HOMOLOGICAL MATH

Algebra 6321

DEFINITION: **RELATIVE CATEGORY**

A relative category is a pair $(\mathcal{C}, W)$
where $\mathcal{C}$ (called the "underlying category")
is a category
and $W \subseteq \mathcal{C}$ (called the "category of weak equivalences")
is a subcategory
such that
$Ob(W) = Ob(\mathcal{C})$
and $Mor(W)$ are designated
to be "weak equivalences" in $\mathcal{C}$.

since $Ob(W) = Ob(\mathcal{C})$
(all the objects are
the same...)
the category W will
contain all the
identity morphisms

These "weak equivalences"
are also closed under
composition.

"A category with a class
of morphisms singled out"
- nLab

(This is essentially taking a category $\mathcal{C}$ and specifying
a subclass of its morphisms to serve a particular
function in new developments)

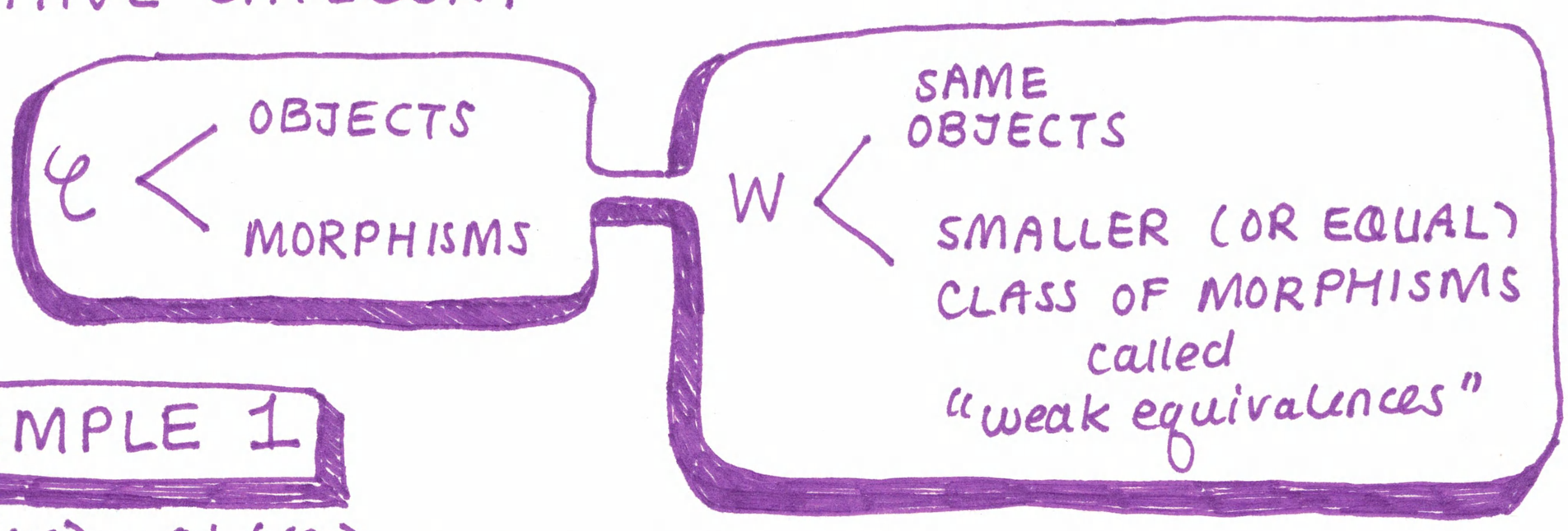
Similarly, a **RELATIVE FUNCTOR** is
a functor between relative categories that
preserves weak equivalences.

Recall, a homotopy equivalence between X and Y
is a morphism $f: X \longrightarrow Y$
such that $\exists\, g: Y \longrightarrow X$
and $g \circ f \sim id_X$, $f \circ g \sim id_Y$
(where "$\sim$" means homotopic,
in the topological sense)

A weak homotopy equivalence between X and Y
is a morphism $f: X \longrightarrow Y$
which induces isomorphisms on
all homotopy groups

(In homological algebra, when the objects X, Y
are chain complexes, a chain map inducing isomorphisms
on all homotopy groups is called a "quasi-isomorphism.")

RELATIVE CATEGORY



EXAMPLE 1

$Ob(W) = Ob(\mathcal{C})$ (necessary)
$Mor(W) = Mor(\mathcal{C})$

This is called the "maximal homotopical category"

EXAMPLE 2

$Ob(W) = Ob(\mathcal{C})$ (necessary)

$Mor(W) = \{ f \in Mor(\mathcal{C}) \mid f \text{ is an isomorphism} \}$

This is called the "minimal homotopical category!"

EXAMPLE 3

$Ob(W) = Ob(\mathcal{C})$ (necessary)

$Mor(W) = \{ f \in Mor(\mathcal{C}) \mid f \text{ is an identity morphism for some object } x \in Ob(\mathcal{C}); f: x \longrightarrow x \}$

$f = id_X$

This is called the "minimal relative category!"

EXAMPLE 4

When $\mathcal{C} = sSet$

$Ob(W) = Ob(\mathcal{C})$ (necessary)

$Mor(W) = \{ f \in Mor(\mathcal{C}) \mid f \text{ is a simplicial homotopy equivalence} \}$

EXAMPLE 5

When $\mathcal{C} = sSet$

$Ob(W) = Ob(\mathcal{C})$ (necessary)

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EXAMPLE 6

When $\mathcal{C} = Ch$

$Ob(W) = Ob(\mathcal{C})$ (necessary)

$Mor(W) = \{ f \in Mor(\mathcal{C}) \mid f \text{ is a chain homotopy equivalence} \}$

MORE! ⟶

EXAMPLE 7 When $\mathcal{C} = Ch$

$Ob(W) = Ob(\mathcal{C})$ (necessary)

$Mor(W) = \{ f \in Mor(\mathcal{C}) \mid f \text{ is a quasi-isomorphism} \}$

EXAMPLE 7' When $\mathcal{C} = Ch$

$Ob(W) = Ob(\mathcal{C})$ (necessary)

$Mor(W) = \{ f \in Mor(\mathcal{C}) \mid f \text{ is a chain isomorphism} \}$

This is the same case as in Example 2, only applied to the category of chain complexes

EXAMPLE 8 When $\mathcal{C}$ = differential-graded algebras (DGA)

$Ob(W) = Ob(\mathcal{C})$ (necessary)

$Mor(W) = \{ f \in Mor(\mathcal{C}) \mid f \text{ is a } \underline{\text{quasi-isomorphism}} \}$

[That is, maps of DGA's that induce isomorphisms on homology groups]

EXAMPLE 9 There is a forgetful functor between

$\mathcal{C} = sGrp := Fun(\Delta^{op}, Grp)$

and $\mathcal{D} = sSet := Fun(\Delta^{op}, set)$

The image of weak equivalences in $\mathcal{C}$ under this functor are weak equivalences in $\mathcal{D}$.

Restricting $\mathcal{C}$ to abelian groups (that is, $\mathcal{C}' = sAb$) there is a correspondence:

$sAb \underset{\text{as relative categories}}{\cong} Ch_{\geq 0}$ [Dold-Kan correspondence]

EXAMPLE 10 When $\mathcal{C} = Top$

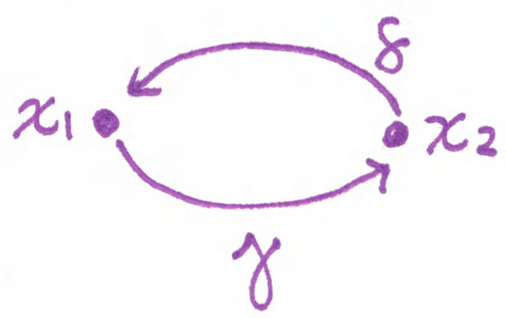
$Ob(W) = Ob(\mathcal{C})$ (necessary)

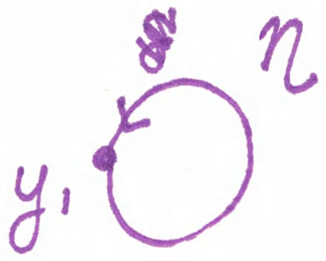
$Mor(W) = \{ f \in Mor(\mathcal{C}) \mid f \text{ is a weak homotopy equivalence} \}$

***NOTE:** Sometimes, it is only possible to define a weak equivalence in one direction (but not the other).

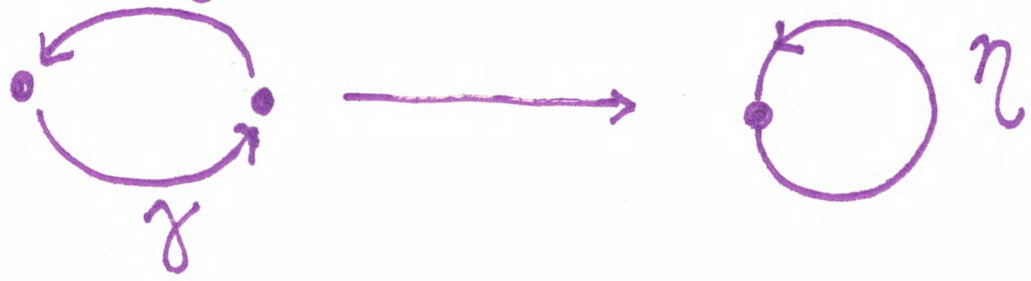
($A \xrightarrow{\sim} B$, but not, $A \xleftarrow{\not\sim} B$)

For example, consider

$X =$ 

and $Y =$ 

There is a weak equivalence $f: X \longrightarrow Y$



(collapse δ to a point, and "stretch" γ (to match η))

but there is no weak equivalence the other direction [a [1]-simplex has to map to a [1]-simplex with the same endpoints, and that is not possible here.]

