LIE GROUPS & THEIR REPRESENTATIONS

* MATH 6333 *

LIE GROUP

DEFINITION A Lie Group is a group object in the category of smooth manifolds.

Diff := the category of smooth manifolds

> ..."where by a "smooth manifold," we mean a finite-dimensional, second-countable, Hausdorff, C^∞ manifold without boundary."
> —nLab

[This will be defined in more detail later...]

group object := $G \in Ob(\mathcal{C})$
an object in the category

(i) a unit map $1 : * \longrightarrow G$
where $*$ is the terminal object in the category $\mathcal{C}$

(ii) an inverse map $(-)^{-1} : G \longrightarrow G$

(iii) a multiplication map $m : G \times G \longrightarrow G$
(that is associative)

LIE ALGEBRA

DEFINITION A Lie Algebra over a field K is a K-vector space $\mathfrak{g}$ together with a bilinear pairing:

$$[-,-] : \mathfrak{g} \otimes_K \mathfrak{g} \longrightarrow \mathfrak{g} \quad \text{(the "Lie bracket")}$$

such that:

1. $[-,-]$ is skew-symmetric:

$$[x,y] = -[y,x], \quad \forall x,y \in \mathfrak{g}$$

2. $[-,-]$ satisfies the Jacobi identity:

$$\forall x,y,z \in \mathfrak{g}$$

$$[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$$

LieAlg := the category of Lie Algebras

A homomorphism between two Lie Algebras:

Lie Algebra #1 : $\mathfrak{g}$ and $[-,-]_{\mathfrak{g}}$
Lie Algebra #2 : $\mathfrak{h}$ and $[-,-]_{\mathfrak{h}}$

homomorphism: $\varphi : \mathfrak{g} \longrightarrow \mathfrak{h}$
linear map

$$\forall x,y \in \mathfrak{g}, \quad \varphi([x,y]_{\mathfrak{g}}) = [\varphi(x), \varphi(y)]_{\mathfrak{h}}$$

A Crash Course IN SMOOTH MANIFOLDS

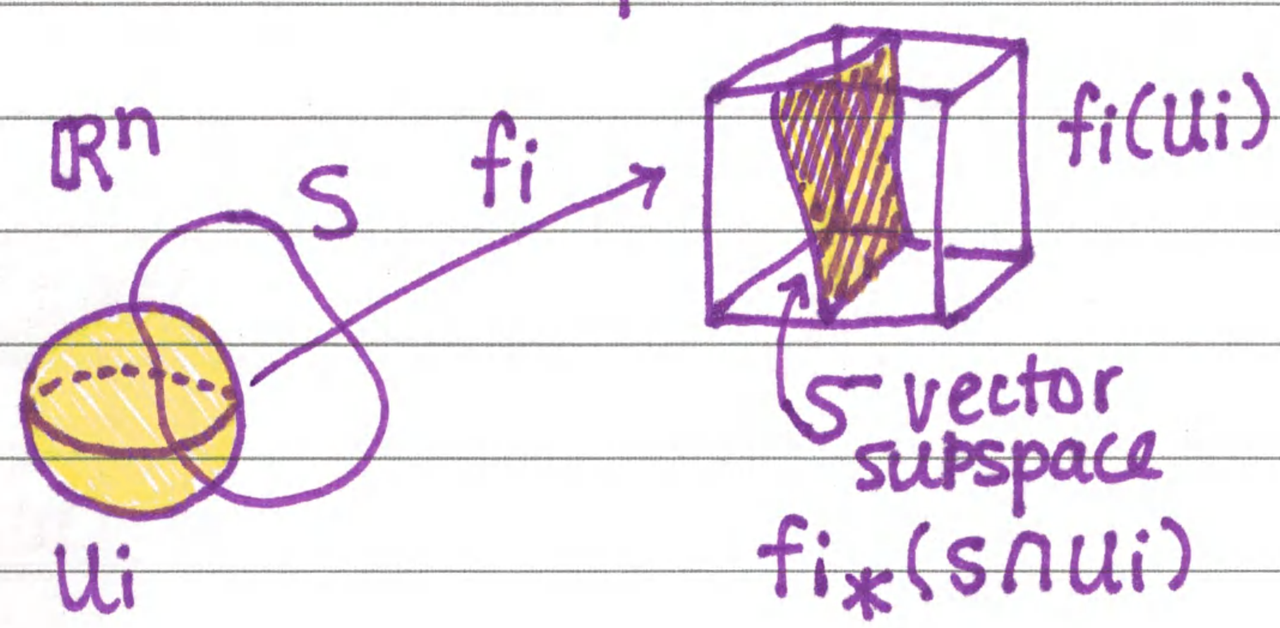
PREDEFINITION 1

A smooth manifold is a subset $S \subset \mathbb{R}^n$
(of dimension $d \geq 0$) (for $n \geq 0$, $n \geq d$)
such that

