the colimit of $Ob(\mathcal{D}(i))$ exists, $\forall i$. ⟶

$i \in I$

$\mathcal{D}(i) \in Grpd$

$I \xrightarrow{\mathcal{D}} Grpd$

$Ob(\mathcal{D}(i)) \in set$

set is cocomplete

$colim(Ob(\mathcal{D}(i)))$ exists

STEP #1

Let $A = colim(Ob(\mathcal{D}(i)))$ specifying the object in the colimit

injection morphisms $Ob(\mathcal{D}(i)) \xrightarrow{L_i} colim(Ob(\mathcal{D}(i)))$ exist

that is, $\{ Ob(\mathcal{D}(i)) \xrightarrow{L_i} A \}$

STEP #2

we want to show that we can specify injection morphisms (for the colimit of Grpd) from the injection morphisms (for the colimit of set)

Generating isomorphisms: $\forall i \in I$

$\forall f: x \longrightarrow y$, $x, y \in \mathcal{D}(i) \in Grpd$ (x and y are objects in groupoid)

STEP #3

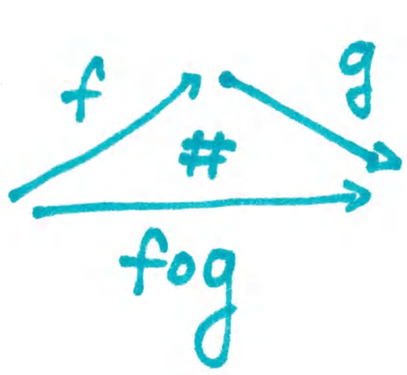
we want:

$$\begin{cases} L_i(f): L_i(x) \longrightarrow L_i(y) \\ L_i: Ob(\mathcal{D}(i)) \longrightarrow A \end{cases}$$

to be the generating isomorphisms $\{L_i\}$

so we specify relations

Relation #1 $\forall i \in I$, $\forall$ commutative diagrams



$L_i(g \circ f) = L_i(g) \circ L_i(f)$

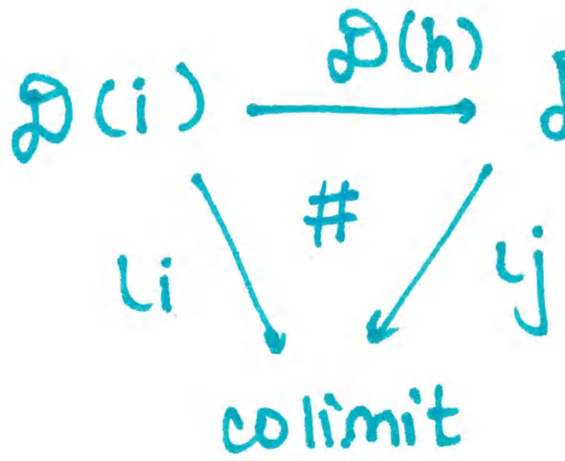
Relation #2

$L_i(id_X) = id_{L_i(x)}$

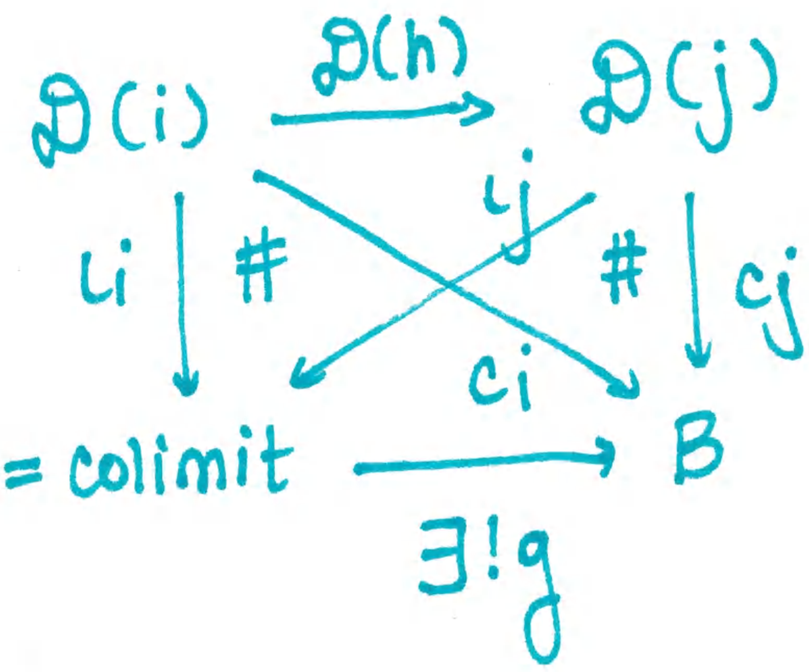
Relations #1 and #2 force maps $\{L_i\}$ to be functors (this is necessary, since they must be morphisms between groupoids)

Relation #3 for $h: i \longrightarrow j$ a morphism in $\mathcal{I}$
and $\forall f: x \longrightarrow y$ morphism in $\mathcal{D}(i)$

$L_i(f) = L_j(\mathcal{D}(h)(f))$ where $\mathcal{D}(h): \mathcal{D}(i) \longrightarrow \mathcal{D}(j)$ is a functor



} and the universal property of colimits is satisfied because we have the universal property of groupoids.



We wish to show $\exists g: A \longrightarrow B$

$C_i = g \circ L_i$

we know $C_i = C_j \circ \mathcal{D}(h)$

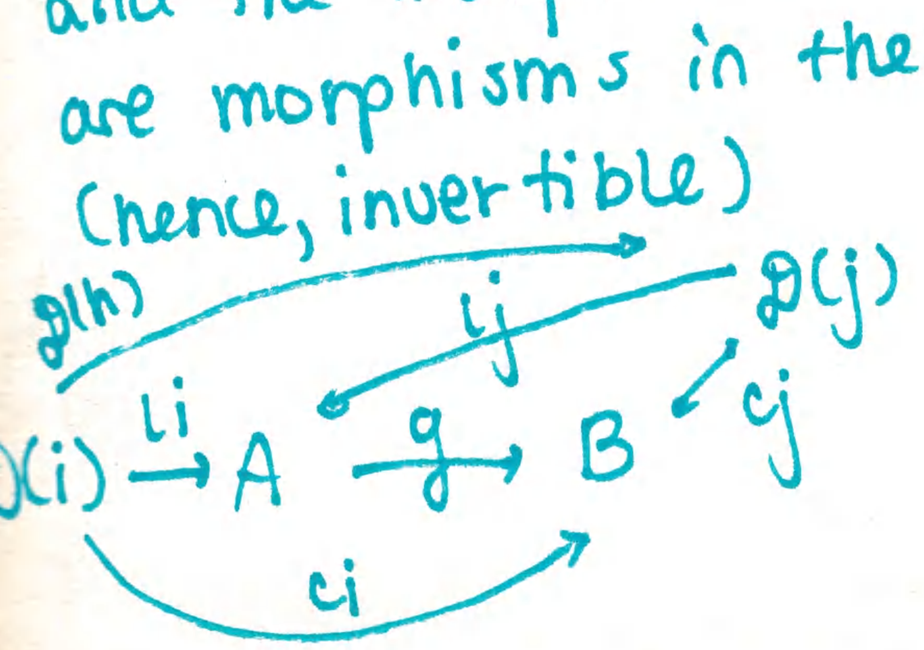
so, it is that we wish to show

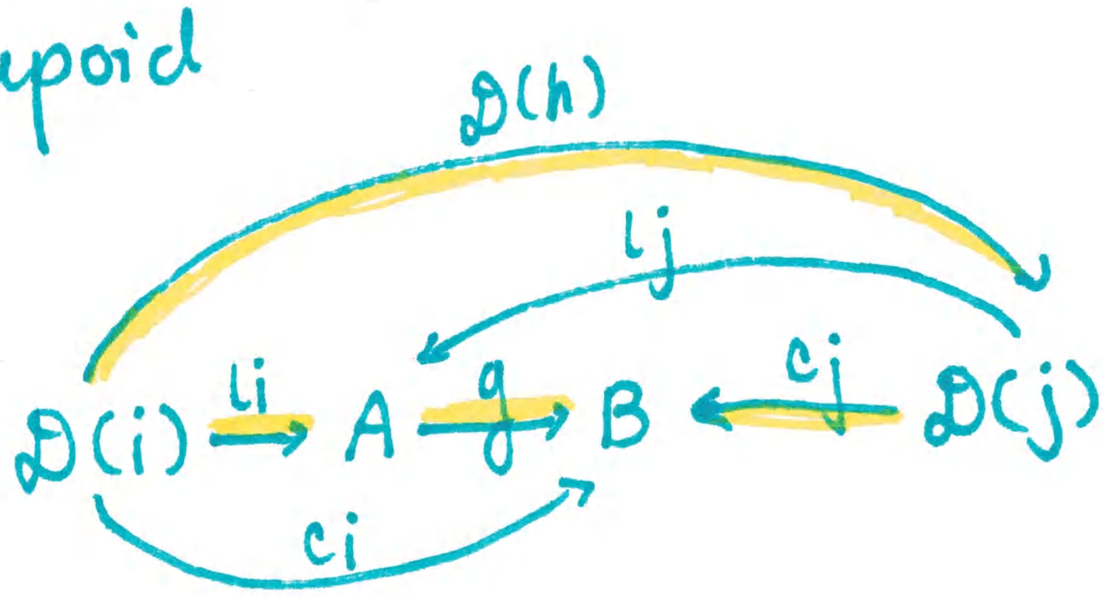
$g \circ L_i = C_j \circ \mathcal{D}(h)$

Here, $\mathcal{D}(i)$, $\mathcal{D}(j)$, A, and B are all objects in the groupoid and the morphisms we see are morphisms in the groupoid (hence, invertible)

and we can do this by the universal property of groupoids.

composition
$\mathcal{D}(h) \circ C_j^{-1} \circ g \circ L_i = id$





} so the universal property of groupoids sets to identity the composition with the same domain and codomain?

* Clarification Required!

Resolution of Category of Groupoids being Cocomplete (continued...)
GRPD

Summary so far: To show that the category of Groupoids is cocomplete, we need to show that colimits in Grpd always exist.

The colimit in Grpd will be an object ($A \in Grpd$) (that is, a groupoid A) together with a family of injection morphisms ($\{L_i\}_{i \in I}$) (that is, a family of morphisms of groupoids)

STEP #1 Specifying the object

For any diagram $\mathcal{D} : I \longrightarrow Grpd$,
$\mathcal{D}(i) \in Grpd$, so $\mathcal{D}(i)$ is itself a groupoid.

The objects of $\mathcal{D}(i)$ is just a set

$$Ob(\mathcal{D}(i)) \in Set$$

and Set is cocomplete (colimits exist)

so the most natural choice for A is

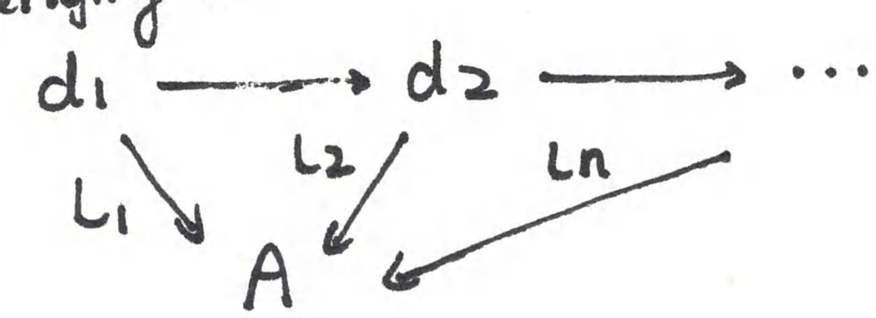
$$A = colim(Ob(\mathcal{D}(i)))$$

STEP #2 Specifying the morphisms

$colim(Ob(\mathcal{D}(i)))$ is the colimit of the set $Ob(\mathcal{D}(i))$

so, there are injection morphisms from the objects of the groupoid $\mathcal{D}(i)$ [the set $\mathcal{D}(i)$] to the colimit, A

$U(\mathcal{D}(i))$ (underlying of)
$Ob(\mathcal{D}(i))$
$=$
$\{d_1, d_2, \dots\}$



$L_i : d_i \longrightarrow A$ is an injection morphism.
This morphism exists, $\forall i$.

We want to show that these maps can double as the injection morphisms in the colimit of Grpd (the colimit that we are constructing) $\longrightarrow$

STEP #3 The colimit A is to be a groupoid, so we will specify it by specifying its generators and relations:

We must specify:

- $\mathcal{O}$: A set of objects.
- $\{f_i : x_i \longrightarrow y_i\}$: A set of generating morphisms.
- relations : A set of necessary relations between generating morphisms.