We now define a **WEAK EQUIVALENCE** in a **RELATIVE CATEGORY**

relative category $(\mathcal{C}, W)$

$X, Y \in Ob(\mathcal{C})$

"X is weakly equivalent to Y"

"zigzag" refers to a chain of maps, not necessarily in the same direction

if there is a zigzag of weak equivalences ($\{f_i\}_{i \in I} \subset Mor(W)$)

such that

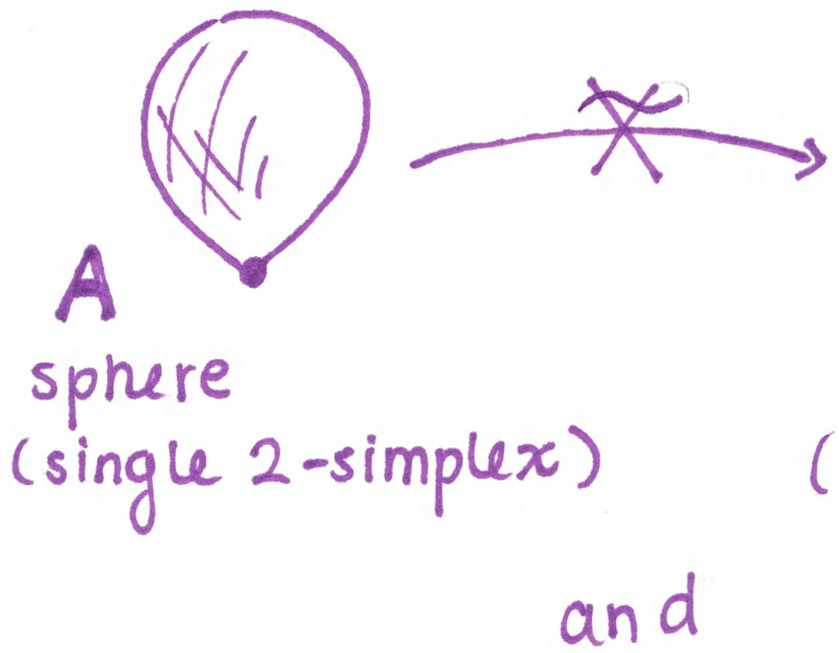
$$X = X_0 \xrightarrow[\sim]{f_0} X_1 \xleftarrow[\sim]{f_1} X_2 \xrightarrow[\sim]{f_2} X_3 \xleftarrow[\sim]{f_3} X_4 \cdots X_n = Y$$

NOTE: In the case of simplicial sets, it turns out that if X is weakly equivalent to Y, they can always be related by a zigzag of length 2 ... Wow!

An illustration of this example is as follows:

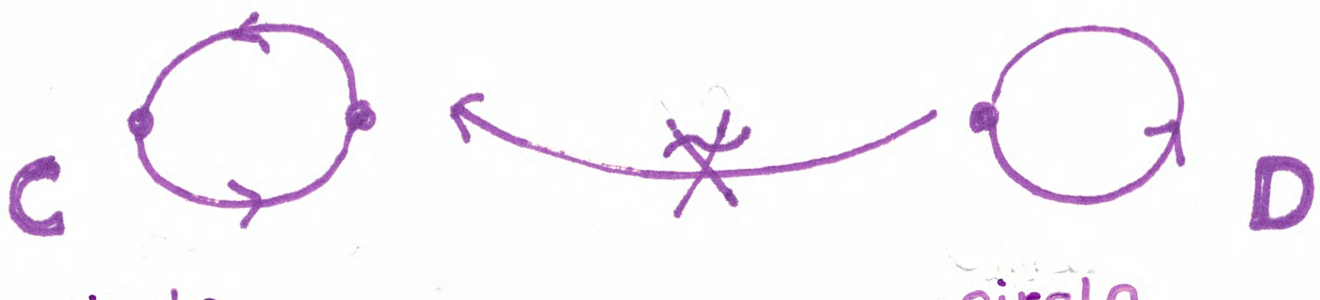
ILLUSTRATION

w.e. = weak equivalence



There is no w.e. from A to B (although there is a w.e. from B to A)

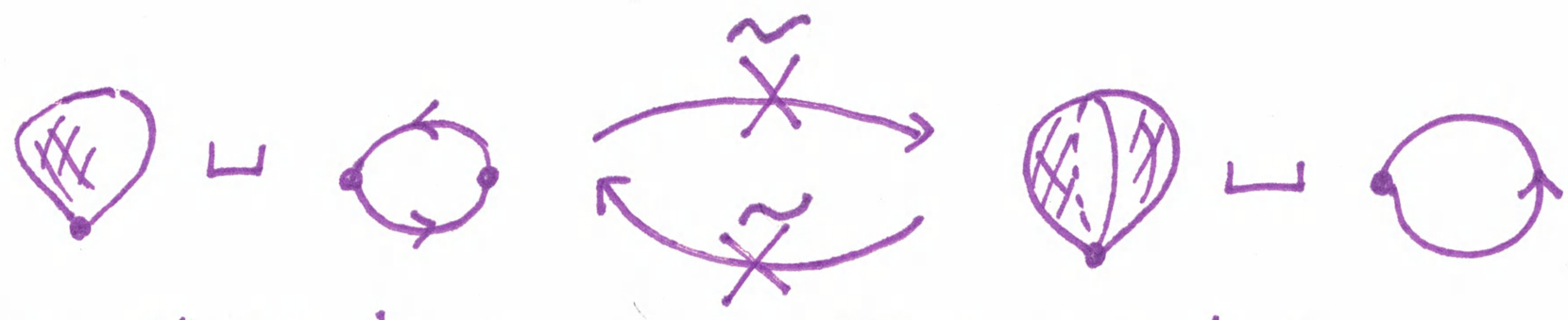
and



There is no w.e. from D to C (although there is a w.e. from C to D)

combine!

Let $X = A \sqcup C$ and $Y = B \sqcup D$



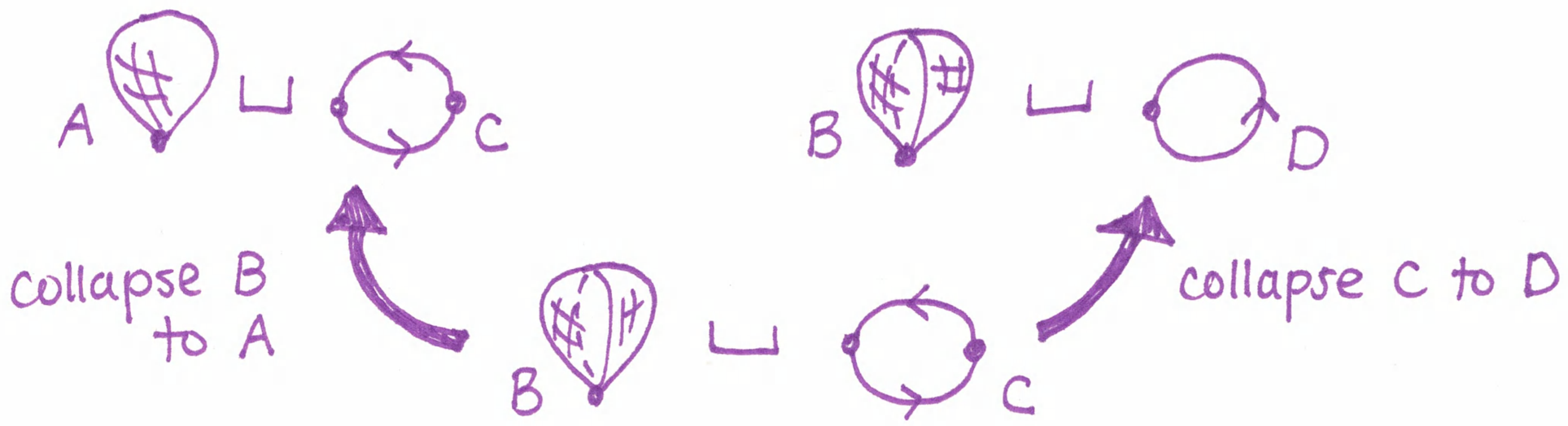
Now there is no w.e. either way! (No weak equivalence)

Let $Z = B \sqcup C$ 

There is no weak equivalence $X \not\to Y$
There is no weak equivalence $X \not\leftarrow Y$
There is no weak equivalence $X \not\to Z$
There is no weak equivalence $Y \not\to Z$

BUT →

There IS a weak equivalence $Z \longrightarrow X$
and a weak equivalence $Z \longrightarrow Y$



so there is a "zigzag" of weak equivalences!

$$X \xleftarrow{\sim} Z \xrightarrow{\sim} Y$$

Hence, X and Y are called "weakly equivalent"!