1. $\exists$ open cover $\{U_i\}_{i \in I}$ of S
2. $\forall i \in I$, $\exists f_i : U_i \longrightarrow \mathbb{R}^n$
   (diffeomorphism ; f_i, f_i^{-1} are C^∞)
3. $f_{i*}(S \cap U_i) = \{ x \in \mathbb{R}^n \mid \forall j > d, x_j = 0 \}$

C^∞ := infinitely differentiable or "smooth" functions

condition ③ says that locally (up to a diffeomorphism) S looks like the embedding of a vector space into another vector space.



deform U_i by f_i such that within U_i, S looks "straight" (locally Euclidean)

A **morphism** (smooth map) between smooth manifolds S and T

$$\begin{array}{ccc} S & \xrightarrow{f} & T \\ \cap & & \cap \\ \mathbb{R}^n & & \mathbb{R}^m \end{array}$$

is a map of underlying sets

$$U(S) \longrightarrow U(T)$$

such that $\exists g : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ $(g \in C^\infty)$

whose restriction to S is f $(g|_S = f)$

[That is, the map of underlying sets can be extended to the map of ambient spaces.]

Man := the category of smooth manifolds

EXAMPLE 1 $\mathbb{R}^n$ is a smooth manifold
$(U = \mathbb{R}^n)$

EXAMPLE 2 A single point $p \in \mathbb{R}^n$ is a smooth manifold.

EXAMPLE 3 $\underset{\text{open}}{U \subset \mathbb{R}^n}$, $U = \bigcup_{i \in I} B_i$ (open balls)

each open ball is diffeomorphic to $\mathbb{R}^n$, and

$$B_i \xrightarrow{\cong} \mathbb{R}^n$$

EXAMPLE 4

For a smooth map $f: \mathbb{R}^n \longrightarrow \mathbb{R}^r$ we can specify a condition under which the preimage of a subset $K \subset \mathbb{R}^r$ under f is a manifold

That is, if $K \subset \mathbb{R}^r$ is a smooth manifold, then satisfying a certain condition means that $f^*(K)$ is a smooth manifold.

CONDITION $\forall x \in \mathbb{R}^n$, $\operatorname{rank}((Df)(x)) = r$

... which means that...

the map has full rank, or rather, the derivative at any point is surjective (there are no critical points, hence, "smooth")

USING THE RANK CONDITION: $(r=1)$

Let $f: \mathbb{R}^n \longrightarrow \mathbb{R}$

$$x \longmapsto \sum_i x_i^2$$

$$(Df)(x) = \begin{pmatrix} 2x_1 & 2x_2 & \cdots & 2x_n \end{pmatrix}$$

$$\operatorname{rank}((Df)(x)) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \neq 0 \end{cases}$$

so, provided we avoid 0 in the image, the rank is the dimension of $\mathbb{R}$ $(\dim = 1)$ $\longrightarrow$

→ The rank condition is satisfied!

So, choose a smooth submanifold of $\mathbb{R} \setminus \{0\}$; its preimage is a smooth submanifold of $\mathbb{R}^n$:

For some $a \in \mathbb{R}$, $a \neq 0$ choose $\{a^2\}$.

$\{a^2\} \subset \mathbb{R}$ is a smooth submanifold, so,

$$f^*\{a^2\} = \{x \in \mathbb{R}^n \mid \sum_i x_i^2 = a^2\}$$

is a smooth submanifold called "THE SPHERE OF RADIUS $a>0$!"

EXERCISE #1 Prove that the torus is a smooth manifold.

(You can use the same technique as above, only, a different map, f.)