~~$\mathcal{O} = \text{colim}(\text{Ob}(\mathcal{D}(i)))$~~ We have already specified:

$\mathcal{O} = \text{Ob}[\text{colim}(\text{Ob}(\mathcal{D}(i)))]$ A set of objects

---

To specify a set of generating morphisms,

take $\{L_i : d_i \longrightarrow A\}$

and add on a few additional morphisms!

Consider $d_{i_1}, d_{i_2} \in \text{Ob}(\mathcal{D}(i))$

and morphism $f_{12} : d_{i_1} \longrightarrow d_{i_2}$, $f \in \text{Mor}(\mathcal{D}(i))$

Add the generating morphism

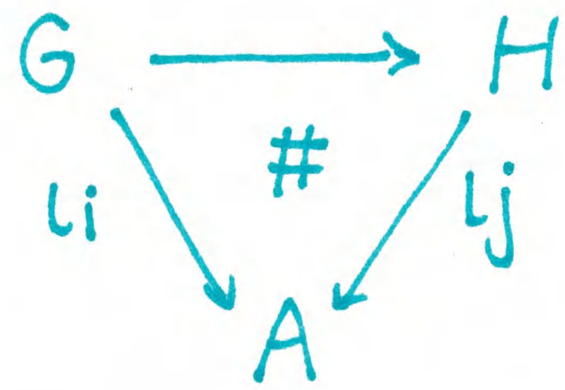
$$L_{f_{12}} : L_i(d_{i_1}) \longrightarrow L_i(d_{i_2})$$

so, now our generating morphisms are:

$$\{L_i : d_i \longrightarrow A,\ L_{f_{12}} : L_i(d_{i_1}) \longrightarrow L_i(d_{i_2})\}$$

To specify the necessary relations, consider what these morphisms must do to produce a commutative colimit diagram (as required)

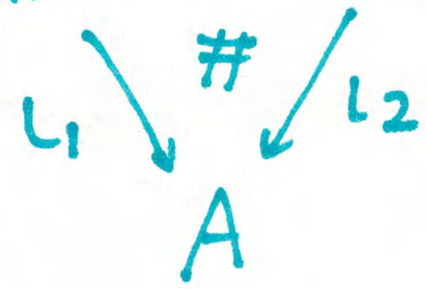
$G, H \in \mathrm{Grpd}$



For particular i, groupoid $\mathcal{D}(i)$

$Ob(\mathcal{D}(i)) = \{d_1, d_2, d_3, \dots\}$

$d_1 \xrightarrow{f_{12}} d_2 \longrightarrow \cdots$



There is a level of comprehension I lack here that has to do with the original choice of i!

Hmm...

$L_1(d_1) \in A \qquad L_2(d_2) \in A$

$f_{12} : d_1 \longrightarrow d_2$

$Lf_{12} : L_1(d_1) \longrightarrow L_2(d_2)$

A choice of i gives me a groupoid $\mathcal{D}(i)$

so changing to j gives me another groupoid $\mathcal{D}(j)$

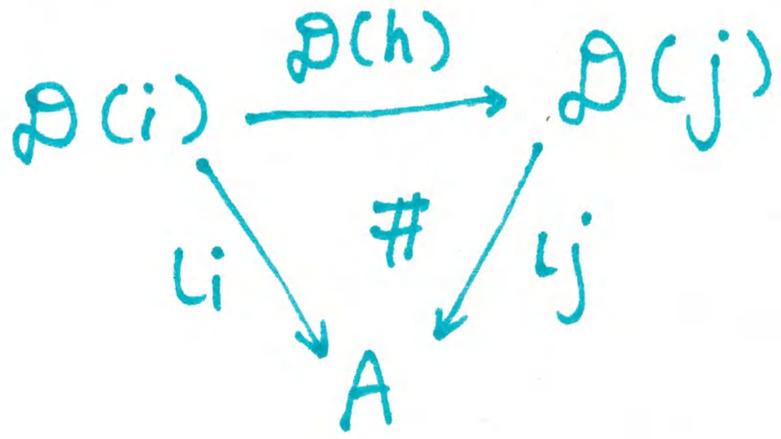
Let $h : i \longrightarrow j$ in the index set I. Then,

$\mathcal{D}(h) : \mathcal{D}(i) \longrightarrow \mathcal{D}(j)$ is a functor that maps objects $\mathcal{D}(i)$ to objects of $\mathcal{D}(j)$ (which will map $\{d_i\}$ to the set $\{d_j\}$ in some way)

$\forall f_{pq} : d_p \longrightarrow d_q$ in $\mathcal{D}(i)$

$L_i(f_{pq}) = L_j(\mathcal{D}(h)(f_{pq}))$

And we somehow form:



whose ~~this~~ universal property (the universal property of colimits) is satisfied by the universal property of groupoids. *

More Resolution of Category of Groupoids being cocomplete!

GRPD

The category Grpd has objects: groupoids and morphisms : Morphisms of groupoids.
To say that Grpd is cocomplete is to say that any diagram $\mathcal{D}$ admits a colimit.
So, given an indexing category I and a diagram $\mathcal{D}: I \longrightarrow$ Grpd, we construct a colimit by specifying its generators and relations.

$i \in I$
$\mathcal{D}(i) \in$ Grpd
so $\mathcal{D}(i)$ is a groupoid.

Groupoids are themselves categories, with their own objects and morphisms.
$\mathcal{D}(i)$ has objects and morphisms

~~The generators of this extends to Grpd over the objects of this~~

$Ob(\mathcal{D}(i))$ is a set (the objects of the groupoid),
and so, it has a colimit. (colim $\mathcal{D}(i)$ exists.)
colim $\mathcal{D}(i)$ is a single object in the groupoid $\mathcal{D}(i)$ along with a family of morphisms from the objects of $\mathcal{D}(i)$ to colim $\mathcal{D}(i)$

$i_1 \in I$
$\mathcal{D}(i_1) \in$ Grpd
colim $\mathcal{D}(i_1)$

$i_2 \in I$
$\mathcal{D}(i_2) \in$ Grpd
colim $\mathcal{D}(i_2)$

, etc... so, we set our generators for the colimits of Grpd

$A = \{\text{colim } Ob(\mathcal{D}(i))\}$ the set of all the colimits

For each i, we have a family of maps
from the other objects of $\mathcal{D}(i)$ to that particular colimit $\neq$ colim $\mathcal{D}(i)$.

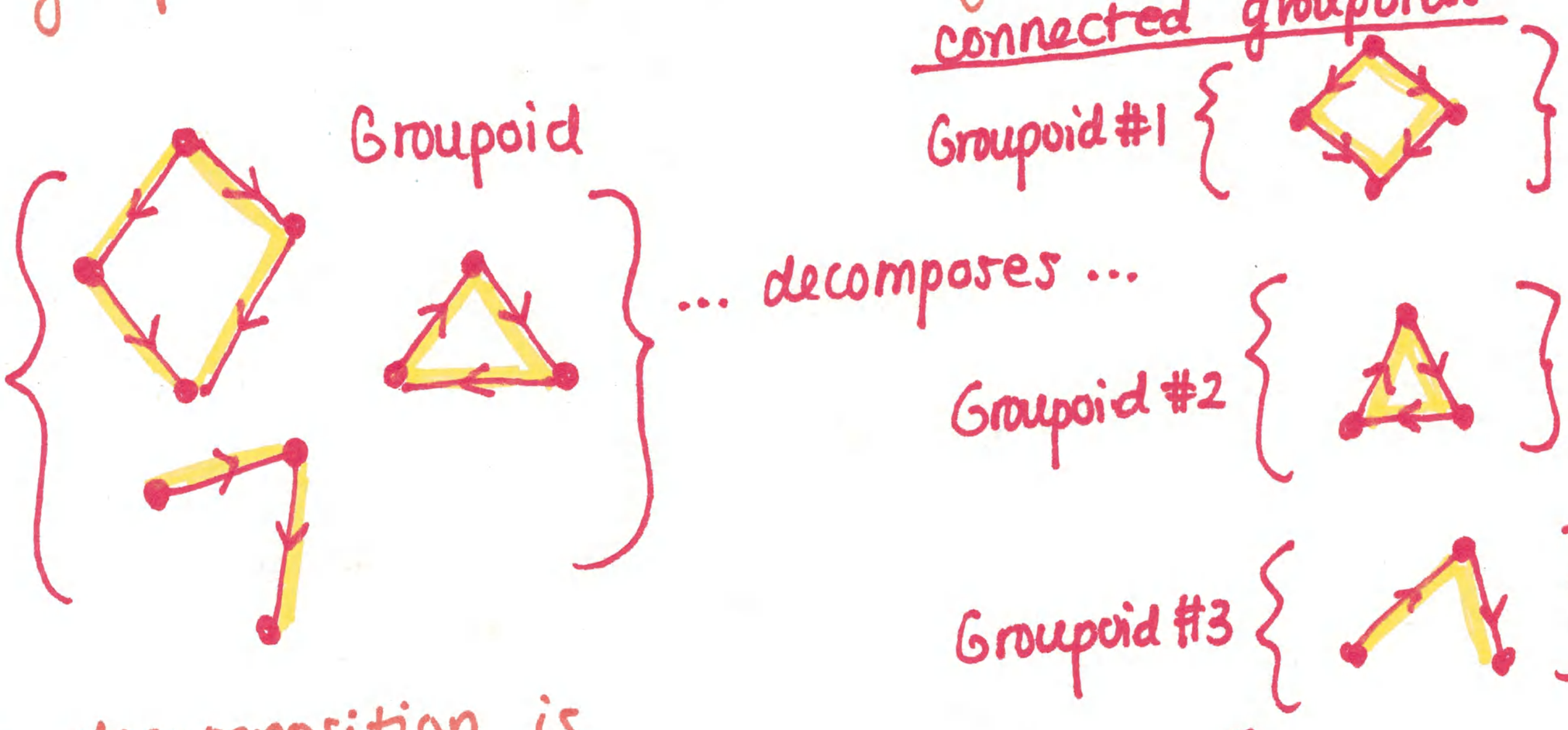
$\{ L_i : U(\mathcal{D}(i)) \longrightarrow \text{colim } U(\mathcal{D}(i)) \}$ injection maps for fixed i.

For a fixed i, and $x, y \in \mathcal{D}(i)$, $f: x \longrightarrow y$,
we define $Lf: L_i(x) \longrightarrow L_i(y)$ we must add morphisms and relations to make this
set of generating morphisms suitable to define a groupoid (which is a category)

CLASSIFYING GROUPOIDS

Every groupoid G decomposes into the coproduct of nonempty groupoids that themselves cannot be further decomposed into such a coproduct with two or more summands.

We refer to these nonempty non-decomposable groupoids as "connected groupoids"



This decomposition is unique up to a unique isomorphism

The indexing set of this decomposition can be computed as the set of objects of G modulo the equivalence relation of isomorphisms

This "indexing set" is denoted $\pi_0(G)$.

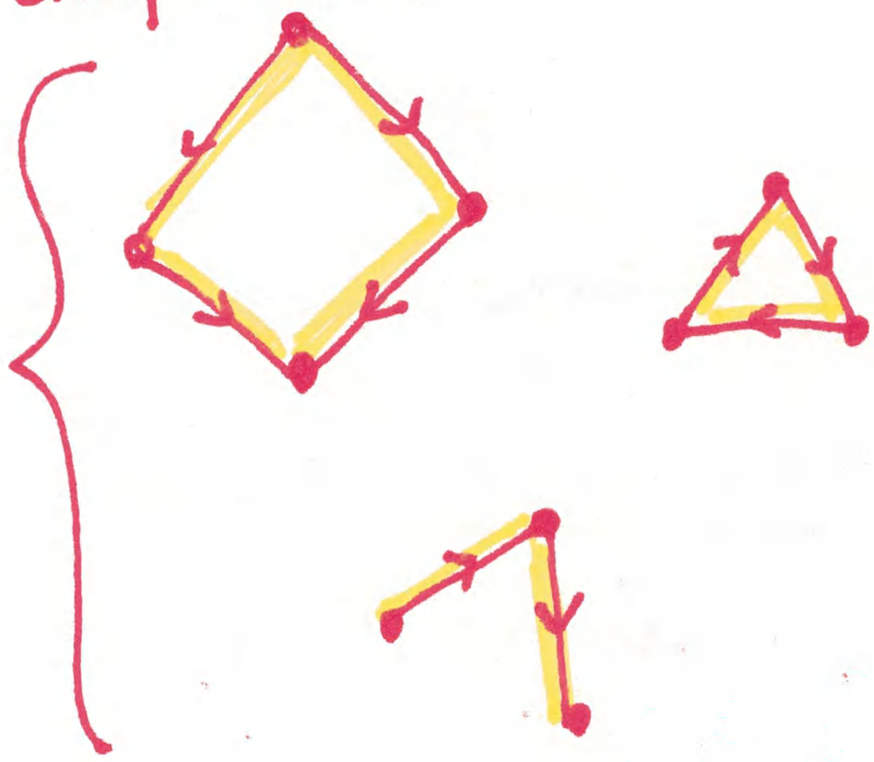
$\pi_0(G)$ is the number of connected components in the groupoid G.

$$\pi_0 : Grpd \longrightarrow Set, \quad \pi_0(G) = Ob(G) / \text{equivalence relation of isomorphisms} \longrightarrow$$

$$\pi_0(G) = \frac{Ob(G)}{\text{eq. relation of isomorphisms}} = coeq\left(Mor(G) \underset{\text{target}}{\overset{\text{source}}{\rightrightarrows}} Ob(G) \right)$$

That is, for any morphism in G, we identify the source and target in the set of objects.