The notion of "weak equivalence" [in RELATIVE CATEGORIES] will stand in for "isomorphisms" [in ORDINARY CATEGORIES]

[That is, rather than having a single map (as with isomonphisms), we will have a zigzag of maps to specify a weak equivalence.]

In the same way that category constructions respect isomorphisms,

relative category constructions respect weak equivalences.

The "correct" way to perform these constructions is referred to by adjectives "derived" or "homotopy"

like "the derived tensor product"

or

"homotopy limits & colimits" $\longrightarrow$

These descriptions imply that the definition has been corrected to respect some property.

[Thus "Homotopical Algebra" is like "corrected Algebra"]

EXAMPLE In ordinary categories, functors preserve isomorphisms (send isomorphisms to isomorphisms).

However, in relative categories, not all functors respect weak equivalences.

For example, $hom: \mathcal{C}^{op} \times \mathcal{C} \longrightarrow^{*} Set$ (a functor)

If $X \in \mathcal{C}^{op}$,
and $Y \in \mathcal{C}$ and there is a $Y' \in \mathcal{C}$ such that

but we could have, $Y \xrightarrow{\sim} Y'$ (weakly equivalent in $\mathcal{C}$)

$$hom(X,Y) \not\longrightarrow hom(Y')$$

NOT weakly equivalent

More specifically...

If $X = \Delta^{\circ}$, $(\mathcal{C} = \Delta)$
and $Y = \Delta^{n}$
$Y' = \Delta^{\circ}$,

$$Y = \Delta^{n} \xrightarrow{\sim} Y' = \Delta^{\circ}$$

(these are weakly equivalent)

but,

$$hom(\Delta^{\circ}, \Delta^{n}) \not\longrightarrow hom(\Delta^{\circ}, \Delta^{\circ})$$

since there is no map (isomorphism)

$$[n] \not\longrightarrow [o]$$

*NOTE:
Weak is Set are isomorphisms.
In an arbitrary relative category, if w.e.'s are not specified, assume w.e.'s are isomorphisms

The hom functor does not respect weak equivalences!

$\longrightarrow$

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*NOTE: Weak equivalences in Set are isomorphisms. In an arbitrary relative category, if w.e.'s are not specified, assume w.e.'s are isomorphisms

More specifically...

If $X = \Delta^0$, $(\mathcal{C} = \Delta)$
and $Y = \Delta^n$
$Y' = \Delta^0$,

$$Y = \Delta^n \xrightarrow{\sim} Y' = \Delta^0$$

(these are weakly equivalent)

but,

$$hom(\Delta^0, \Delta^n) \not\longrightarrow hom(\Delta^0, \Delta^0)$$

since there is no map (isomorphism)

$$[n] \not\longrightarrow [0]$$

The hom functor does not respect weak equivalences!

→

EXAMPLE The Internal hom functor has a similar problem.

$$\mathrm{Hom}: sSet^{op} \times sSet \longrightarrow sSet$$

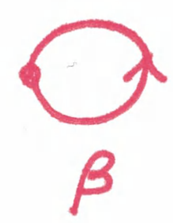
does not respect weak equivalences.

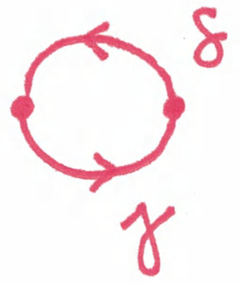
However, we can not see this from the same inputs as the previous example,

since $\mathrm{Hom}(\Delta^0, \Delta^n) \longrightarrow \mathrm{Hom}(\Delta^0, \Delta^0)$

simply picks out a map $\Delta^n \longrightarrow \Delta^0$.
(that is, it is isomorphic to the map $\Delta^n \to \Delta^0$)

Consider another example:

$X =$ 

$Y_1 =$ 

$Y_2 =$ 

There is a weak equivalence $Y_1 \longrightarrow Y_2$

(obtained by collapsing δ to a point, and "stretching" γ to η...)

but $\mathrm{Hom}(X, Y_1) \not\longrightarrow \mathrm{Hom}(X, Y_2)$

is <u>NOT</u> a weak equivalence.

[NOTE that in Y_2, there is one map (the identity map id,  that goes around the circle once. In Y_1, no map goes completely around the circle.]

EXERCISE 1 In the example above, fill in the details as to why $\mathrm{Hom}(X, Y_1) \longrightarrow \mathrm{Hom}(X, Y_2)$ is NOT a weak equivalence.

DEFINITION A category $\mathcal{C}$ is called

"cartesian closed" (c.c.)

if $\forall X \in \mathcal{C}$, the functor:

$$\mathcal{C} \longrightarrow \mathcal{C}$$

$$W \longmapsto W \times X$$

has an adjoint (right adjoint) functor:

$$\mathcal{C} \longleftarrow \mathcal{C}$$

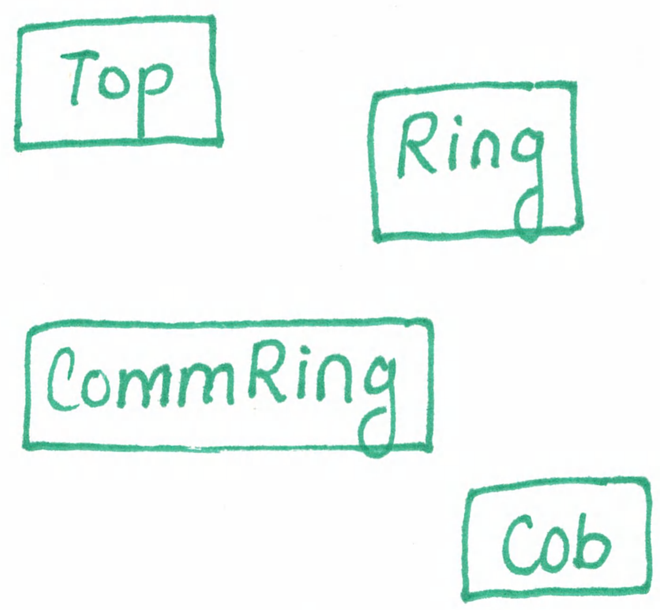
$$\mathrm{Hom}(X,Y) \longleftarrow\!\shortmid Y$$

[That is, maps $W \longmapsto W \times X$ are associated with maps $Y \longmapsto \mathrm{Hom}(X,Y)$, so called "hom-adjunction", "hom-product adjunction"]

CARTESIAN CLOSED



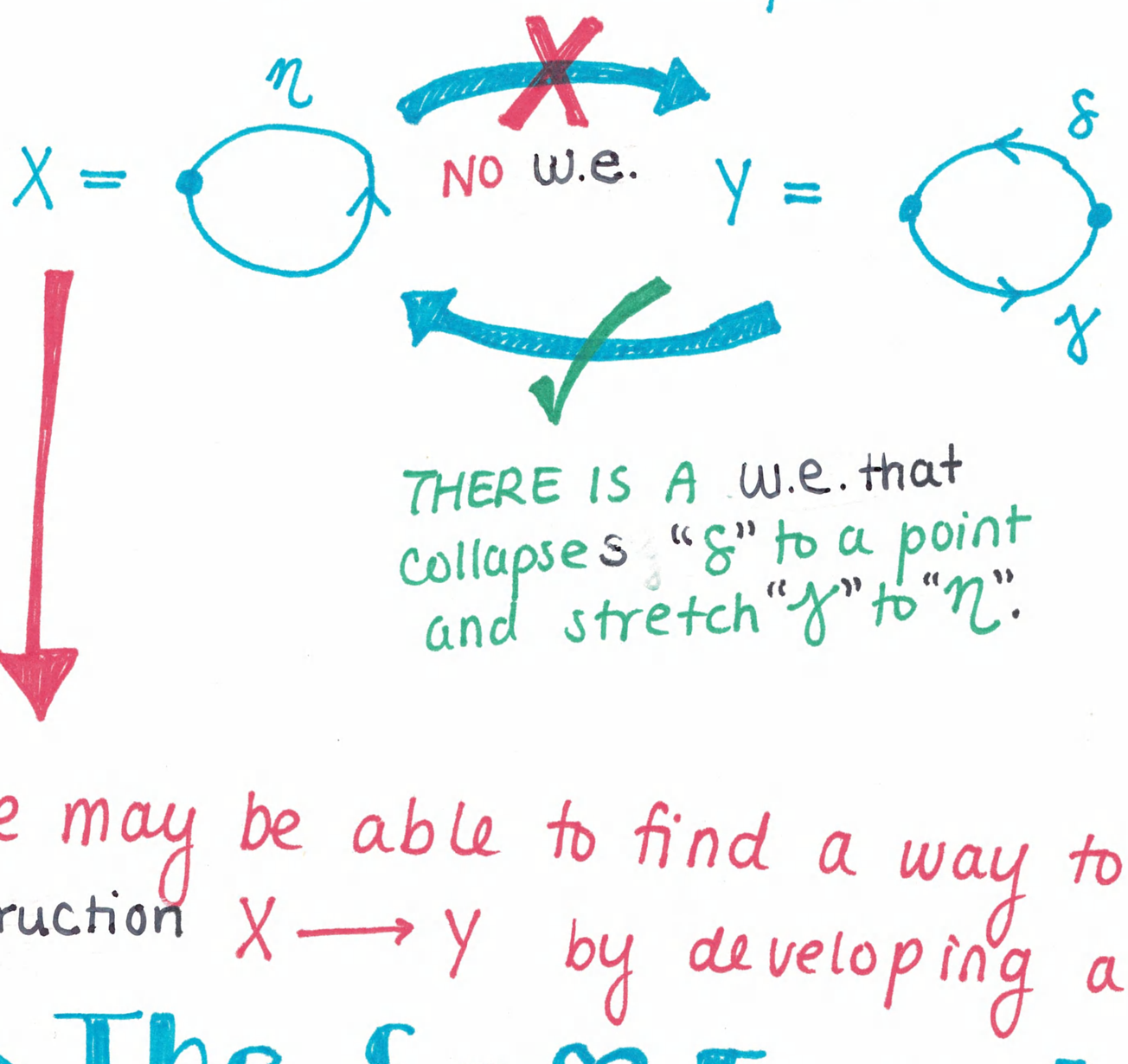
NOT CARTESIAN CLOSED



EXERCISE 2

Show that Group is NOT cartesian closed.