→ PREDEFINITION 1: **PROS & CONS**

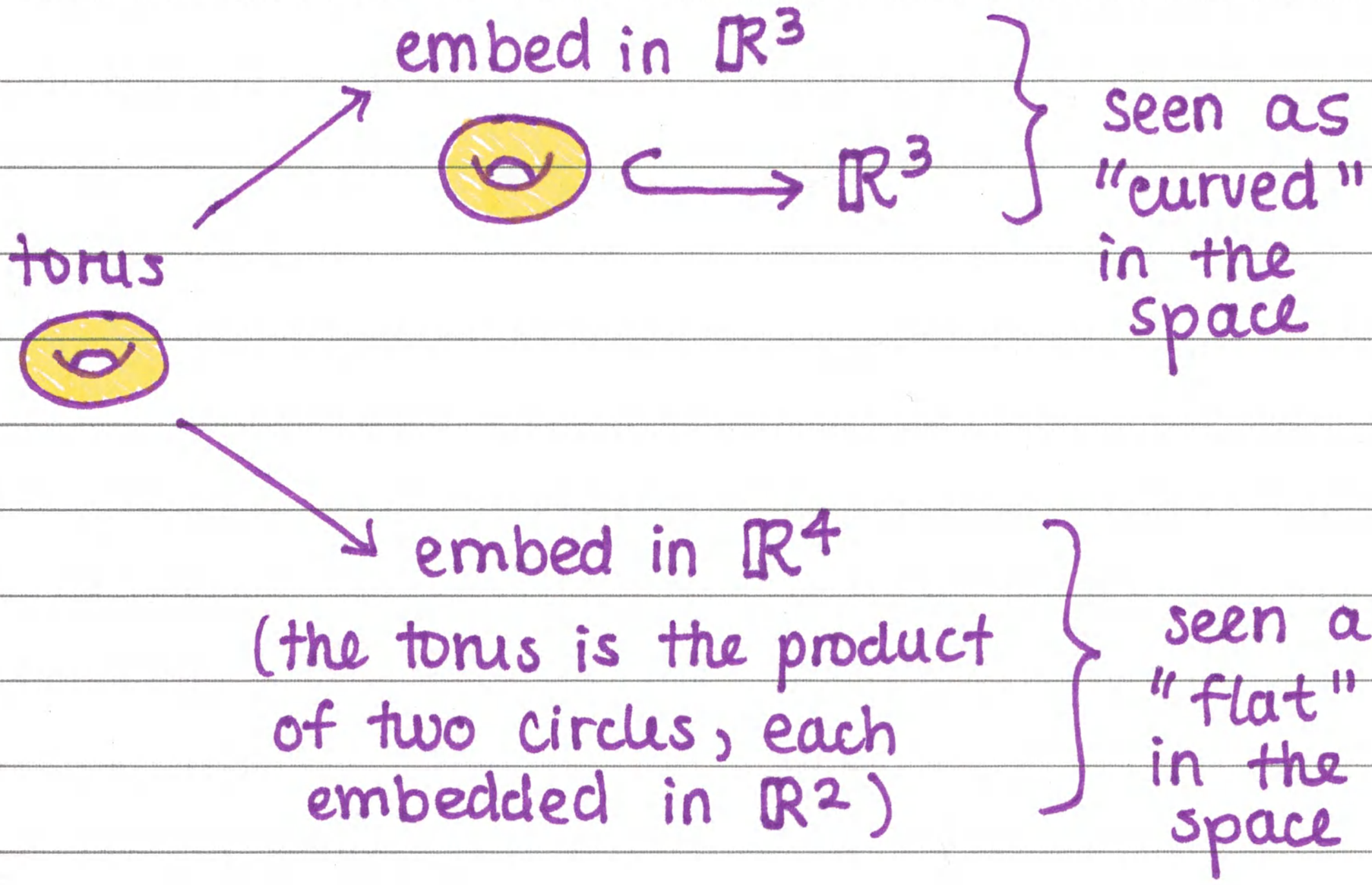
PROS This definition is easy to visualize, and interpret/understand!

CONS It is difficult to separate properties internal to the manifold from those specific to a given ambient space.

For example...

[Cons of Predefinition 1 continued...]

Consider two embeddings of the torus:



These two embeddings are isomorphic!
(And, it is difficult to determine what is specific to the torus and what is specific to the ambient space)

$\boxed{\text{CONS}}$ (continued...)

- Constructing open cover $\{U_i\}$ and maps, f_i, can be difficult.

- The embeddings of some manifolds in $\mathbb{R}^n$ may not be obvious to see.