Groupoid (G)



$\pi_0(G) = \{S_1, S_2, S_3\}$

Given $G \in Grpd$,

$x, y \in Ob(G)$

$f: x \longrightarrow y$, $f \in Mor(G)$, $f: \underset{\text{source}}{x} \longrightarrow \underset{\text{target}}{y}$

$\pi_0(G)$ identifies source and target, $x \sim y$ in G

Given $G \longrightarrow H$ a functor in Grpd

$\pi_0(G \longrightarrow H) = Ob(G) \longrightarrow Ob(H)$

(a functor in Grpd) (a functor in set)

[That is, a functor in Grpd induces a functor in set]

(functors preserve isomorphisms because functors preserve composition)

ALL GROUPOIDS DECOMPOSE AS A UNIQUE COPRODUCT OF CONNECTED GROUPOIDS:

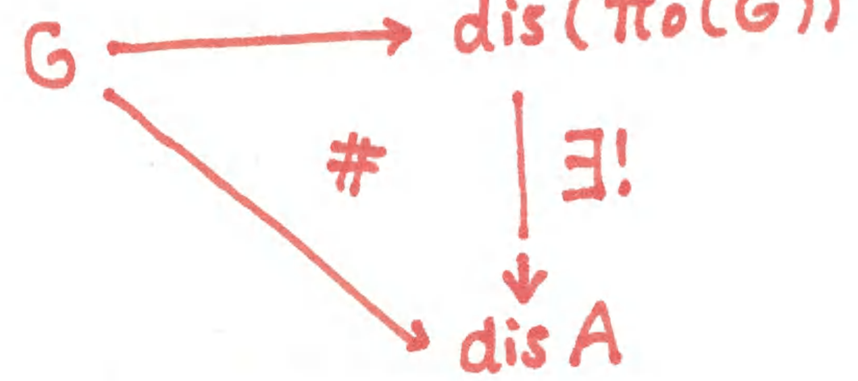
$$G = \coprod_{i \in \pi_0(G)} G_i$$

CLASSIFYING CONNECTED GROUPOIDS

The groupoid G is connected if $\pi_0(G) = \{*\}$
(there is only one connected groupoid in the coproduct decomposition, and that is G itself)

THE UNIVERSAL PROPERTY OF π_0 :

Given an arbitrary set A
and a functor $G \longrightarrow A$,
there exists a unique map
$\mathrm{dis}(\pi_0(G)) \longrightarrow A$
such that the triangle commutes:



PROPOSITION Suppose G is a connected groupoid.

[NOTE: We lose no generality is restricting to a connected groupoid, as any groupoid can be realized as unique coproduct of connected groupoids; this proposition can be applied to each component in the coproduct.]

Choose $x \in G$ [Connected groupoids are by definition nonempty]

Consider $\mathrm{Aut}_G(x) := \{ f : x \longrightarrow x \}$
[consider the set of morphisms in G which have the chosen point x at both source/target]

$\mathrm{Aut}_G(x)$ is the automorphism group of x in G.

Construct ~~a groupoid~~ a category (specifically, a groupoid)

$B(\mathrm{Aut}_G(x))$: Object : $\{*\}$ Morphisms : $\{ f : * \longrightarrow * \}$





... The set of "loops" with basepoint $*(x)$

We have a functor $B(\mathrm{Aut}_G(x)) \longrightarrow G$
that sends $* \longmapsto x$ and $f : * \to * \longmapsto f : x \to x$.

CLAIM: The functor $B(Aut_G(x)) \longrightarrow G$

$$* \longmapsto x$$
$$f: * \to * \longmapsto f: x \to x$$

is an equivalence of groupoids !

$B(Aut_G(x))$ the classifying simplicial set (space) of automorphisms of $x \in G$.

G the groupoid

Groupoid G: The groupoid G is specified by a system of generators and relations:

a set of objects $\mathcal{O}$

a set of generating isomorphisms $f_i: x_i \to y_i$ for $x_i, y_i \in \mathcal{O}$

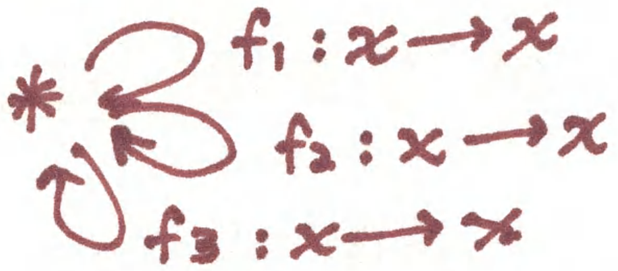
with relations

For $x \in G$

$Aut_G(x) = \{ f: x \to x \}$

The classifying space

$B(Aut_G(x))$



The map $B(Aut_G(x)) \xrightarrow{\Psi} G$ is specified by

sending $* \longmapsto x$

$f: * \to * \longmapsto f: x \to x$

We now specify the reverse map:

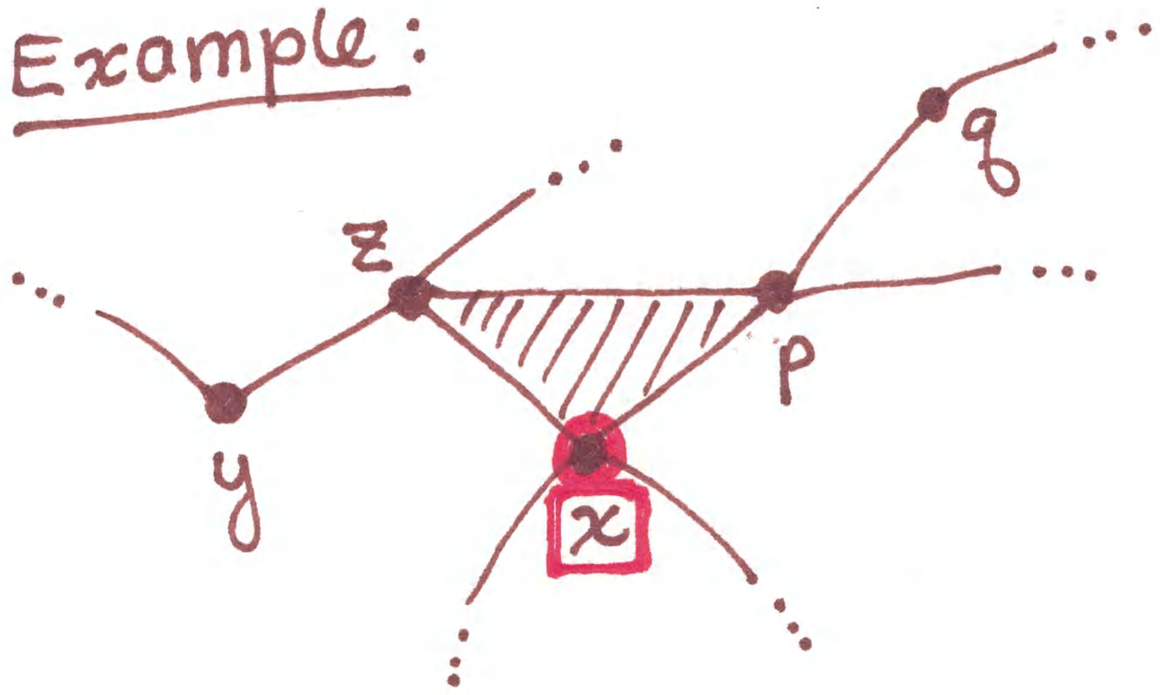
$G \xrightarrow{\varphi} B(Aut_G(x))$

$\forall y \in G$, choose $p_y: x \to y$ (G is connected, so p_y exists in all cases)

For $x \in G$, $p_x: x \to x$
choose the identity morphism

(Illustrated on next page...) →

Example:



We can choose isomorphisms *

For $y \in G$, define

$$p_y : x \longrightarrow y$$

For $z \in G$, define

$$p_z : x \longrightarrow z$$

For $p \in G$, define

$$p_p : x \longrightarrow p$$

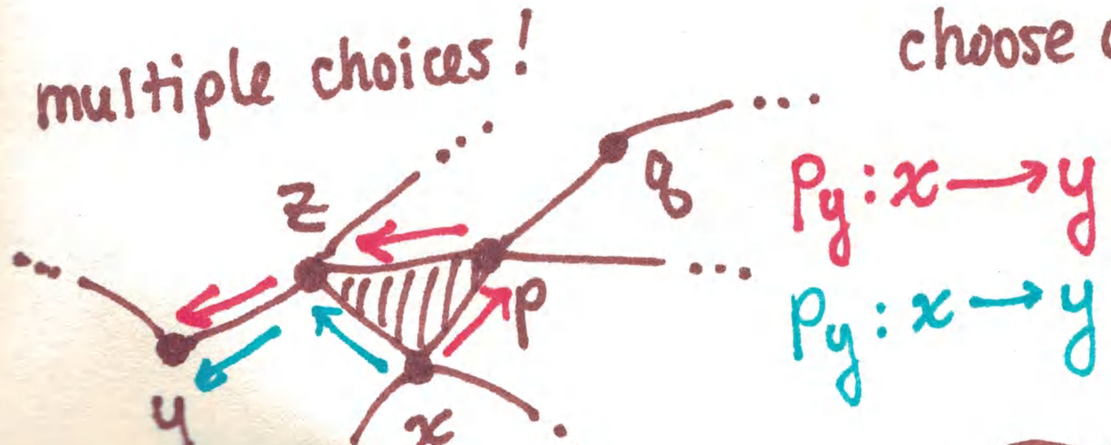
For $q \in G$, define

$$p_q : x \longrightarrow q$$

* We can do this because G is connected; $\pi_0(G) = \{*\}$, and all morphisms are in the same isomorphism class. Also, remember that since G is a groupoid, all morphisms are isomorphisms.

If there are multiple morphisms, we choose one:

multiple choices!



choose one!

$p_y : x \longrightarrow y$

$p_y : x \longrightarrow y$

To show an equivalence of categories, we need to define a morphism:

$$\varphi : G \longrightarrow B(\mathrm{Aut}_G(x))$$

by specifying where to send objects

$(y \in G)$

and morphisms

$(f : y \longrightarrow p)$

etc...