Recall the previous example...



We may be able to find a way to repair this construction $X \longrightarrow Y$ by developing a new notion...

The Ex^∞ Functor

Recall... the "subdivision functor"

$$sd : sSet \longrightarrow sSet$$

is a cocontinuous functor induced by

$$\Delta \longrightarrow sSet$$

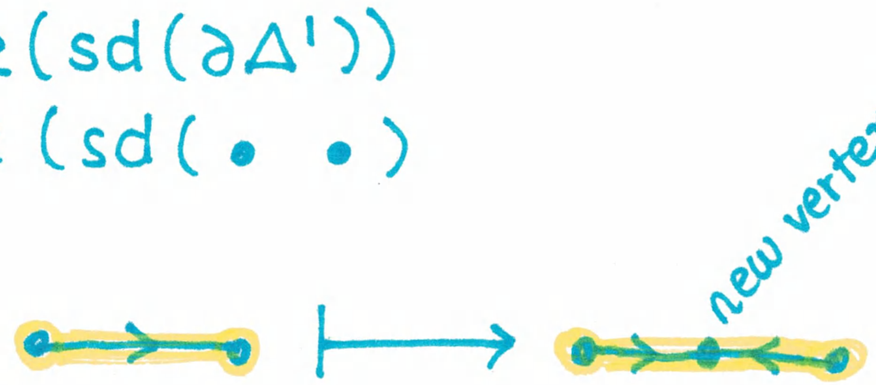
(that is, once you specify the functor on individual n-simplices, you can extend it to simplicial sets.)

0-simplices : $sd\,\Delta^0 = \Delta^0$



1-simplices : $sd\,\Delta^1 = cone(sd(\partial\Delta^1))$

$= cone(sd(\bullet\ \bullet))$

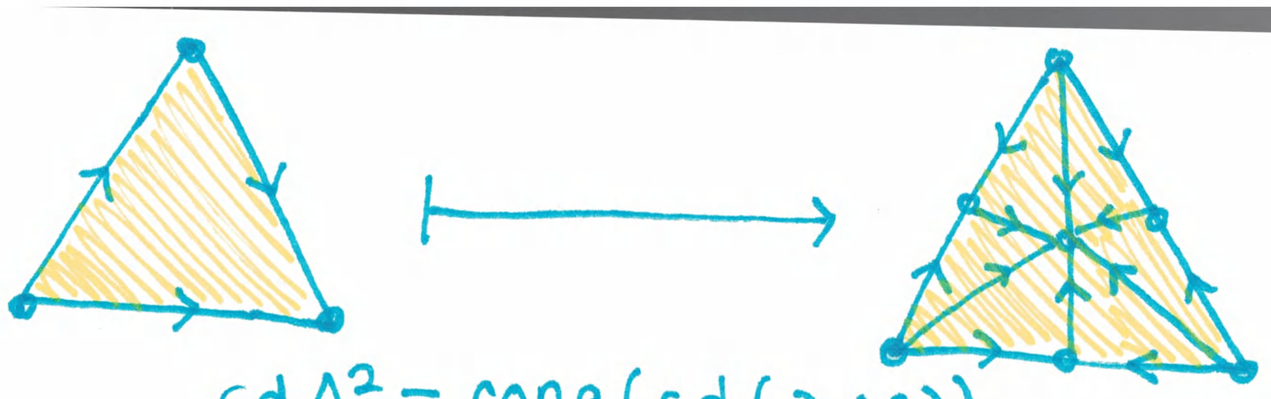


2-simplices : $sd\,\Delta^2 = cone(sd(\partial\Delta^2))$

$= cone\left(sd\left(\begin{array}{l}\rightarrow\\ \rightarrow\\ \rightarrow\end{array}\right)\right)$

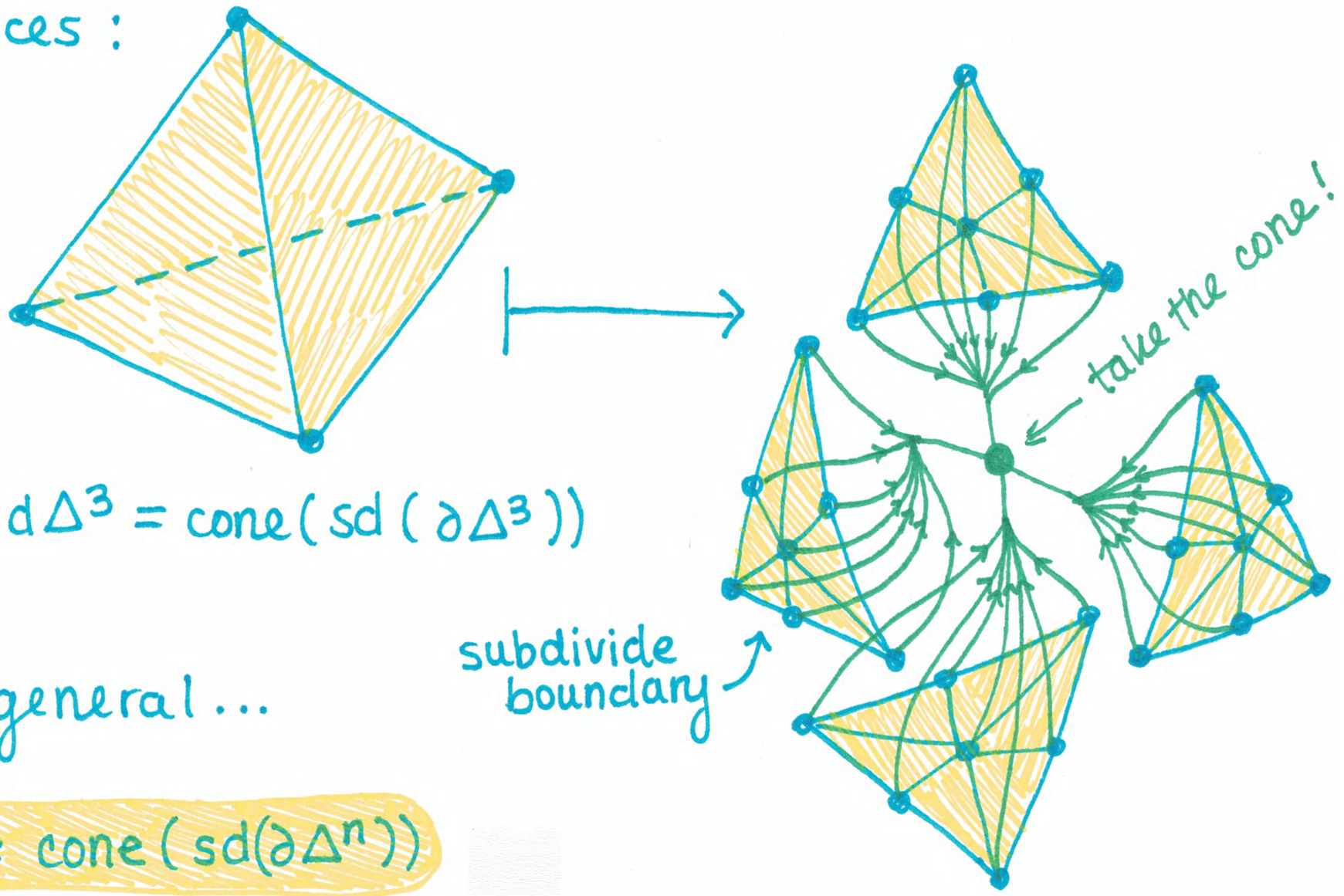
continued...

2-simplices:



$$\mathrm{sd}\,\Delta^2 = \mathrm{cone}(\mathrm{sd}(\partial\Delta^2))$$

3-simplices:



$$\mathrm{sd}\,\Delta^3 = \mathrm{cone}(\mathrm{sd}(\partial\Delta^3))$$

and in general...

$$\mathrm{sd}\,\Delta^n = \mathrm{cone}(\mathrm{sd}(\partial\Delta^n))$$

RECALL... "adjoint functor"

If the functor L is left adjoint to the functor R
(or the functor R is right adjoint to the functor L)

$$L \dashv R \qquad \begin{array}{l} L : \mathcal{C} \longrightarrow \mathcal{D} \\ R : \mathcal{D} \longrightarrow \mathcal{C} \end{array}$$

then morphisms

$L(c) \longrightarrow d$ in the category $\mathcal{D}$

can be identified with morphisms

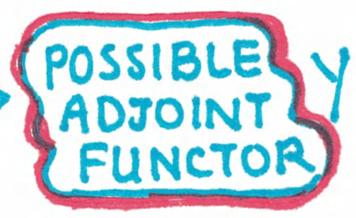
$c \longrightarrow R(d)$ in the category $\mathcal{C}$.

