Mathematics 5325 (Topology II)

Final exam

1. Compute $\mathrm{H}_1(\mathrm{B}G, \mathbf{Z})$ for an arbitrary group G (abelian or not). (The answer should be expressed in terms of G.)

2. Compute the Poincaré duality isomorphism for $\mathbf{RP^3}$, the three-dimensional real projective space, which can be defined by taking the 3-skeleton of $\mathrm{B}\mathbf{Z}/2$ (meaning we take the simplicial subset of $\mathrm{B}\mathbf{Z}/2$ generated by simplices in dimension 3 and lower).

3. Does the orientable surface of genus $g \geq 0$ admit a nontrivial local system with typical fiber $\mathbf{Z}$? What about $\mathbf{Z}/n$, $n \geq 1$? (Give an individual answer for each possible value of $g \geq 0$ and $n \geq 1$ and make sure to treat the case $g = 0$.)

4. The *lens space* $L(p, n)$ is defined as follows: fix an integer $n > 0$ and a generator $p \in \mathbf{Z}/n$ (meaning $\gcd(p, n) = 1$). Both n and p are fixed parameters in this problem. There are $n + n$ generating 3-simplices, denoted by c_i and d_i ($i \in \mathbf{Z}/n\mathbf{Z}$). The simplicial set looks like an orange sliced along the equatorial plane, with c_i above and d_i below, the 0th vertices being the center and the 1st vertices being the poles. The relations are as follows: $\mathrm{d}_1(c_i) = \mathrm{d}_1(d_i)$ (gluing along the equatorial plane), $\mathrm{d}_2(c_i) = \mathrm{d}_3(c_{i+1})$ and $\mathrm{d}_2(d_i) = \mathrm{d}_3(d_{i+1})$ (gluing along the lateral sides of carpels). To these relations we add the following: $\mathrm{d}_0(c_i) = \mathrm{d}_0(d_{i+p})$ (the lens space relation).

Compute the fundamental group of the lens space. Does this space admit a connected covering of degree $d \geq 1$? Does it admit a connected Galois covering of degree $d \geq 1$? Does it admit a connected covering with S_3 (the symmetric group of order 3) as its deck transformation group?

5. Determine which of the following topological spaces are (a) homeomorphic and (b) homotopy equivalent to each other, and which ones are not: $[0, \infty)^n$, $\mathbf{R}^n$, $\mathbf{R}^n \setminus \{0\}$ ($n \geq 0$). (You may cite known computations of homology.) More precisely, you must divide these spaces (for all possible values of parameters) into groups such that spaces in different groups are not homotopy equivalent, whereas spaces in the same group are. Each group must be further subdivided into subgroups such that spaces in different subgroups are not homeomorphic, whereas spaces in the same subgroup are.