PREDEFINITION 2

A smooth manifold (of dimension $d \geq 0$)

is a set S together with
an atlas (a collection of charts
with compatibility conditions
between charts)

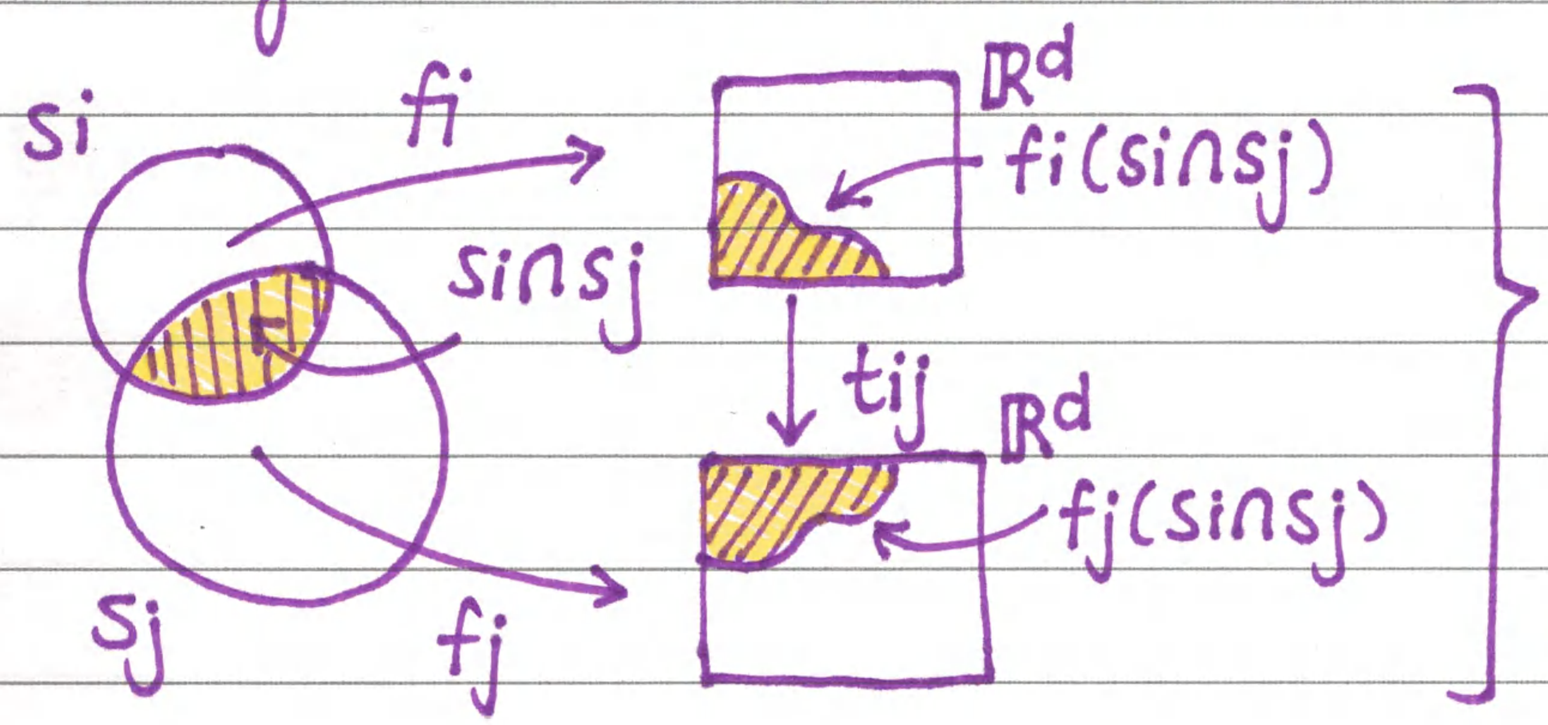
That is, subsets $S_i \subset S$ $(i \in I)$
and maps $f_i : S_i \longrightarrow \mathbb{R}^d$ $(i \in I)$

(these maps providing "local coordinates")

such that (compatibility conditions)

$\forall i, j \in I$ (where I is the indexing set of the charts)

there is a "transition map" (t_{ij}) defined by
its agreement on the intersection of charts:



and...
$t_{ji} = (t_{ij})^{-1}$,
so this map is
a diffeomorphism

$t_{ij} = f_j|_{S_i \cap S_j} \circ (f_i|_{S_i \cap S_j})^{-1}$, and t_{ij} is C^∞
(infinitely differentiable)

EXERCISE #2 Show that a smooth manifold (defined as in Predefinition 1) is a smooth manifold in the sense of Predefinition 2. Further, show that the "forgetful functor" between Predefinition 1 smooth manifolds and Predefinition 2 smooth manifolds induces a fully faithful functor of categories.

→ Given Predefinition 1, Predefinition 2, and the "rank condition" for smooth manifolds, we have enough to show a given group G is a Lie Group (as defined ...)

LIE GROUP

(i) Given a group G with unit (identity), associative binary operation (here, multiplication), and inverses (for each element, another element whose multiplication gives the identity element)

(ii) The group G is a smooth manifold (VERIFY!)