- Left adjoint functors preserve colimits
  (called "cocontinuous")
- Right adjoint functors preserve limits
  (called "continuous")

NOTE: The subdivision functor is cocontinuous, so it seems likely that it is left adjoint to some other functor.

This, however, is not guarenteed.

If the subdivision functor $sd : sSet \longrightarrow sSet$ had a right adjoint functor,
then we could identify morphisms

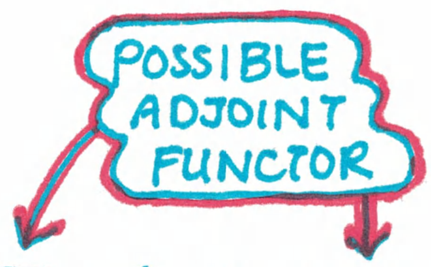
$sd\,X \longrightarrow Y$ with morphisms $X \longrightarrow$ 

Recall, the subdivision functor is induced by a map

$$\Delta \longrightarrow sSet$$

so, we need only determine what happens on an arbitrary n-simplex: Δ^n

LET $X = \Delta^n$

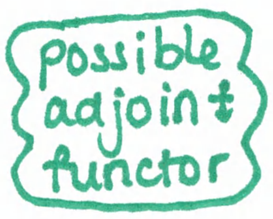


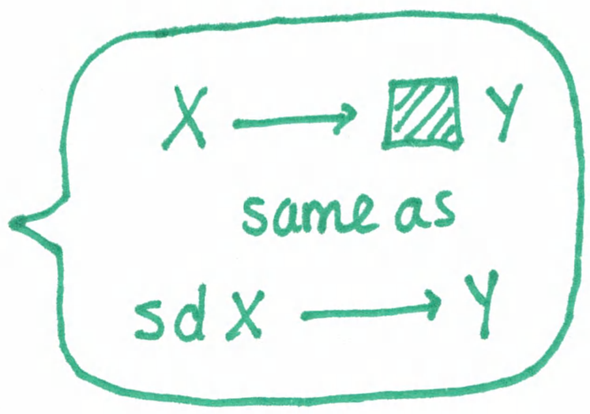
We want to define a map: $hom(X, \blacksquare\, Y) \cong (\blacksquare \times Y)_n$
(isomorphism)

In the example given previously, where

$X =$  and $Y =$ 

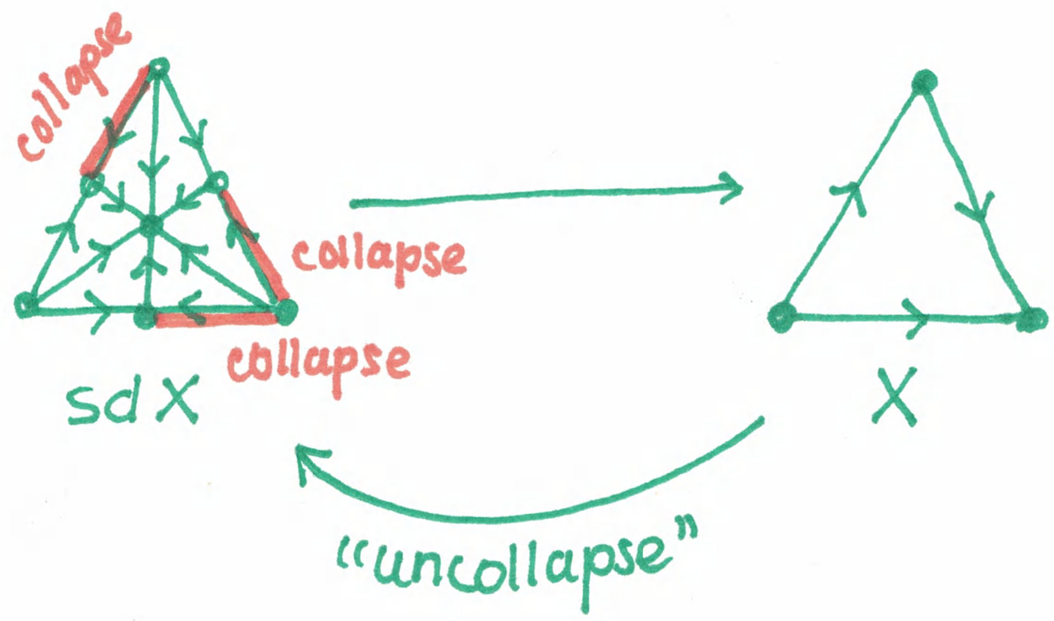
we want to map

$X \longmapsto$ 



and the most natural adjoint functor would be something that effectively subdivides X to make it possible to map to Y.

We want to define a sort of "uncollapsing map" that uncollapses X in the same way that we would collapse sdX to recover the original space



We can construct this map inductively; if we determine what happens to the cone vertex, the rest of the map will be uniquely determined from that.

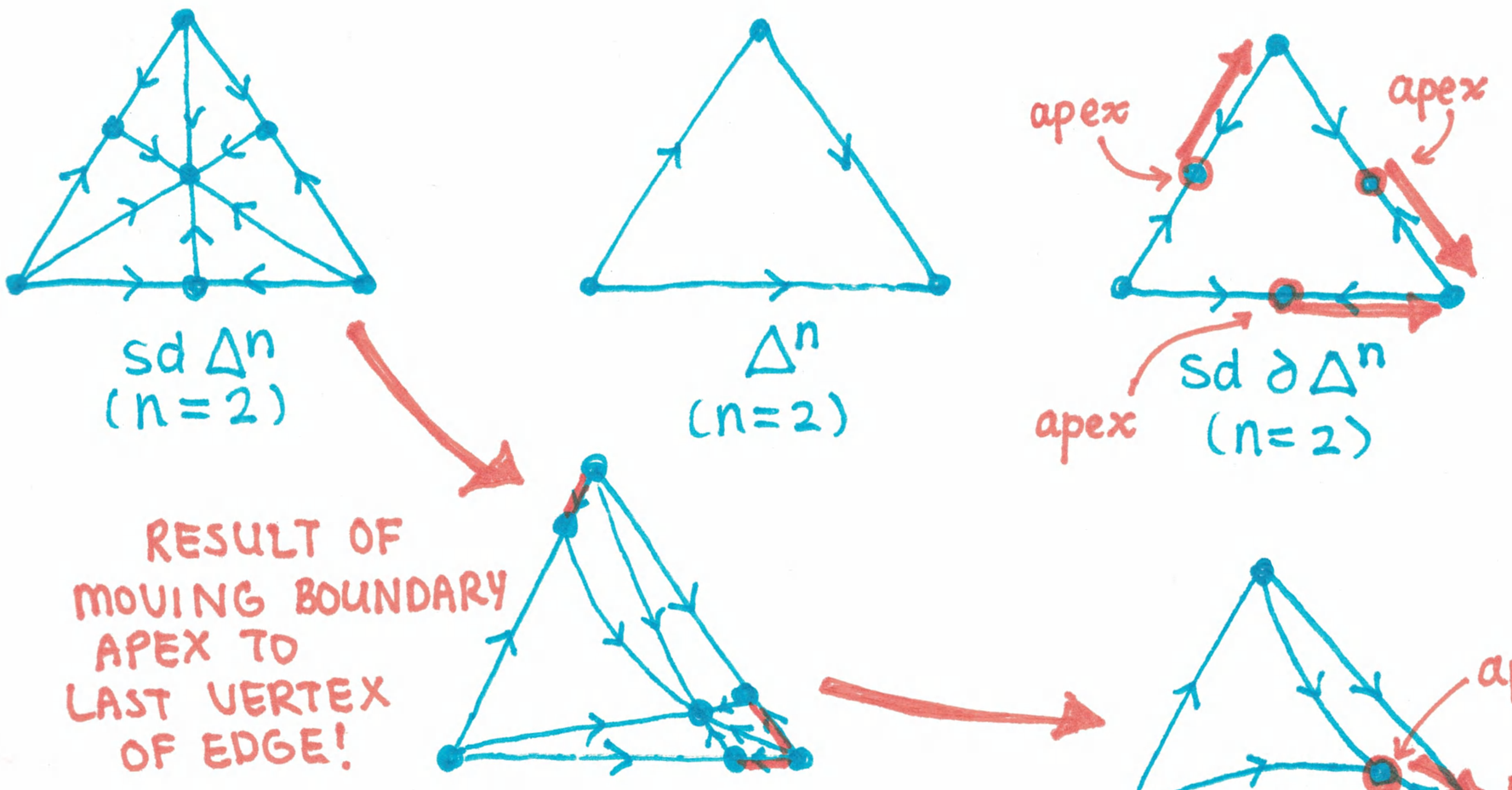
We can extend the map

$$sd\,\partial\Delta^n \longrightarrow \partial\Delta^n$$

to a map

$$\text{cone}(sd\,\partial\Delta^n) \longrightarrow \partial\Delta^n$$

by sending the apex of the cone to the last vertex of Δ^n!