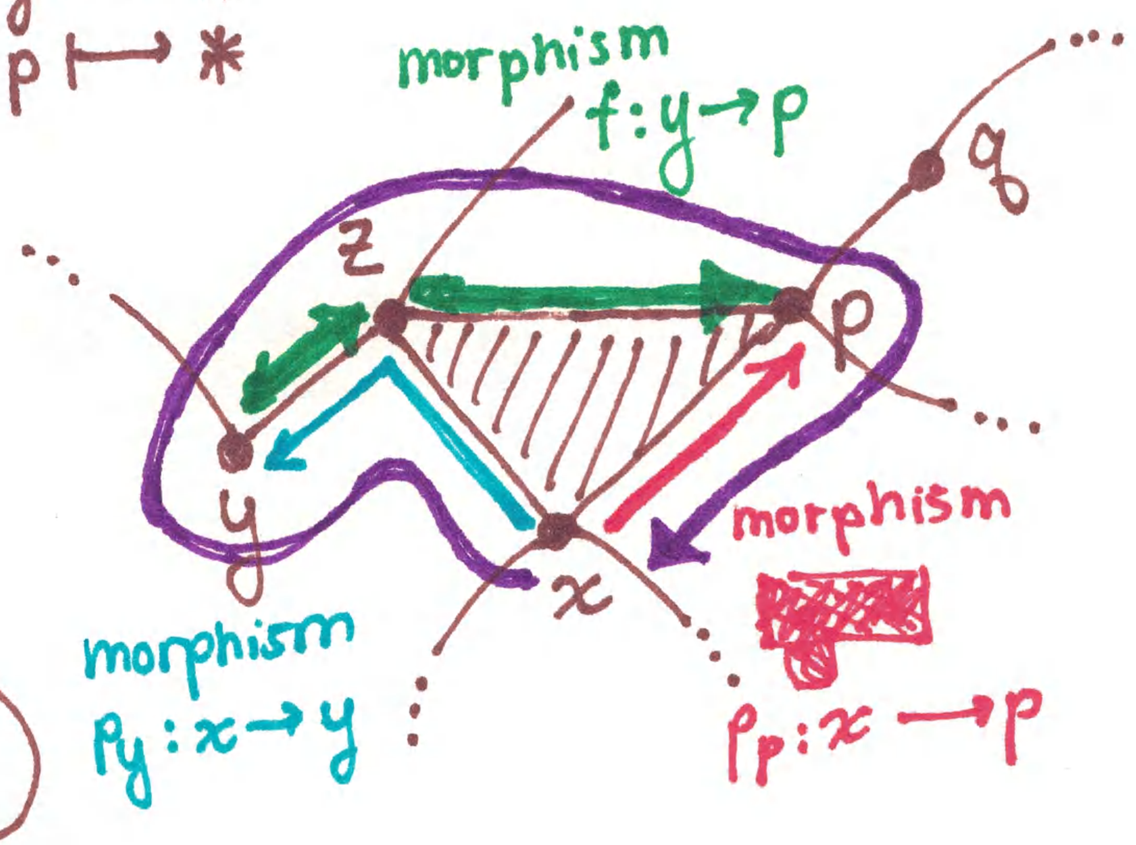
$$\varphi : G \longrightarrow B(\mathrm{Aut}_G(x))$$

$$g \in G \longmapsto * \in B(\mathrm{Aut}_G(x))$$

$$f : y \to p \longmapsto p_p^{-1} f p_y : * \to *$$

$$y \longmapsto *$$

$$p \longmapsto *$$



morphism $p_p^{-1} f p_y : x \longrightarrow x$

$\longrightarrow$

* NOTE: For $x \in G$, define $p_x : x \to x$ to be the identity morphism

⟶ so, in general

$$\varphi : G \longrightarrow B(Aut_G(x))$$

is defined by choosing $\forall y \in G$ a morphism (isomorphism) $p_y : x \longrightarrow y$,

sending all objects to $*$ in $B(Aut_G(x))$,

and sending all morphisms [redacted]

$f : y \longrightarrow y'$ in G

to the morphism $p_{y'}^{-1} f p_y$ in $B(Aut_G(x))$

To show that this is an equivalence of categories, we need to show that:

[1] $B(Aut_G(x)) \xrightarrow{\Psi} G \xrightarrow{\varphi} B(Aut_G(x))$ is the identity morphism in $B(Aut_G(x))$

and

[2] $G \longrightarrow B(Aut_G(x)) \longrightarrow G$ is the identity morphism in G.

We already defined $p_x = id_x$ VERIFY [1]

$\ell \in B(Aut_G(x))$; $\Psi(\ell) = \ell$

$\ell : * \to *$ $\quad$ $\ell : x \to x$

and $\varphi(\ell) = p_x^{-1} \ell p_x = id_x^{-1} \ell id_x = \ell$

so $\varphi \circ \Psi(\ell) = \ell \Rightarrow \varphi \circ \Psi = id_{B(Aut_G(x))}$

and now, we VERIFY [2] ⟶ continued...

$$G \xrightarrow{\varphi} B(\mathrm{Aut}_G(x)) \xrightarrow{\psi} G$$

$y \in G$; $\varphi(y) = *$

$\psi(*) = x$

$f: y \to y' \in \mathrm{Mor}(G)$

$\varphi(f) = p_y^{-1} f p_y$ and $\psi(p_y^{-1} f p_y) = p_y^{-1} f p_y$

$p_y^{-1} f p_y : x \to x$ $\qquad$ $p_y^{-1} f p_y : x \to x$

so $\psi \circ \varphi(f) = p_y^{-1} f p_y$

and since $p_y : x \to y$ $\quad p_{y'}^{-1} : y' \to x$

$$x \xrightarrow{p_y} y \xrightarrow{f} y' \xrightarrow{p_{y'}^{-1}} x$$

and $p_{y'}^{-1} f p_y : x \to x$

so it is the identity on G. ▨
(isomorphic)

So... The functor $B(\mathrm{Aut}_G(x)) \longrightarrow G$

$$* \longmapsto x$$

$$f: * \to * \longmapsto f: x \to x$$

is an equivalence of categories, ...

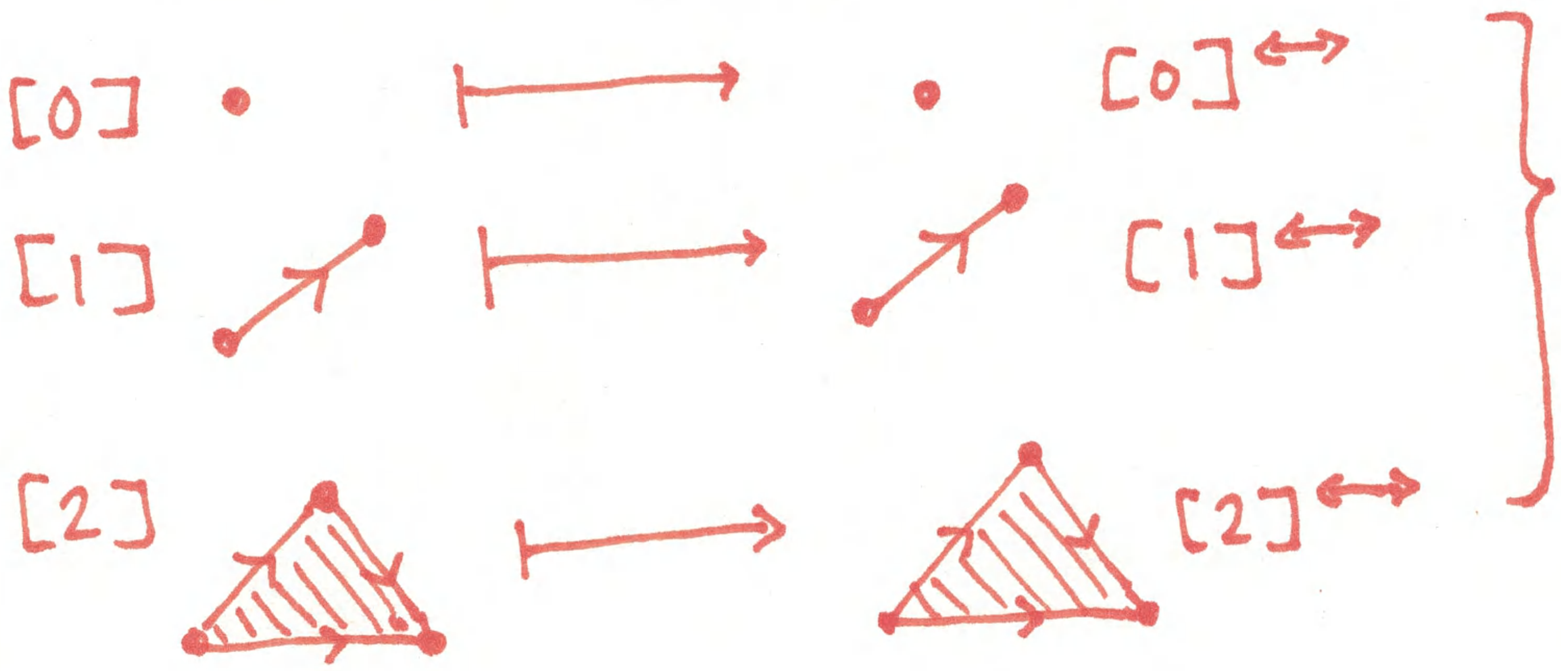
... that is...

(X)
the Groupoid can be defined by the classifying space of automorphisms of a single vertex $x \in X_0$

CONSTRUCTION OF THE FUNCTOR

Consider the functor: $\Delta \longrightarrow \mathrm{Grpd}$

$[m] \longmapsto [m]^{\leftrightarrow}$



$[m]^{\leftrightarrow}$ has $U([m])$ as its objects, and exactly one morphism between any pair of objects

☆ This functor realizes a simplex as a groupoid

THE FUNDAMENTAL GROUPOID FUNCTOR:

$$\Pi_{\leq 1} : \mathrm{sSet} \longrightarrow \mathrm{Grpd}$$

is the realization functor of $\Delta \longrightarrow \mathrm{Grpd}$

---

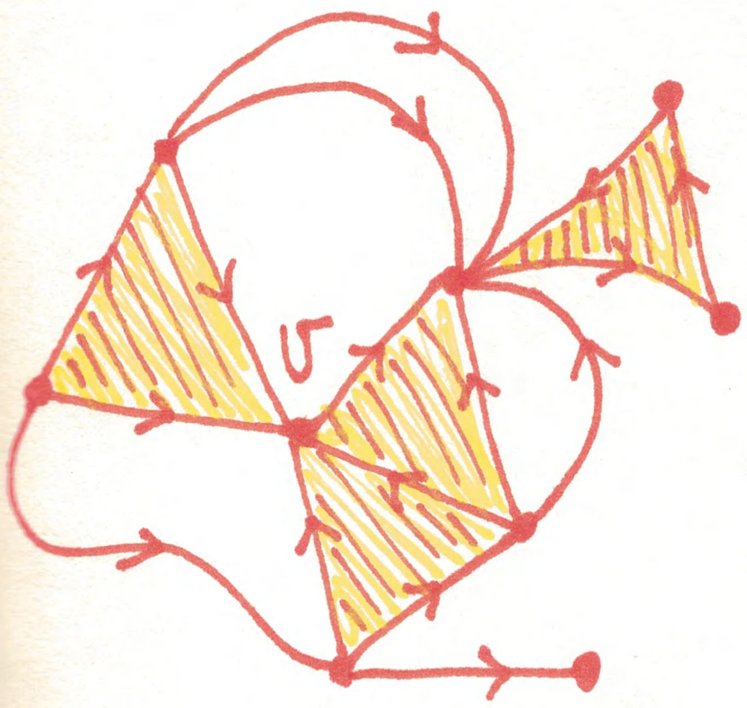
* What is a realization functor?

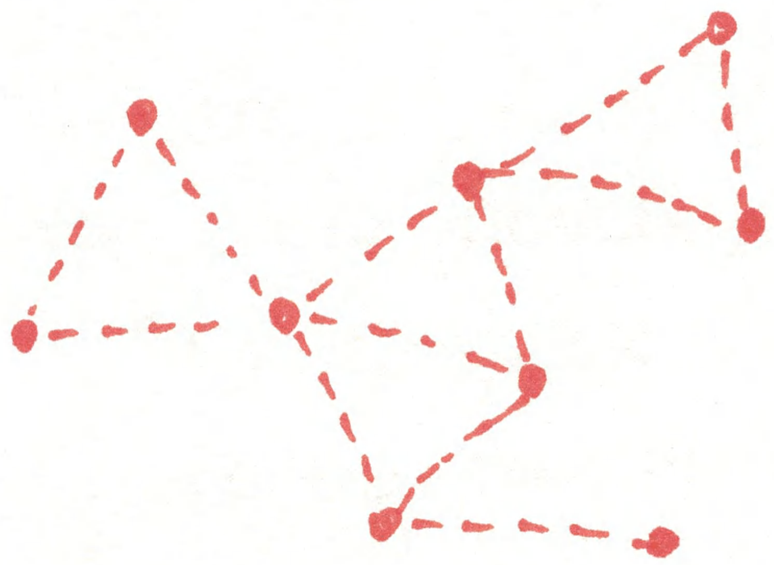
→ Given a vertex v in a simplicial set X