(iii) The maps $1 : * \longrightarrow G$ (identity)

$(-)^{-1} : G \longrightarrow G$ (inverses)

and $m : G \times G \longrightarrow G$ (multiplication)

are smooth maps (VERIFY!)

EXAMPLE 5 $G = GL(\mathbb{R}^n)$ is a Lie group

(i) $G = GL(\mathbb{R}^n)$ is a group

verify (called the "general linear group",
the group of invertible $n \times n$ matrices ($\det \neq 0$)
under the operation of matrix multiplication)

(ii) $G = GL(\mathbb{R}^n)$ is a smooth manifold

verify (we can use the "rank condition" to verify)

let $f : \mathbb{R}^{n^2} \longrightarrow \mathbb{R}$ $\qquad G \subset \mathbb{R}^{n \times n} = \mathbb{R}^{n^2}$

$M \longmapsto \det(M)$

since $GL(\mathbb{R}^n)$ has only matrices where $\det \neq 0$
(we skip 0 in the image), the image of
$GL(\mathbb{R}^n)$ has full rank;

the preimage of nonzero real numbers
under this map is open in $\mathbb{R}^{n^2}$,
so $GL(\mathbb{R}^n) = f^*(\{a \in \mathbb{R} \mid a \neq 0\})$
is a smooth submanifold of $\mathbb{R}^{n^2}$.

(iii) inverse map $(-)^{-1} : G \longrightarrow G$

$M \longmapsto M^{-1}$

verify

and since $M^{-1} = \frac{1}{\det M} B^t$,

where $B_{ij} = (-1)^{i+j} \det(\text{minor}(M)_{i,j})$

[$\det M$ is smooth and nonvanishing]

this map $(-)^{-1}$ is smooth (C^∞).

continued...

multiplication map $m : G \times G \longrightarrow G$

verify

and a map

$$m : \mathbb{R}^{n^2} \times \mathbb{R}^{n^2} \longrightarrow \mathbb{R}^{n^2}$$
$$(x_{ij}, y_{kl}) \longmapsto z_{pq}$$

where $z_{pq} = \sum_h x_{ph} \cdot y_{hq}$

a polynomial of degree 2

[and polynomials are smooth (C^∞),]

this map m is smooth (C^∞).

EXERCISE #3 Show that $SL(\mathbb{R}^n)$

[$SL(\mathbb{R}^n) = \{ g \in GL(\mathbb{R}^n) \mid \det g = 1 \}$] is a Lie group.

→ Predefinition 2 (continued...)