SO, the map $sdX \longrightarrow X$ is called the **LAST VERTEX MAP**

We will call this new adjoint functor
(right adjoint to subdivision)

$$\square = \mathcal{E}x$$

so now maps

$sd\,X \longrightarrow Y$ correspond to maps $X \longrightarrow \mathcal{E}x\,Y$

Repeating this construction...

$$X \ldots \quad \mathcal{E}xX \ldots \quad \mathcal{E}x(\mathcal{E}xX) \ldots$$

we can form a diagram:

$$X \longrightarrow \mathcal{E}xX \longrightarrow \mathcal{E}x^2X \longrightarrow \mathcal{E}x^3X \longrightarrow \ldots$$

THE COLIMIT OF THIS DIAGRAM

$$\operatorname*{colim}_{n\geq 0}\left(X \longrightarrow \mathcal{E}xX \longrightarrow \mathcal{E}x^2X \longrightarrow \mathcal{E}x^3X \longrightarrow \ldots\right)$$

IS CALLED $\mathcal{E}x^\infty X$

and the $\mathcal{E}x^\infty$- functor is the functor:

$$\mathcal{E}x^\infty : sSet \longrightarrow sSet$$
$$X \longmapsto \mathcal{E}x^\infty X$$

} THIS FUNCTOR IS ALSO CALLED "KAN'S FIBRANT REPLACEMENT FUNCTOR"

PROPOSITION:

$X \longrightarrow \mathcal{E}x^\infty X$ is a natural weak equivalence

(To be proved in the following discussion!)

PROPERTIES OF Ex^∞

1. $Ex^\infty(X)$ is a Kan complex, $\forall X \in sSet$.
2. Ex^∞ preserves "fibrations"
3. Ex^∞ preserves finite limits.
4. Ex^∞ preserves $[0]$-simplices.

TO BE PROVED LATER!

BUT FOR NOW...

5. $X \longrightarrow Ex^\infty X$ is a weak equivalence.

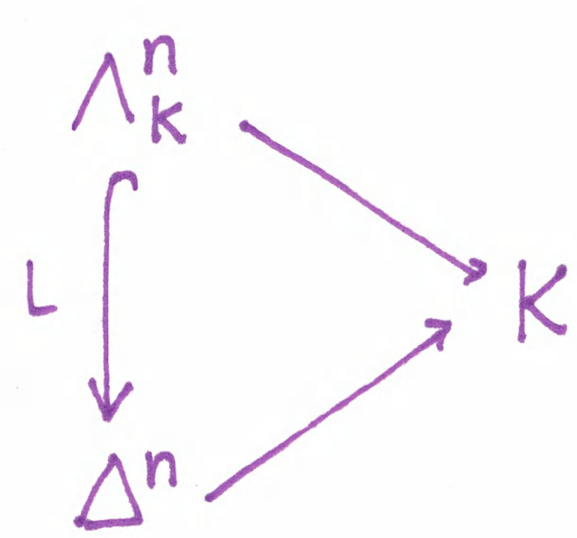
In order to fully understand these properties, we can recall several definitions:

KAN COMPLEX

A Kan complex is a simplicial set K that has a (nonunique) lifting property with respect to horn inclusions.

$\forall n \geq 1$, $\forall (0 \leq k \leq n)$, and $\forall$ map $\Lambda^n_k \longrightarrow X$,
$\exists$ map (noncanonical, nonunique) $\Delta^n \longrightarrow X$

such that the following diagram commutes:



NOTE: The map $\Lambda^n_k \longrightarrow K$ need not be injective. $\longrightarrow$

→ This is to say that...

> ...the existence of retractions of any geometric simplex to any of its horns means that all horns in the singular complex can be filled with genuine simplices, the Kan filler condition.
> — nLab

Now, suppose we have a simplicial set K and a "lifting map" $\Lambda^n_k \longrightarrow K$

we have a diagram

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & K \\ \downarrow & & \\ \Delta^n & & \end{array}$$

and the question is...

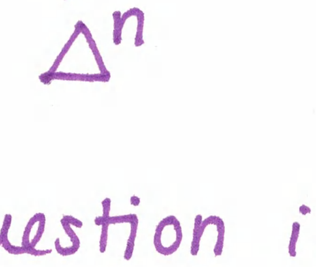
Can we obtain a lift?

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & K \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

n=1

$$\begin{array}{ccc} \Lambda^1_k & \longrightarrow & K \\ \downarrow & & \\ \Delta^1 & & \end{array}$$

$\Lambda^1_0 =$ 

$\Lambda^1_1 =$ 

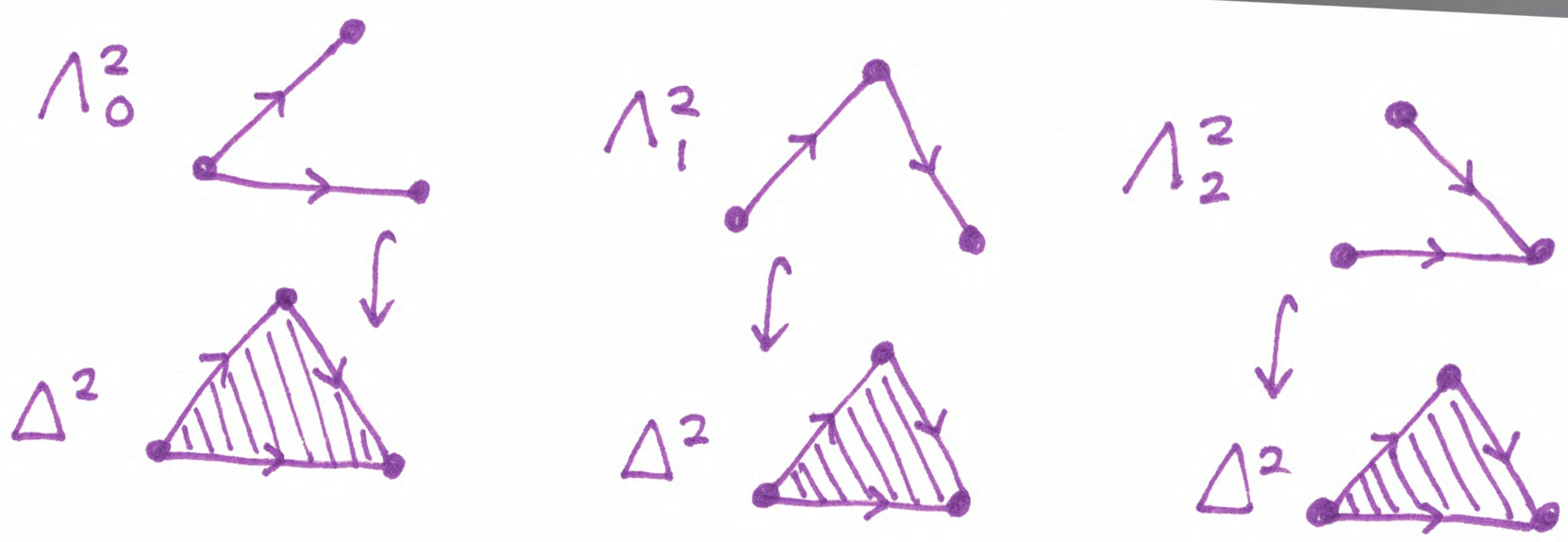
The inclusion $|\Lambda^1_0| \hookrightarrow |\Delta^1|$ has a retraction

so.... $\begin{array}{ccc} \Lambda^1_0 & \longrightarrow & K \\ \downarrow & \nearrow & \\ \Delta^1 & & \end{array}$ has a lift! →

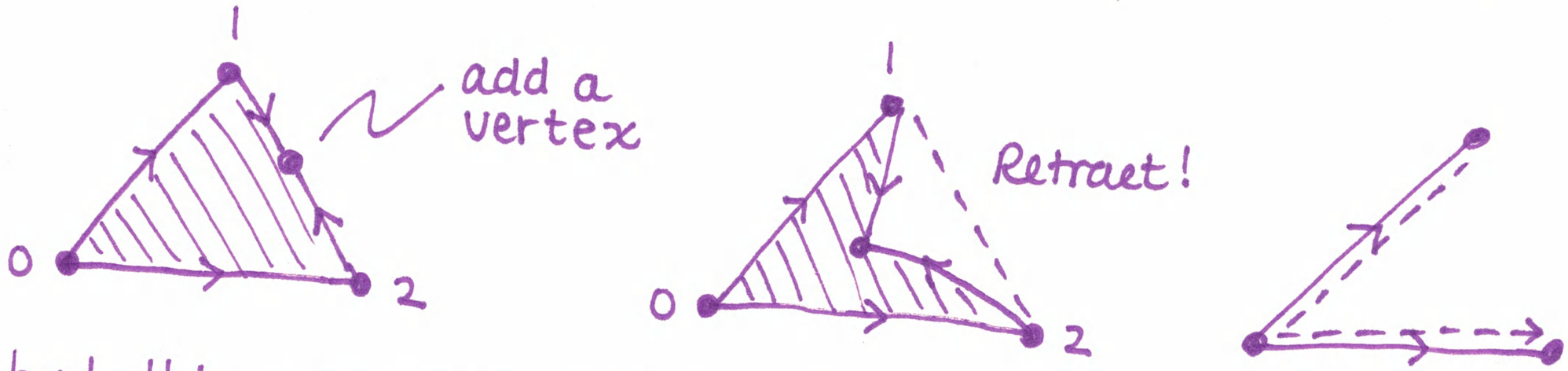
also... $|\Lambda^1_1| \hookrightarrow |\Delta^1|$ and