the **fundamental group** of v in X is the group

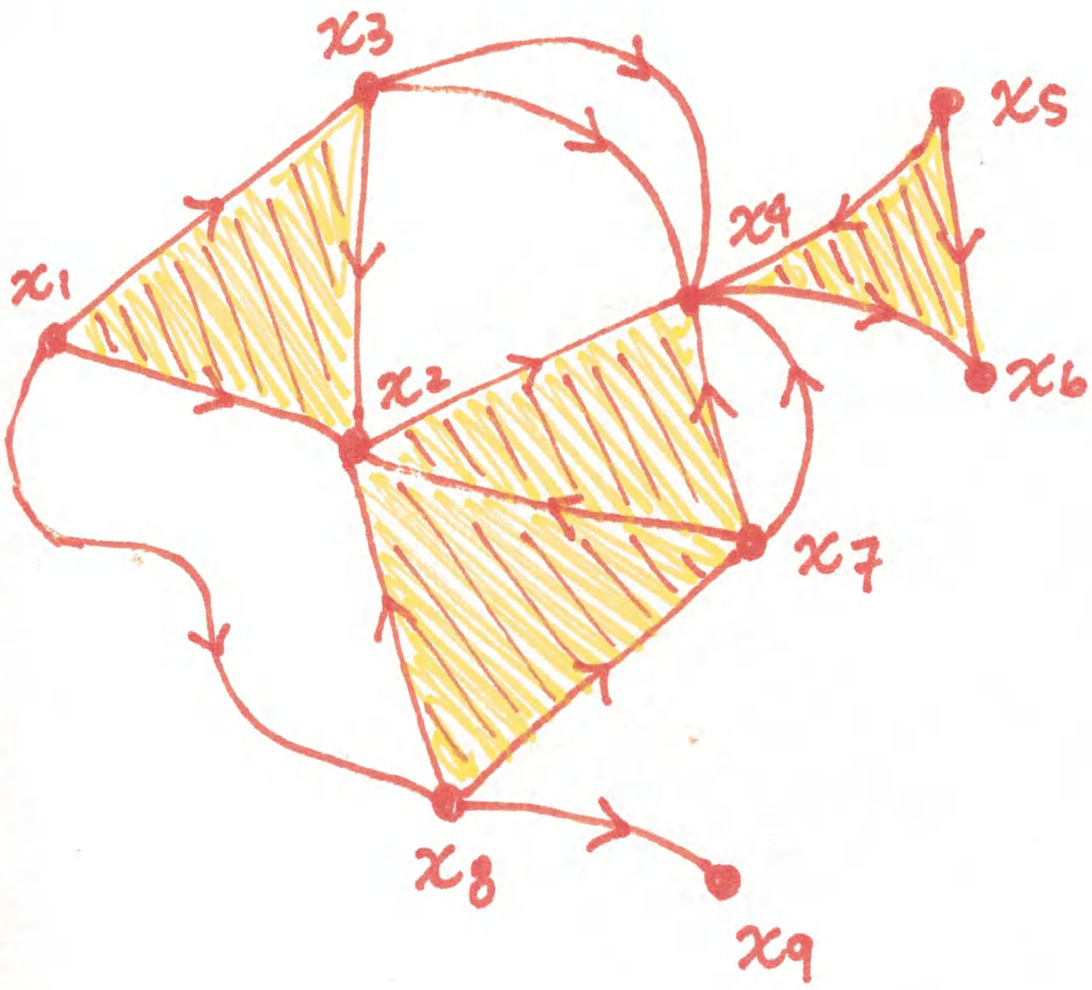
$$Aut_{\pi_{\leq 1}(X)}(v)$$

Automorphisms of $v \longrightarrow v$ in $\pi_{\leq 1}(X)$





GENERATORS AND RELATIONS FOR THE FUNDAMENTAL GROUPOID OF A SIMPLICIAL SET



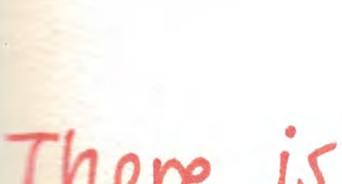
The set of objects is X_0
(the set of vertices)

$\{x_i\}_{i \in \{1,2,\ldots,9\}}$

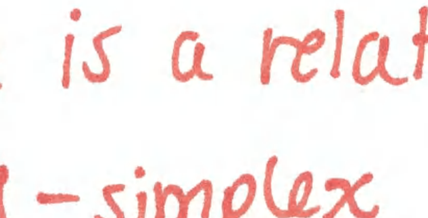
There is a generator
for every nondegenerate
[1]-simplex
(generating morphism)



There is a relation for every nondegenerate
[2]-simplex (relation between generating morphisms)



Relation
: $(12)=(32)\circ(13)$



Relation
: $(56)=(46)(54)$

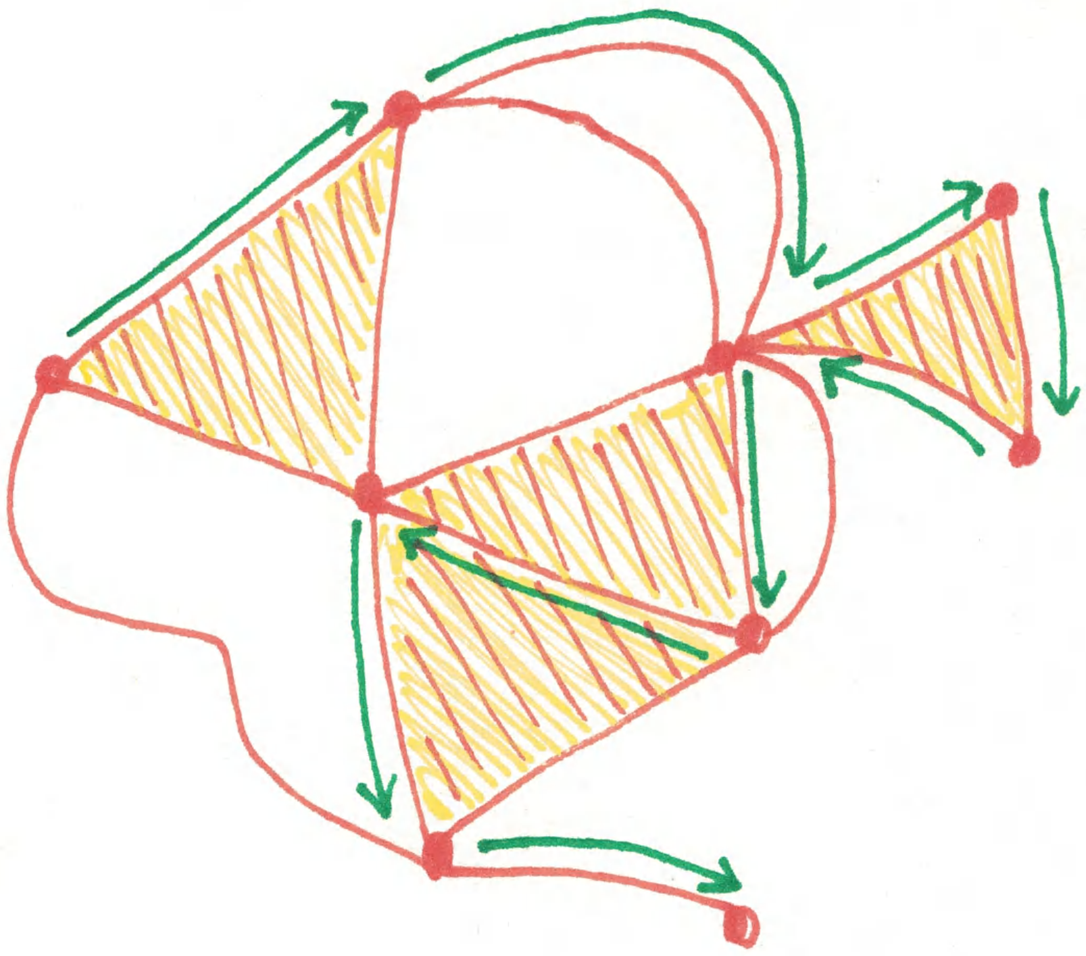


Relation
$(74)=(24)(72)$

Relation
$(82)=(72)(87)$

PRACTICAL CONSTRUCTION OF GENERATORS AND RELATIONS FOR THE FUNDAMENTAL GROUP OF A SIMPLICIAL SET AFTER FIXING A BASE VERTEX:

Construct $\pi_1(X, x)$



① Choose a "spanning tree"

$T \subset X$
generated by a
subset of X_1 such that
$\pi_0(T) \to \pi_0(X)$
is an isomorphism
and $\pi_0(T)$ is a
discrete category

② The set of generators is X_1.

③ For each $[1]$-simplex in $T_1 \subset X_1$, (σ)
introduce a relation $(\sigma = 1)$

④ For each nondegenerate $[2]$-simplex (τ)
introduce a relation $(d_1\tau = d_0\tau\, d_2\tau)$

Example: The Fundamental Group of S^1:

X)



Objects: $\{x\}$ ([0]-simplices)

generating isomorphisms: $\{a\}$ ([1]-simplices)

relations: none ([2]-simplices)



$\pi_1 = \mathbb{Z}$

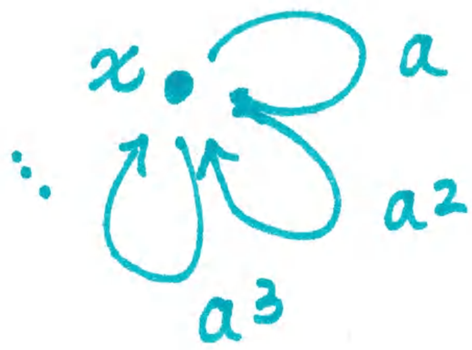
Fundamental Group:

$\pi_1(X, x) = \pi_1(S, x) = \langle a \rangle \cong \mathbb{Z}$

The Fundamental group is $\mathbb{Z}$.

The Fundamental Groupoid

$\langle a : x \to x \rangle$



$\{a : x \to x,$
$a^2 : x \to x,$
$a^3 : x \to x, \dots\}$

The fundamental groupoid is $B\mathbb{Z}$

$\{x,\ a : x \to x,\ a^2 : x \to x,\ a^3 : x \to x, \dots\}$

Example: The Fundamental Group of S^m



$\pi_1 = \emptyset$

When $m = 1$, we have the example above

When $m > 1$, we have only a nondegenerate [0]-simplex and a nondegenerate $[m]$-simplex.

Hence: objects: $\{x\}$

generating isomorphisms: none

relations: $\begin{cases} \{\alpha\} \text{ if } m = 2 & \text{(produces no relations of one-simplices)} \\ \text{none if } m > 2 \end{cases}$ because $[d_i \alpha$ is degenerate, $\forall i]$

The Fundamental Groupoid

$\{x\}$

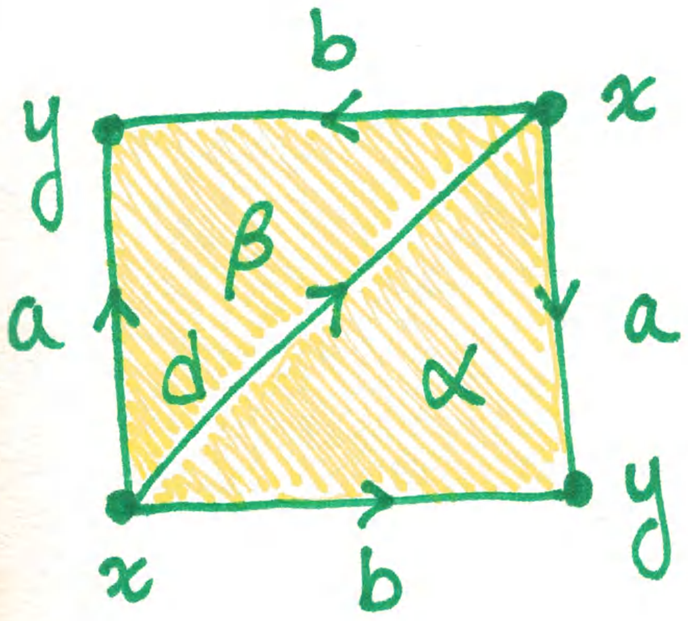
The Fundamental Group

$\emptyset$

the trivial group

* The Fundamental groupoid is the set of objects ([0]-simplices) and generating isomorphisms ([1]-simplices); the fundamental group is the restriction of morphisms to a single objects (investigate the morphisms that start and end at that object)