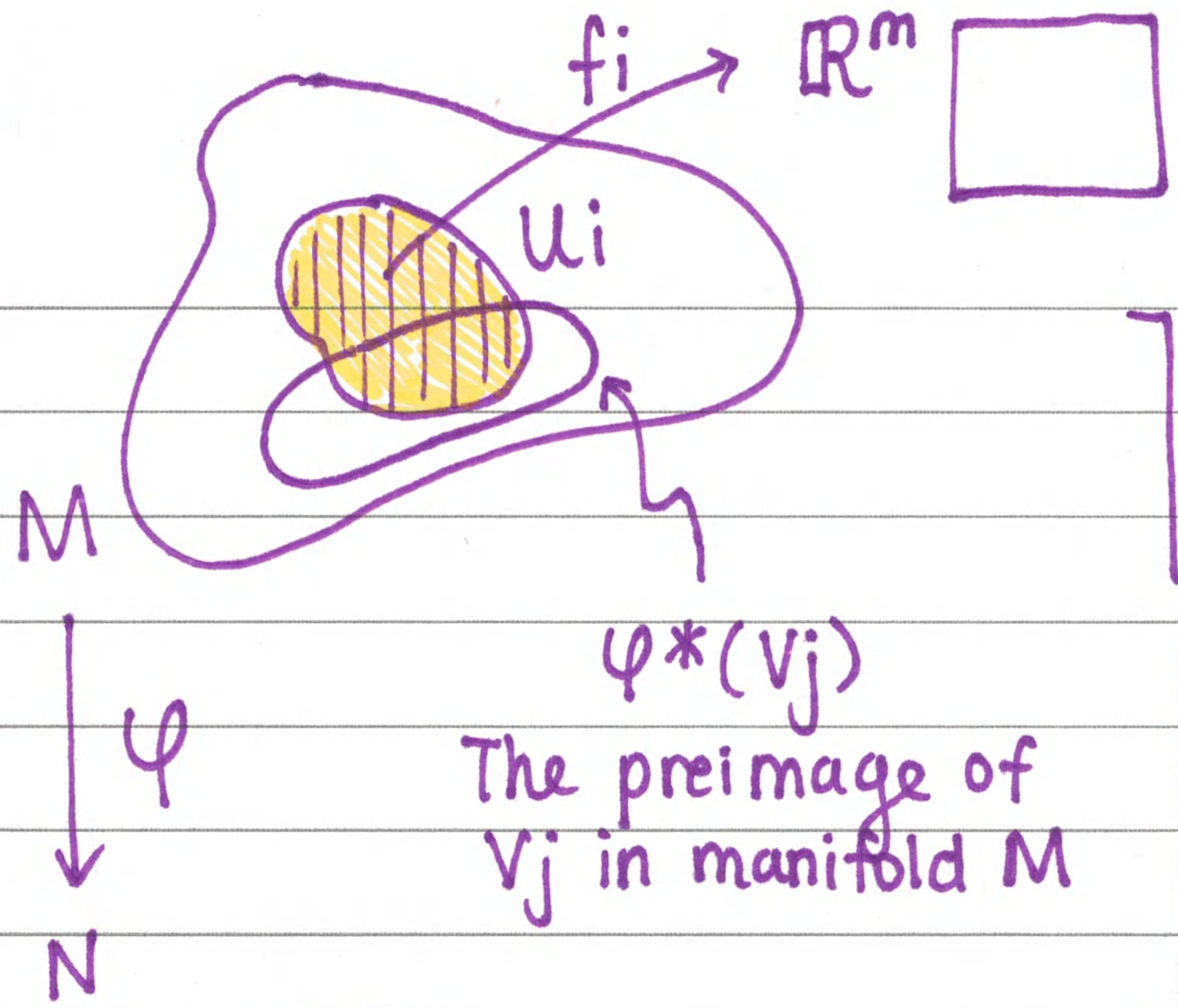
A morphism of smooth manifolds

$$\varphi : M \longrightarrow N$$

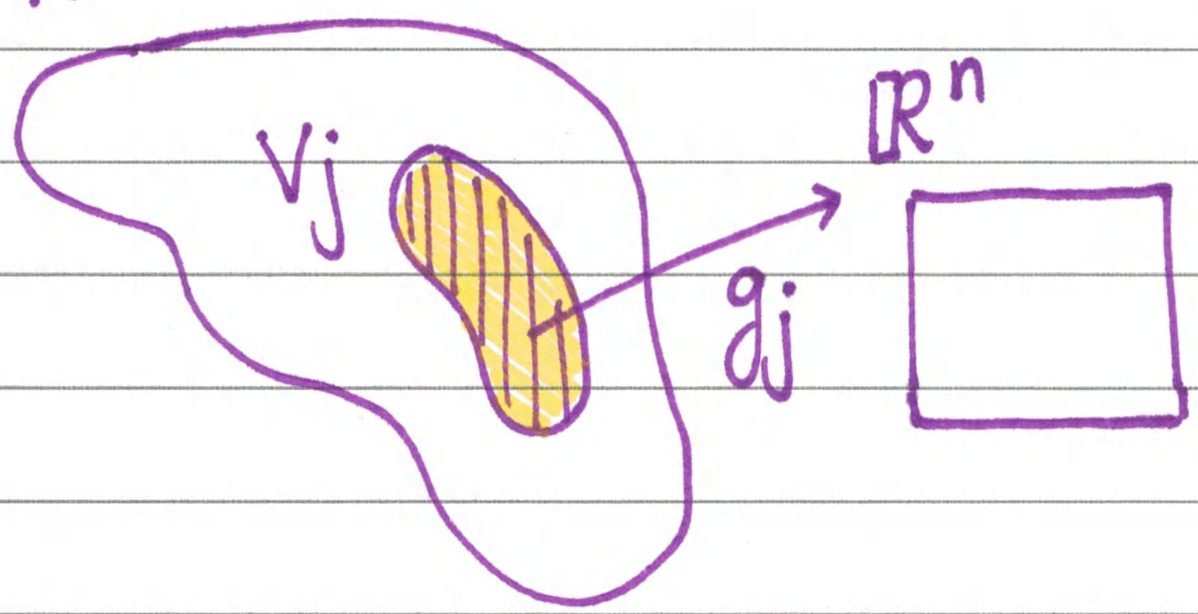
is a map of underlying sets:

$$U(M) \longrightarrow U(N)$$

and $\forall$ charts (U_i, f_i) in M
and (V_j, g_j) in N, ...