$\begin{array}{ccc} \Lambda^1_1 & \longrightarrow & K \\ \downarrow & \nearrow & \\ \Delta^1 & & \end{array}$

n=2



The inclusion $|\Lambda^2_0| \hookrightarrow |\Delta^2|$ has a retraction:



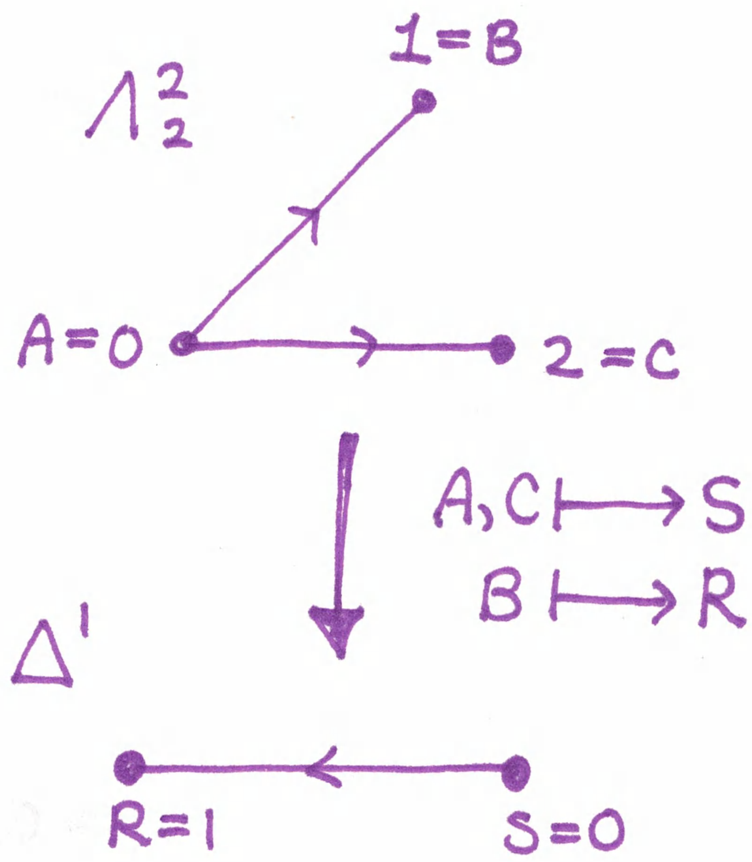
but this requires subdivision along a face of Δ^2, hence, the lifting condition can be related to the subdivision...

EXERCISE #3 Show that Δ^n is Kan $\Leftrightarrow n=0$

Δ^1 is not Kan!

Let $n=2$ then we have a diagram

$$\begin{array}{ccc} \Lambda^2_k & \longrightarrow & \Delta^1 \\ \downarrow & & \\ \Delta^2 & & \end{array}$$



Does this diagram have a lift?

However, there is no [1]-simplex from B to C in the image

$$\begin{array}{ccc} \downarrow & & \downarrow \\ R & \longrightarrow & S \end{array}$$

(Kan filling condition fails!)

EXERCISE #4 If $\mathcal{C} \in Cat$

[$N\mathcal{C} \in sSet$, where $N\mathcal{C}$ is the "nerve" of category $\mathcal{C}$]

Prove that $N\mathcal{C}$ is Kan if and only if $\mathcal{C}$ is a groupoid.

Recall another useful definition:

DEFINITION $Y \in sSet$ is "compact" if Y has finitely many nondegenerate simplices in each dimension.

LEMMA $\forall$ chains W:

$$W_0 \longrightarrow W_1 \longrightarrow W_2 \longrightarrow \cdots \longrightarrow W_n \longrightarrow W_{n+1} \longrightarrow \cdots$$

[such as an inclusion of subsets $W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_n \subset \cdots$]

Consider the set of maps $Y \longrightarrow W_n$:

$$hom(Y, W_n)$$

Y is compact iff $hom(Y, W_{n+1})$ can be defined by composing $hom(Y, W_n)$ with the next morphism in the chain

$$W_n \xrightarrow{f} W_{n+1}$$

In terms of categories, this gives us an isomorphism:

$$\operatorname*{colim}_{n \geq 0} (hom(Y, W_n)) \xrightarrow{\cong} hom(Y, \operatorname*{colim}_{n \geq 0} W_n)$$

1 $Ex^\infty(X)$ is a Kan complex, $\forall X \in sSet$

To show that $Ex^\infty(X) = \operatorname{colim}_{k\geq 0} Ex^k(X)$ is Kan, we need to show that $\forall$ maps $\Lambda^n_\ell \longrightarrow Ex^\infty(X)$

$\exists$ map $\Delta^n \longrightarrow Ex^\infty(X)$

so that the following diagram commutes:

$$\begin{array}{ccc} \Lambda^n_\ell & \xrightarrow{\forall} & Ex^\infty(X) \\ \downarrow & \nearrow_{\exists} & \\ \Delta^n & & \end{array}$$

Since $Ex^\infty(X) = \operatorname{colim}_{k\geq 0} Ex^k(X)$, we have by definition of colimit the maps $l_k : Ex^k(X) \longrightarrow \operatorname{colim}_{k\geq 0} Ex^k(X)$

so, from our diagram ...

$$\begin{array}{ccc} \Lambda^n_\ell & \longrightarrow & Ex^\infty(X) \\ \downarrow & & \\ \Delta^n & & \end{array}$$

NOTE: since Λ^n_ℓ is compact, we have

$$\begin{array}{ccccc} & & Ex^k X & & \\ & \nearrow & \# & \searrow & \\ \Lambda^n_\ell & & & & Ex^\infty X \\ & \searrow & Ex^{k+1} X & \nearrow & \end{array}$$

That is, the diagram will commute at some finite stage

... we have a diagram:

$$\begin{array}{ccccc} & & Ex^k(X) & & \\ & \nearrow & \# & \searrow^{l_k} & \\ \Lambda^n_\ell & & & & Ex^\infty(X) \\ \downarrow & \searrow & & \nearrow_{l_{k+1}} & \\ \Delta^n & & \dashrightarrow & & Ex^{k+1}(X) \end{array}$$

If we can construct a map $\Delta^n \longrightarrow Ex^{k+1}(X)$, we can define the desired map $\Delta^n \longrightarrow Ex^\infty(X)$ by the composition of it and l_{k+1}. $\longrightarrow$

so, we reduce the problem to finding a map $\Delta^n \longrightarrow Ex^{k+1}(X)$ such that the following diagram commutes:

$$\begin{array}{ccc} \Lambda^n_\ell & \longrightarrow & Ex^k(X) \\ \downarrow & & \downarrow \\ \Delta^n & \dashrightarrow & Ex^{k+1}(X) \end{array}$$

If we (very cleverly) substitute $Ex^{k-1}(X)$ for X, we can reduce the diagram further:

$$Ex^k(E^{k-1}(X)) = Ex(X)$$
$$Ex^{k+1}(E^{k-1}(X)) = Ex(Ex(X))$$

$$\begin{array}{ccc} \Lambda^n_\ell & \longrightarrow & Ex(X) \\ \downarrow & & \downarrow \\ \Delta^n & \dashrightarrow & Ex\,Ex(X) \end{array}$$

Now, recall that $sd \dashv Ex$ (adjoint to subdivision functor)

Rewriting the composition:

amounts to defining a map

$$\begin{array}{ccc} \Lambda^n_\ell & \longrightarrow & Ex(X) \\ \downarrow & \searrow & \downarrow \\ \Delta^n & & Ex\,Ex(X) \end{array}$$

$$\begin{array}{ccccc} & & sd(\Lambda^n_\ell) & & \\ & \nearrow^{*} & & \searrow & \\ sd(sd(\Lambda^n_\ell)) & & & & X \end{array}$$

* WE DEFINE THIS MAP AS THE "LAST VERTEX MAP"

Also, rewriting the composition:

would amount to defining a map

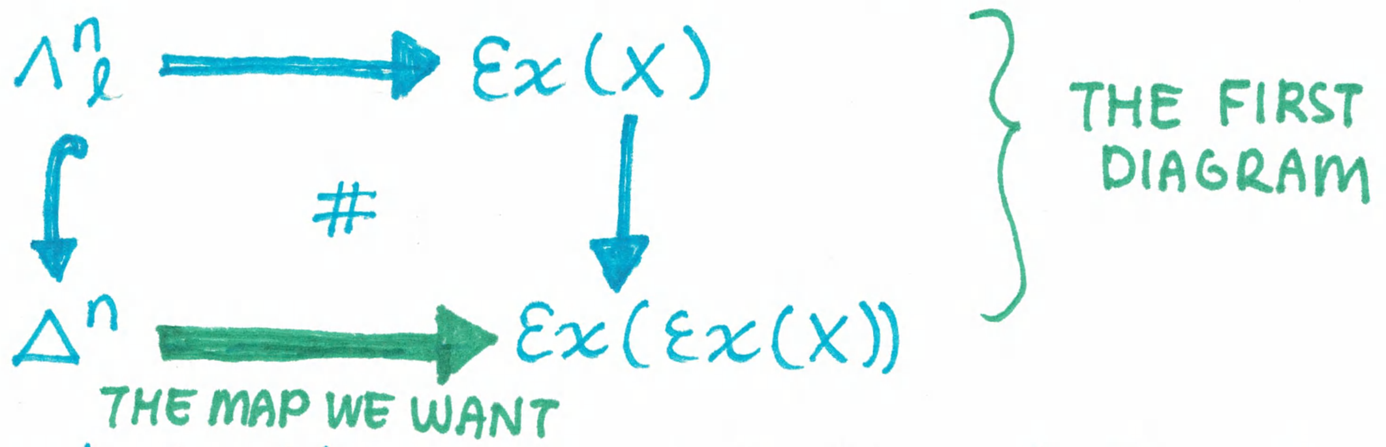
$$\begin{array}{ccccc} sd(sd(\Lambda^n_\ell)) & & & & \\ & \searrow^{*} & & & \\ & & sd(sd(\Delta^n)) & \nearrow & X \end{array}$$