Example: The Fundamental Group of the Real Projective Plane



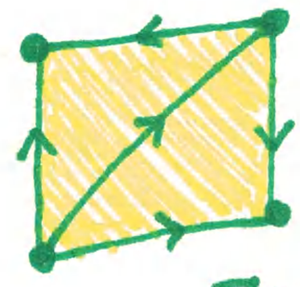
Objects : $\{x, y\}$

generating isomorphisms : $\{a, b, d\}$

$a : x \longrightarrow y$

$b : x \longrightarrow y$

$d : x \longrightarrow x$



$\pi_1 = \mathbb{Z}/2\mathbb{Z}$

The fundamental groupoid :

relations : $\alpha \Rightarrow a \circ d = b$

$\beta \Rightarrow b \circ d = a$

so, the fundamental groupoid :

$$< x, y, a, b, d \mid a \circ d = b,\ b \circ d = a >$$

substitute; a can be generated by b and d

$$< x, y, \cancel{a}, b, d \mid (b \circ d) \circ d = b >$$

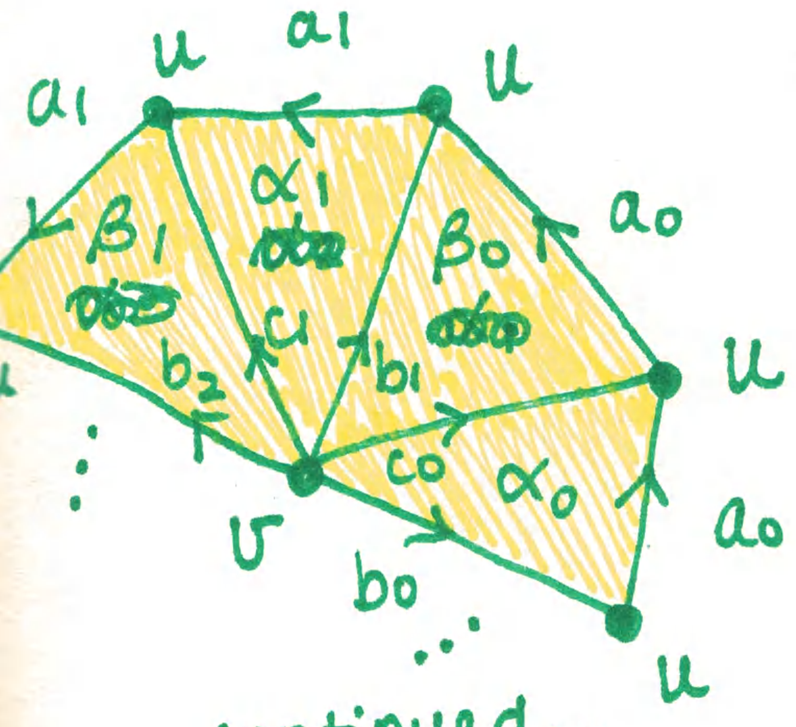
$$< x, y, b : x \to y, d : x \to x \mid d^2 = id >$$

The Fundamental group $\pi_1(X, x)$

$$\pi_1(x, x) = < d : x \to x \mid d^2 = id > \cong \mathbb{Z}/2\mathbb{Z}$$

$\pi_1(x, y) \equiv$ trivial group

Example: The Fundamental Group of the nonorientable surface of g crosscaps:



Objects: $\{u, v\}$

generating isomorphisms: $\{a_i, b_i, c_i\}$

relations: From $\alpha_i \Rightarrow$

$$a_i \circ b_i = c_i$$

From $\beta_i \Rightarrow$

$$a_i \circ c_i = b_{i+1}$$

The Fundamental groupoid:

$$\langle u, v, \{a_i : u \to u\}, \{b_i : v \to u\}, \{c_i : v \to u\} \mid \forall i,\ a_i b_i = c_i,\ a_i c_i = b_{i+1} \rangle$$

investigate relations:

$a_0 b_0 = c_0$, $a_0 c_0 = b_1 \longrightarrow a_0^2 b_0 = b_1$

$a_1 b_1 = c_1$, $a_1 c_1 = b_2 \longrightarrow a_1 a_0^2 b_0 = c_1$

$a_2 b_2 = c_2$, $a_2 c_2 = b_3 \longrightarrow a_1^2 a_0^2 b_0 = b_2$

$\vdots \qquad \vdots$

$a_2 a_1^2 a_0^2 b_0 = c_2$

spanning tree: $b_0 = 1$ $\qquad a_2^2 a_1^2 a_0^2 b_0 = b_3$

$\vdots$

when $j = g-1$: $a_j^2 \cdots a_2^2 a_1^2 a_0^2 b_0 = b_0$

$a_j^2 \cdots a_2^2 a_1^2 a_0^2 = id$

$$\langle u, v, \{a_i : u \to u\}, \{b_i : v \to u\}, \{c_i : v \to u\} \mid a_j^2 \cdots a_2^2 a_1^2 a_0^2 = 1 \rangle$$

can be expressed in terms of $\{a_i\}$ $\quad j = g-1$

The Fundamental group (w/ respect to the baspoint u)

$$\pi_1(X, u) = \langle \{a_i : u \to u\} \mid a_j^2 \cdots a_2^2 a_1^2 a_0^2 = 1 \rangle \quad j = g-1$$

The Fundamental Group is independento of the choice of "base point"

PROPOSITION

HOW DOES THE FUNDAMENTAL GROUP DEPEND ON THE BASE POINT?

IF $X \in sSet$ IS CONNECTED, THEN ALL FUNDAMENTAL GROUPS ARE ISOMORPHIC!

For any simplicial set X, there is a functor:

$$\pi_{\leq 1}(X) \longrightarrow \text{Group}$$

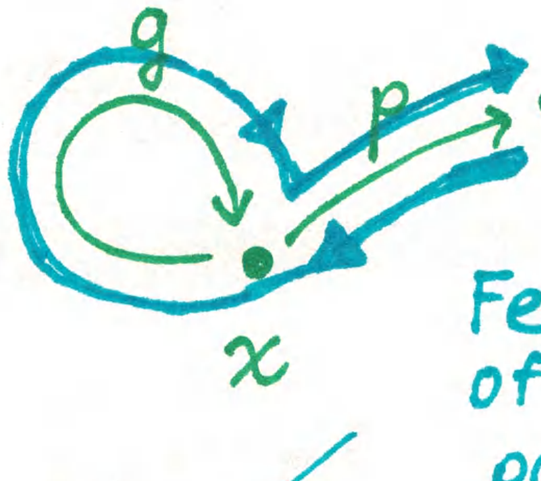
Fundamental Groupoid — Group

$$x \longmapsto \pi_1(X, x)$$

vertex $x \in X_0$ — fundamental group with that vertex as a base point

$$p : x \to y \longmapsto \pi_1(X,x) \longrightarrow \pi_1(X,y)$$

$$g : x \to x \longmapsto pgp^{-1} : y \to y$$



That is, ...

Feeds all automorphisms of x through the path from x to y to create automorphisms of y.

Rerouting preserves fundamental group

ADJOINT FUNCTORS

DEFINITION: ($\mathcal{C}$ and $\mathcal{D}$ are categories)

$L: \mathcal{C} \to \mathcal{D}$ is a functor
$R: \mathcal{D} \to \mathcal{C}$ is a functor

L is "left adjoint" to R > L ⊣ R
R is "right adjoint" to L

There is a natural isomorphism

$$u \longrightarrow v$$

where u, v are both functors of the form

$$u: \mathcal{C}^{op} \times \mathcal{D} \longrightarrow \text{set}$$
$$v: \mathcal{C}^{op} \times \mathcal{D} \longrightarrow \text{set}$$

and $u(c,d) = \hom_{\mathcal{D}}(L(c), d)$
$v(c,d) = \hom_{\mathcal{C}}(c, R(d))$

INFORMAL EXPLANATION:

$\hom_{\mathcal{D}}(L(c), d) \longrightarrow \hom_{\mathcal{C}}(c, R(d))$
"natural isomorphism"

$L(c) \longrightarrow d$ morphism in $\mathcal{D}$ can be identified with $c \longrightarrow R(d)$ morphism in $\mathcal{C}$

The Universal Property of Adjoint Functors

PROPOSITION

$\mathcal{C}, \mathcal{D}$ cocomplete categories

$L: \mathcal{C} \to \mathcal{D}$
$R: \mathcal{D} \to \mathcal{C}$ $\rangle$ $L \dashv R$

Left adjoint functor pair!
right adjoint

L is a cocontinuous functor
(L preserves colimits)

THEOREM The "Weak Yoneda Lemma"

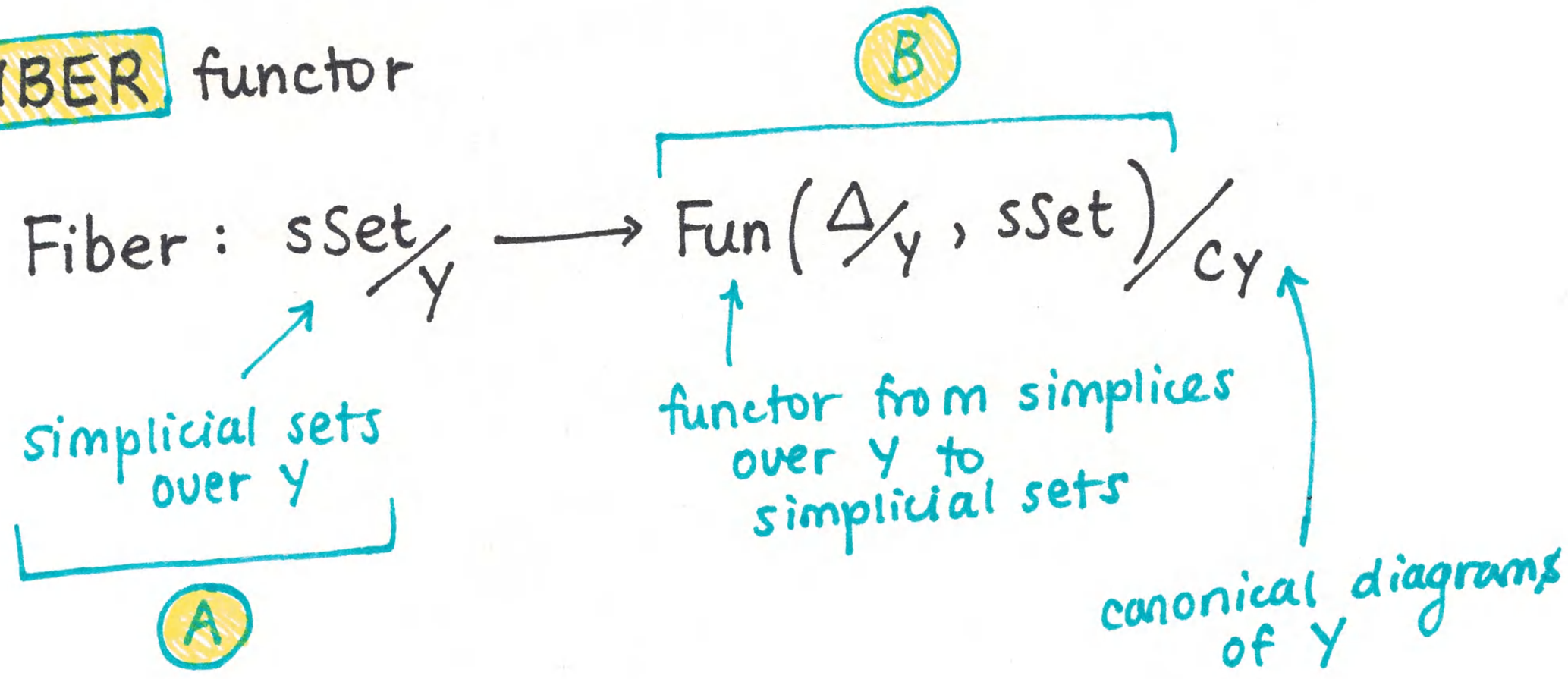
This Theorem is about fully faithful functors.

$$Y: \mathcal{C} \longrightarrow Fun(\mathcal{C}^{op}, set)$$
$$c \longmapsto d \mapsto hom_{\mathcal{C}}(d, c)$$

$$Y: \mathcal{C}^{op} \longrightarrow Fun(\mathcal{C}, set)$$
$$c' \longmapsto d' \mapsto hom_{\mathcal{C}}(c', d')$$

FIBER FUNCTORS

The **FIBER** functor



(A) $sSet/Y$] $f \in sSet/Y$, $f: X \longrightarrow Y$ simplicial map

An object of $sSet/Y$ is a simplicial map $f: X \longrightarrow Y$

(B) $Fun(\Delta/Y, sSet)$] $f \in Fun(\Delta/Y, sSet)$ f is a natural transformation

An object of $Fun(\Delta/Y, sSet)$ is a natural transformation $f: F \longrightarrow CY$ of two functors $\Delta/Y \longrightarrow sSet$ where CY is the canonical diagram of Y and F is a functor that sends a simplex $S: \Delta^m \longrightarrow Y$ to the pullback of $\Delta^m \times_Y X$ and a morphism of simplices $h: \Delta^m \longrightarrow \Delta^{m'}$ to the induced map $h \times_Y X: \Delta^m \times_Y X \longrightarrow \Delta^{m'} \times_Y X$.

The natural transformation f on a simplex $S: \Delta^m \longrightarrow Y$ is given by the projection $\rho_S: \Delta^m \times_Y X \longrightarrow \Delta^m$.