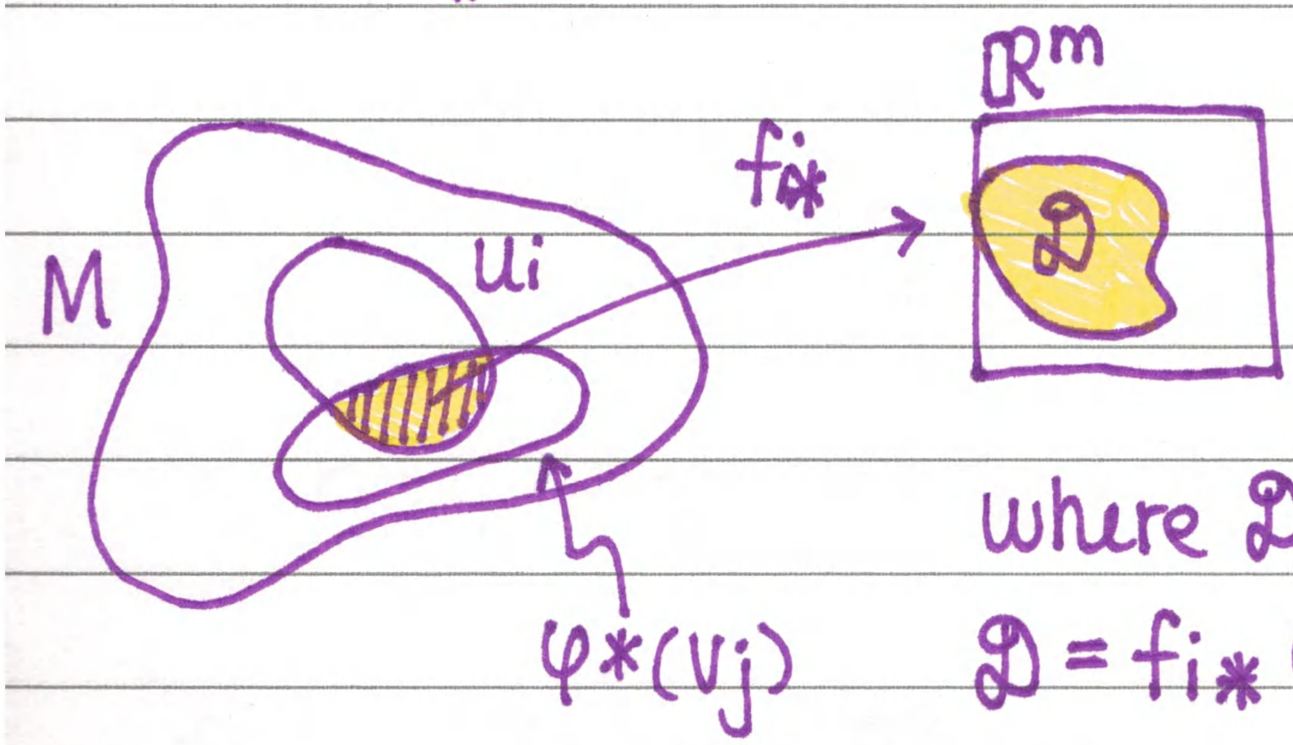


Consider $U_i \cap \varphi^*(V_j)$



To express local coordinates, we define a map:

$$f_{i*} : U_i \cap \varphi^*(V_j) \longrightarrow \mathbb{R}^m$$



where $\mathcal{D}$ is an open subset of $\mathbb{R}^m$

$\mathcal{D} = f_{i*}(U_i \cap \varphi^*(V_j))$

and then we map this open set $\mathcal{D} \subset \mathbb{R}^m$ to $\mathbb{R}^n$:

that is...

$$g_j \circ \varphi \circ f_i^{-1}\big|_{\mathcal{D}} : \mathbb{R}^m \longrightarrow \mathbb{R}^n \text{ is } C^\infty.$$

EXAMPLE 6 In Example 4, we defined the sphere of radius $a>0$ as a smooth manifold using the rank condition.

Here,
we define S^n (the n-dimensional sphere, radius $=1$) in the language of Predefinition 2.

Underlying set : $S = \mathbb{R}^n \sqcup \{\infty\}$

The atlas consists of two charts :

chart #1: $\mathcal{U}_1 = \mathbb{R}^n$
& map $f_1 : \mathcal{U}_1 \xrightarrow{id} \mathbb{R}^n$

chart #2: $\mathcal{U}_2 = \mathbb{R}^n \setminus \{0\} \sqcup \{\infty\}$
& map $f_2 : \mathcal{U}_2 \longrightarrow \mathbb{R}^n$
$\infty \longmapsto 0$
$x \longmapsto x/\|x\|_2$

and we verify the compatibility of chart #1 and chart #2.

$\mathcal{U}_1 \cap \mathcal{U}_2 = \mathbb{R}^n \setminus \{0\}$

$f_1 : \mathcal{U}_1 \cap \mathcal{U}_2 \longrightarrow \mathbb{R}^n$ by inclusion
$f_2 : \mathcal{U}_1 \cap \mathcal{U}_2 \longrightarrow \mathbb{R}^n$ by $x \longmapsto x/\|x\|_2$

That is, verify that the map $f_2 \circ f_1^{-1} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ is smooth.

EXERCISE #4 Show that $\mathbb{R}P^n$ $[\mathbb{R}P^n := S^n/x \sim -x]$ is a smooth manifold.

Further, show that the quotient map $S^n \longrightarrow \mathbb{R}P^n$ is smooth (C^∞).

DEFINITION There is a forgetful functor from the category of manifolds to the category of topological spaces:

$$U^{Top} : Man \longrightarrow Top$$

A manifold $M \in Man$ maps to a set $U^{Top}(M)$, and we equip this set with a subbase:

$$\{ f_i^* V \mid i \in I,\ V \text{ open in } \mathbb{R}^d \}$$

The closure of this set under arbitrary unions and finite intersections are the open sets of the topological space.

NOTE From now on, assume topological spaces are paracompact and Hausdorff.

* For smooth manifolds *

paracompact $\Leftrightarrow$ every connected component is second-countable

DEFINITION There is a functor from the category of sets to the category of smooth manifolds in that every set can be identified with a 0-dimensional manifold where each element in the set is its own connected component.

$\mathrm{Set} \longrightarrow \mathrm{Man}$

PREDEFINITION 2: PROS & CONS

PROS The ambient space (and its complications) are gone!

CONS

- Local coordinate systems can be difficult to work under (everything must be verified in local coordinates, and the preservation under transition maps must be checked)

- Geometric ideas can be obscured by local coordinates

- The construction of "good" (easy to work with ...) charts and atlases may not be easy or obvious.

NOTION 1: PRESHEAF

DEFINITION A "presheaf" over a category A is a functor of the form:

$$\mathcal{F} : A^{op} \longrightarrow \mathrm{Set}$$

(a contravariant functor)

"fiber"

$a \in Ob(A)$

The set $\mathcal{F}(a)$ is called the "fiber" of the presheaf.

"section"

The elements of $\mathcal{F}(a)$ are called "sections" of $\mathcal{F}$ over a.

...ON A TOPOLOGICAL SPACE...

the category $A = \mathrm{open}(T)$

(the open sets of topological space T)

so the presheaf on T is...

$$\mathcal{F} : \mathrm{open}(T)^{op} \longrightarrow \mathrm{set}$$

objects: open sets of T
morphisms: inclusions

NOTE: The target category set can be replaced by other arbitrary target categories.

- $\mathrm{Open}(T)$ is the "poset of open sets".
- There is a category of presheaves (called $\mathrm{Set}^{C^{op}}$, for category C)

■ THE PRESHEAF OF CONTINUOUS REAL FUNCTIONS ON T.

EXAMPLE 7 For topological space $T = \mathbb{R}^n$ or M $\forall M \in \mathrm{Man}$ (the category of smooth manifolds)

A presheaf on T

$$\mathcal{F} : \mathrm{Open}(T)^{op} \longrightarrow \mathrm{Alg}_{\mathbb{R}}$$

$$\underset{\text{open}}{U \subset M} \longmapsto \underset{\substack{\text{smooth real} \\ \text{valued functions} \\ \text{on } U}}{C^\infty(U, \mathbb{R})}$$

and for an inclusion of open sets $U \subset V$ we have a map between smooth functions on V to smooth functions on U that is a restriction:

$$U \subset V \longmapsto C^\infty(V, \mathbb{R}) \longrightarrow C^\infty(U, \mathbb{R})$$

(contravariant)

> Intuitively, this is a good way to think of a presheaf (or sheaf; will be defined shortly...) A presheaf is the set of functions of some type defined on the open sets of the space.

We can similarly define:

- THE PRESHEAF OF SMOOTH FUNCTIONS ON T,
- THE PRESHEAF OF HOLOMORPHIC FUNCTIONS ON T,
- THE PRESHEAF OF MEASURABLE FUNCTIONS ON T,
- THE PRESHEAF OF BOUNDED FUNCTIONS ON T.