ASSEMBLE FUNCTOR

The **ASSEMBLE** functor

$$\text{Assemble} : \text{Fun}(\Delta/_Y, \text{sSet})/_{cY} \longrightarrow \text{sSet}/_Y$$

$$f : F \to cY \longmapsto \text{colim}(f)$$

(A) (B)

(A) $f \in \text{Fun}(\Delta/_Y, \text{sSet})$

$f : F \to cY$ is a natural transformation

(B) $\text{colim}(f) \in \text{sSet}/_Y$

$$\text{colim}(f) : \text{colim}(F) \longrightarrow \underbrace{\text{colim}(cY)}_{\text{Proposition 27.4}} \cong Y$$

Assemble is left adjoint to **Fiber**

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§31

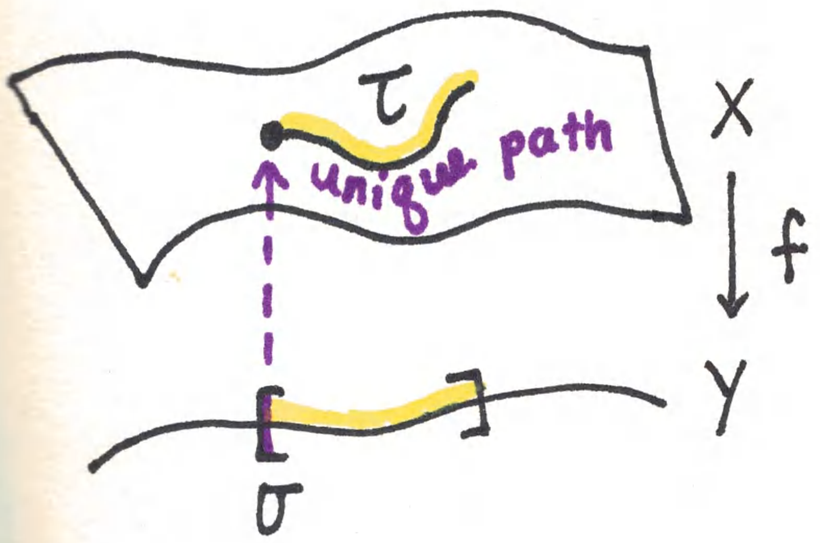
COVERINGS

* $\mathrm{Cov}/_Y$ is the category of coverings of a simplicial set Y

Definition: A covering is a simplicial map

$$f: X \longrightarrow Y$$

with a unique "lifting property".



[0]-simplices

Δ^0

$\Delta^m \longrightarrow Y$

[m]-simplices in Y

We know how to construct [m]-simplices in Y

$\Delta^0 \longrightarrow X$

we know how [0]-simplices sit in X

X
f
Y

We have a covering X of Y (the covering is actually the simplicial map f)

LIFTING PROPERTY

Any commutative square:

$$\begin{array}{ccc} \Delta^0 & \longrightarrow & X \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle f} \\ \Delta^m & \longrightarrow & Y \end{array}$$

has a diagonal "filler" map:

$$\begin{array}{ccc} \Delta^0 & \longrightarrow & X \\ {\scriptstyle k}\downarrow & \overset{\tau}{\nearrow} & \downarrow{\scriptstyle f} \\ \Delta^m & \xrightarrow{\sigma} & Y \end{array}$$

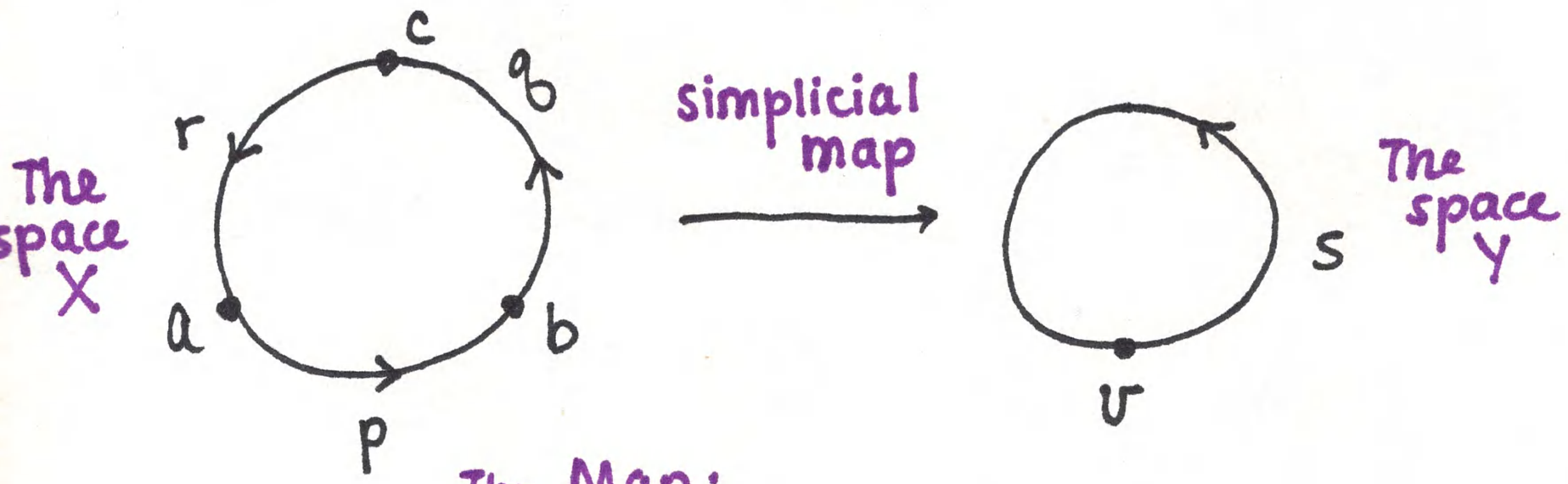
that makes both triangles commute!

UNIQUE LIFTING PROPERTY

$$\begin{array}{ccc} \Delta^0 & \longrightarrow & X \\ {\scriptstyle k}\downarrow & \overset{\exists!\tau}{\nearrow} & \downarrow{\scriptstyle f} \\ \Delta^m & \xrightarrow{\sigma} & Y \end{array}$$

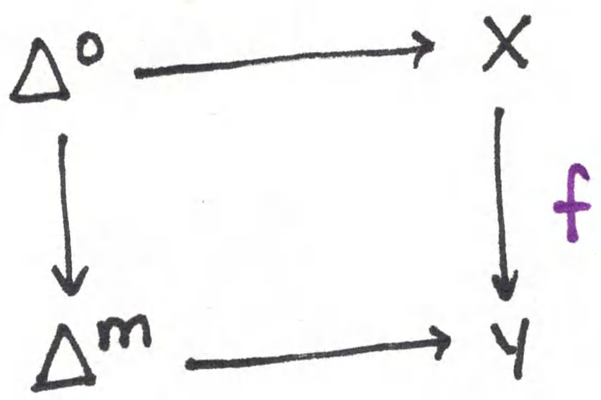
There is a unique way to construct [m]-simplices in X.

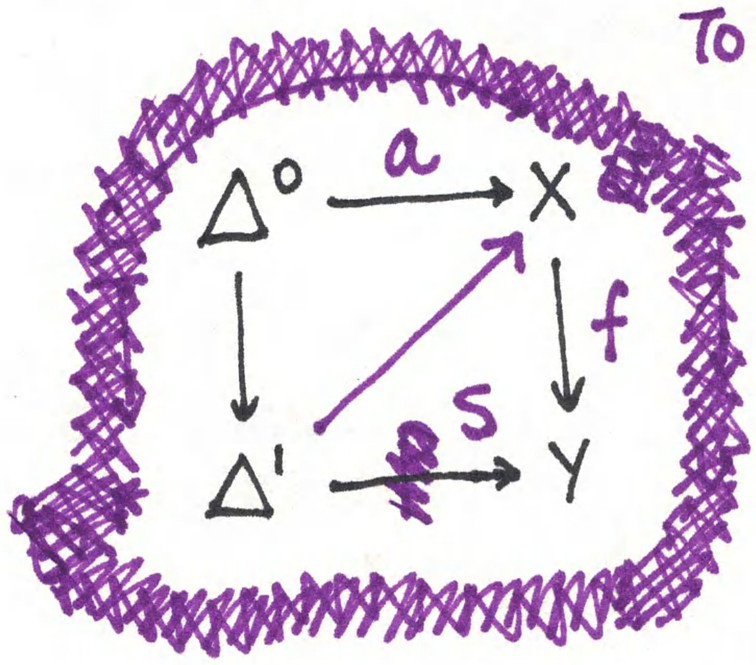
Example:



The Map:

The simplicial map $f: X \to Y$ { $a, b, c \longmapsto v$ map [0]-simplices; $p, q, r \longmapsto s$ map [1]-simplices





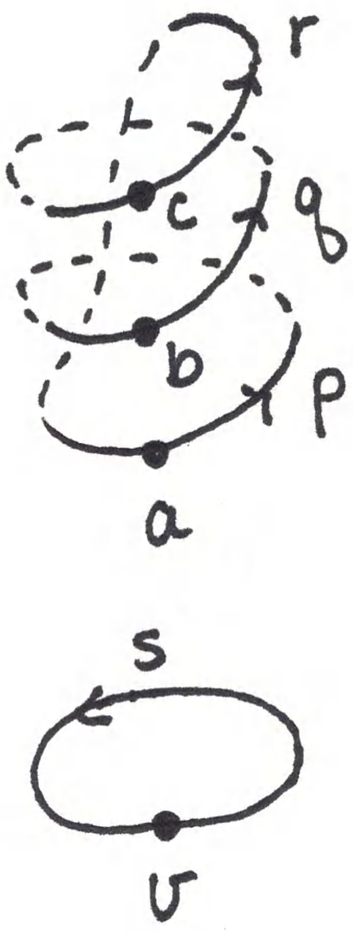
To Illustrate the "lifting property" in this example:

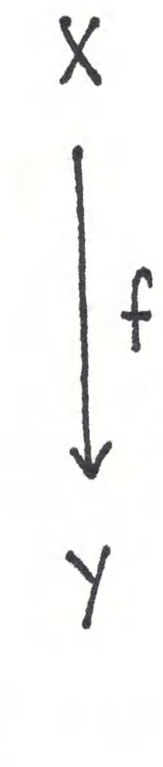
Let m=1

Then

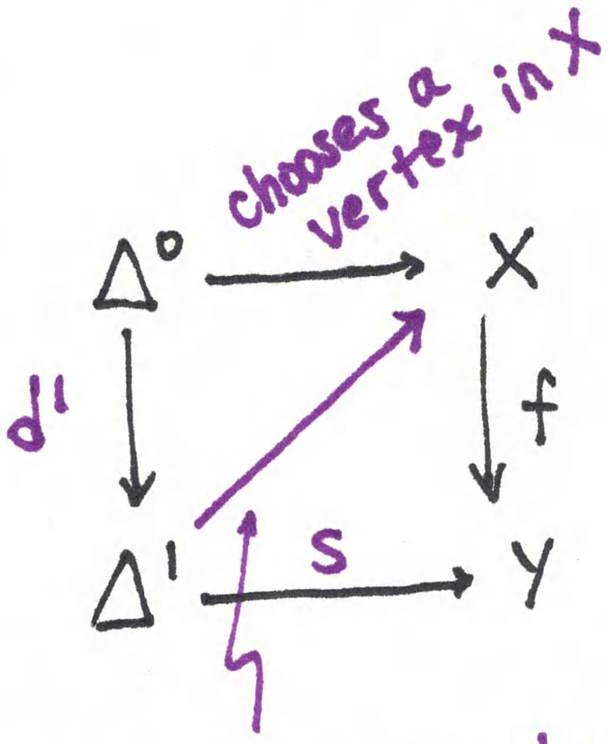
$\Delta^1 \longrightarrow Y$

is a [1]-simplex:





X can be used to cover Y

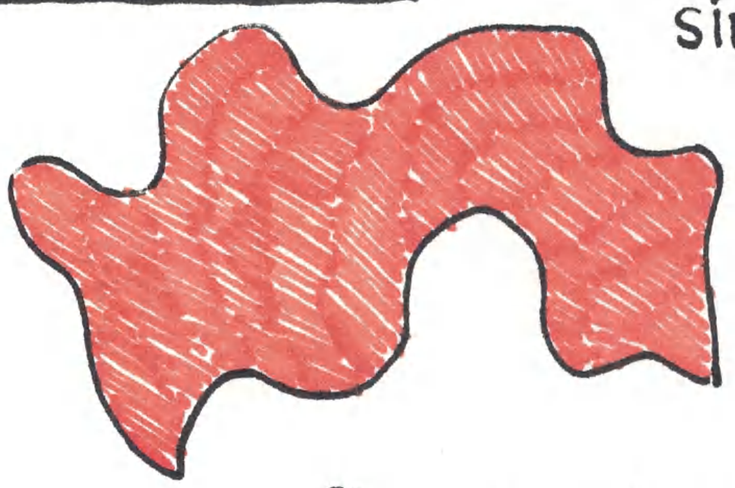


We need to show that this map exists (this map is the [1]-simplex in X starting at this vertex)



Example: "lifting" the [1]-simplex

Illustration:



simplicial fabric



simplicial set



determine how to "fold" the simplicial fabric to cover the simplicial set

(create fibers over vertices)



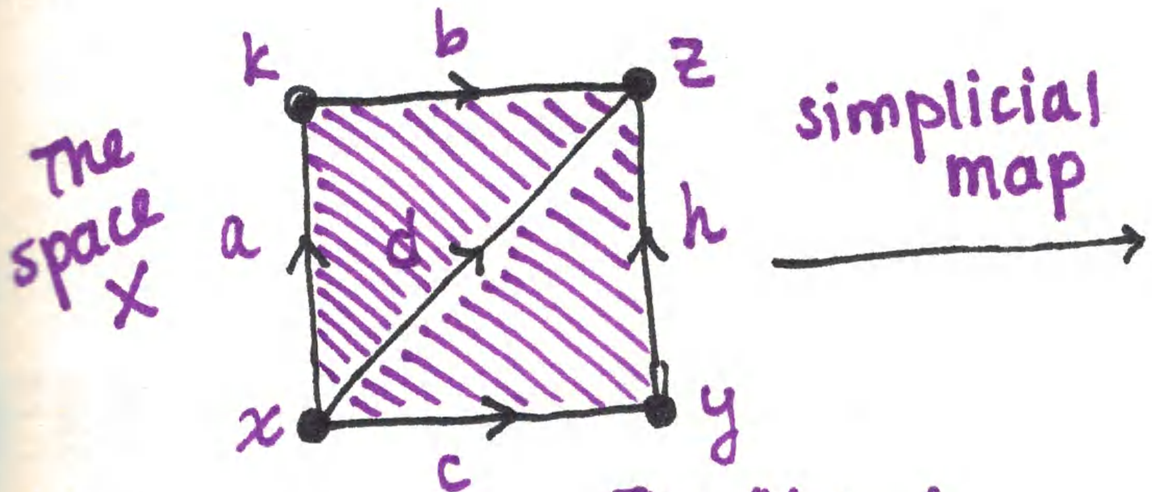


Does [m]-simplex in the simplicial set determine a unique [m]-simplex in the simplicial fabric?

If so, the simplicial fabric is a "cover" of the simplicial set!

A covering is like a "vestment", clothing the space in multiple "layers!"

Nonexample:



The space X

$\Delta^1 \times \Delta^1$

The space Y

The Map:

* projection onto s *

$k, a, x \longmapsto t_1$

$b, d, c \longmapsto s$

$z, h, y \longmapsto t_2$

$x \longmapsto t_1$ $\quad s : t_1 \longrightarrow t_2$ but $f(d): t_1 \longrightarrow t_2$ $\quad f(d): f(x) \longrightarrow t_2$

$f(c): t_1 \longrightarrow t_2$ $\quad f(c): f(x) \longrightarrow t_2$

Here, there are two different [1]-simplices that start at the vertex $x \in X_0$. S (a [1]-simplex in Y) does not have a unique "lifting" in X!

so, the space $\Delta^1 \times \Delta^1$ is NOT a cover of Δ^1 (actually, the projection map)

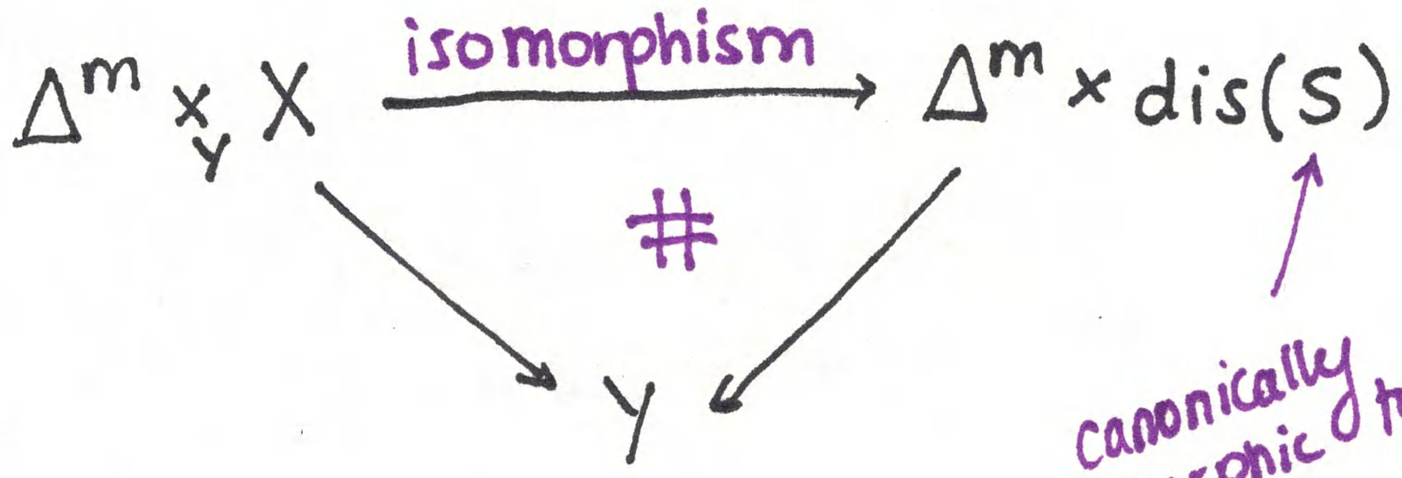
chooses a vertex in X, $x \in X_0$

$$\begin{array}{ccc} \Delta^0 & \longrightarrow & \Delta^1 \times \Delta^1 = X \\ \downarrow & \nearrow \text{NOT UNIQUE!} & \downarrow \\ \Delta^1 & \xrightarrow{S} & \Delta^1 = Y \end{array}$$

PROPOSITION

A simplicial map $f: X \longrightarrow Y$ is a covering ... if and only if... for any [m]-simplex $s: \Delta^m \longrightarrow Y$ in the space Y,

there is a commutative triangle:



* I do not understand what this proposition means ∴

... but it produces the following lemma...

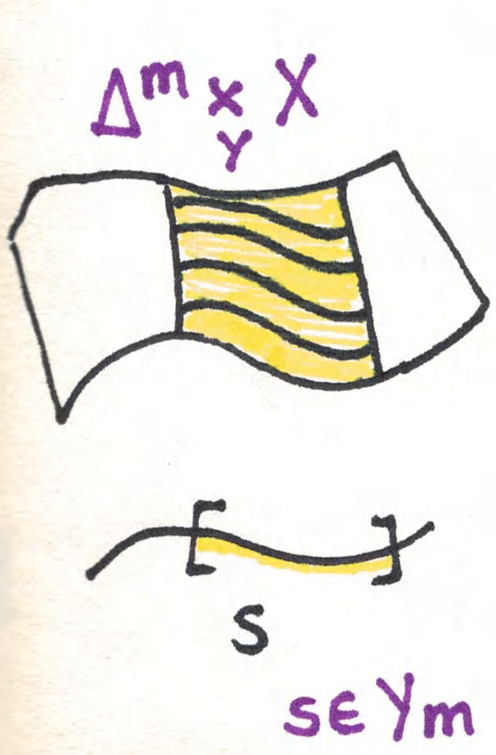
LEMMA

If $f: X \longrightarrow Y$ is a covering, $\longrightarrow$ then $\rightarrow$ for any $\Delta^m \longrightarrow \Delta^n \longrightarrow Y$, the map of sets

$\pi_0(\Delta^m \times_Y X) \longrightarrow \pi_0(\Delta^n \times_Y X)$

is an isomorphism.

FIBER PRODUCT

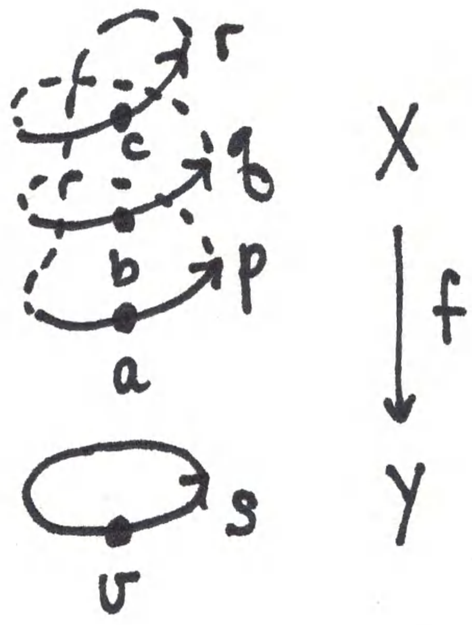


The fiber product is the join of all preimages of s in the space X

The fiber product $\Delta^m \times_Y X$

Example

Consider the simplicial map



we know (from the previous example) that the simplicial map $f: X \to Y$ is a covering.

By our, proposition,
$\forall$[m]-simplices $\in Y$
the triangle should commute:

$$\begin{array}{ccc} \Delta^m \underset{Y}{\times} X & \longrightarrow & \Delta^m \times dis(S) \\ & \searrow \quad \swarrow \# & \\ & Y & \end{array}$$

CASE 1 $m=0$ $\quad \Delta^0 \xrightarrow{v} Y$
(a [0]-simplex in Y)

$\Delta^m \underset{Y}{\times} X = \Delta^0 \underset{Y}{\times} X$ is the fiber product of ~~preimages~~ preimages of v in the space X

$$\Delta^0 \underset{Y}{\times} X = \begin{array}{l} \bullet c \\ \bullet b \\ \bullet a \end{array}$$

the fiber product

$S = \pi_0(\Delta^m \underset{Y}{\times} X)$
is the functor which takes the simplicial set and realizes it as a set

$\Delta^m \times dis(S) = \Delta^0 \times dis(S)$
is extracting a set from a simplicial set

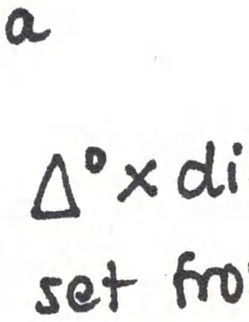
$\Delta^m \times dis(S) = \bullet_v$

dis(S) is the discrete simplicial set, a functor

$dis: Set \longrightarrow sSet$

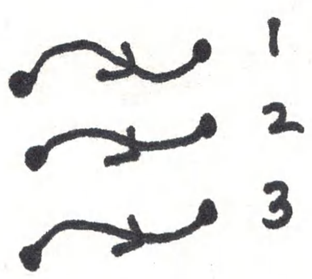
By definition, $(dis\, S)_m = S$
for any simplex [m] and
$(dis\, S)_f = id_S$
for any map of simplices f.

$dis(\pi_0(\Delta^m \underset{Y}{\times} X))$



$\pi_0(\ \ ) =$ the number of connected components in the fiber product
(the number of fibers)

$\longrightarrow$ $dis(\pi_0(\ast\!\ast\, \Delta^m \underset{y}{\times} X))$



$\pi_0(\Delta^m \underset{y}{\times} X) = \{s_1, s_2, s_3\}$ ← a set

$dis(\pi_0(\Delta^m \underset{y}{\times} X)) =$ • s_1 • s_2 • s_3 } a simplicial set (when $m=0$)

continuing when $m=0$:

$\Delta^0 \underset{y}{\times} X$ = fiber product in X { • c • b • a

$\Delta^0 \underset{y}{\times} dis(\pi_0(\Delta^0 \underset{y}{\times} X))$

$\pi_0 \begin{pmatrix} \bullet c \\ \bullet b \\ \bullet a \end{pmatrix}$

$\Delta^0 \underset{y}{\times} dis(\{a, b, c\})$ produces { • w_3 • w_2 • w_1

$m=0$

• c
• b
• a $\Delta^0 \underset{y}{\times} X$ —isomorphism→ $\Delta^0 \times dis(\pi_0(\Delta^0 \underset{y}{\times} X))$ • w_3 • w_2 • w_1

$\#$

•
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CASE 2 $m=1$ $\Delta^1 \xrightarrow{s} \ast\!\ast\, Y$
(a [1]-simplex in Y)

$\Delta^1 \underset{y}{\times} X$ is the fiber product of preimages of s in X ⟶