THIS MAP IS THE INCLUSION

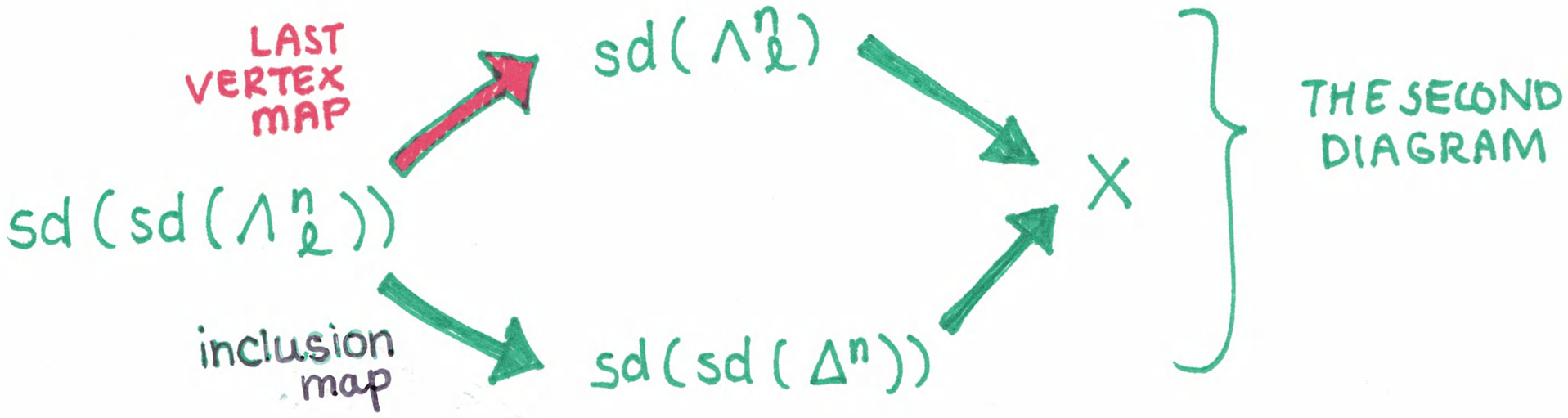
$$\begin{array}{ccc} \Lambda^n_\ell & & \\ \downarrow & \searrow & \\ \Delta^n & \longrightarrow & Ex\,Ex(X) \end{array}$$

THIS IS THE MAP WE WANT TO CONSTRUCT!

So, the commutativity of the diagram:

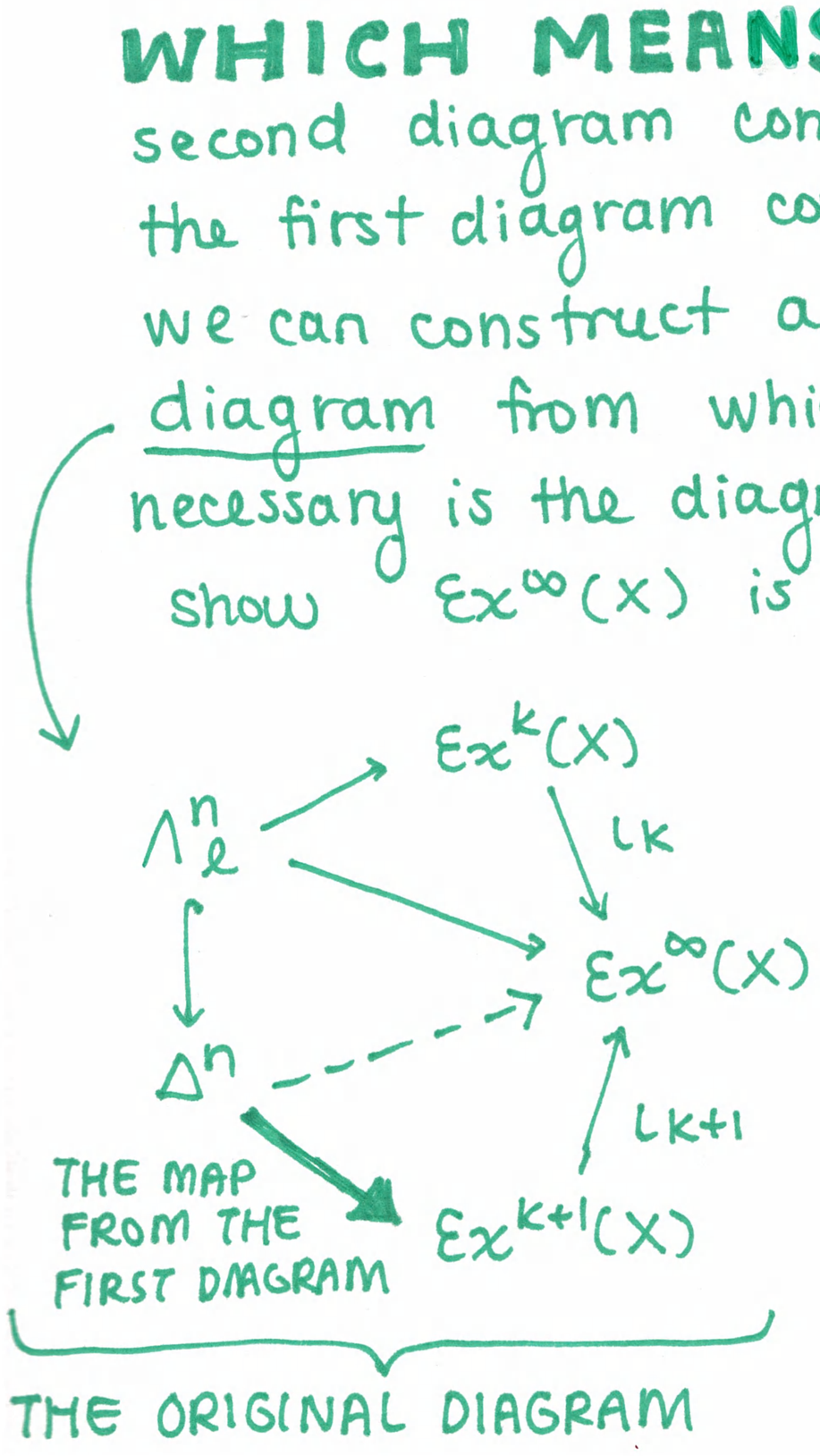


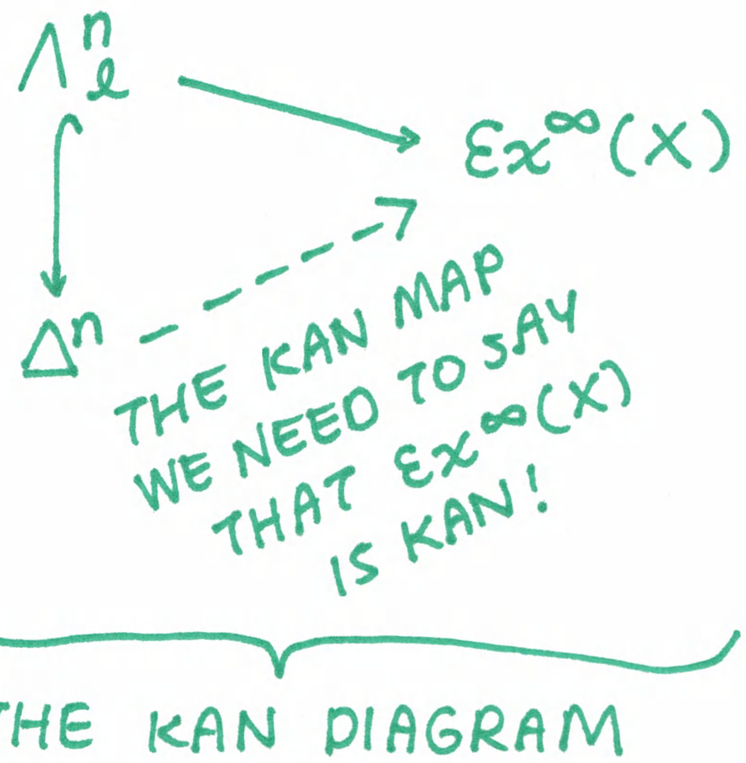
is equivalent to the commutativity of the diagram:



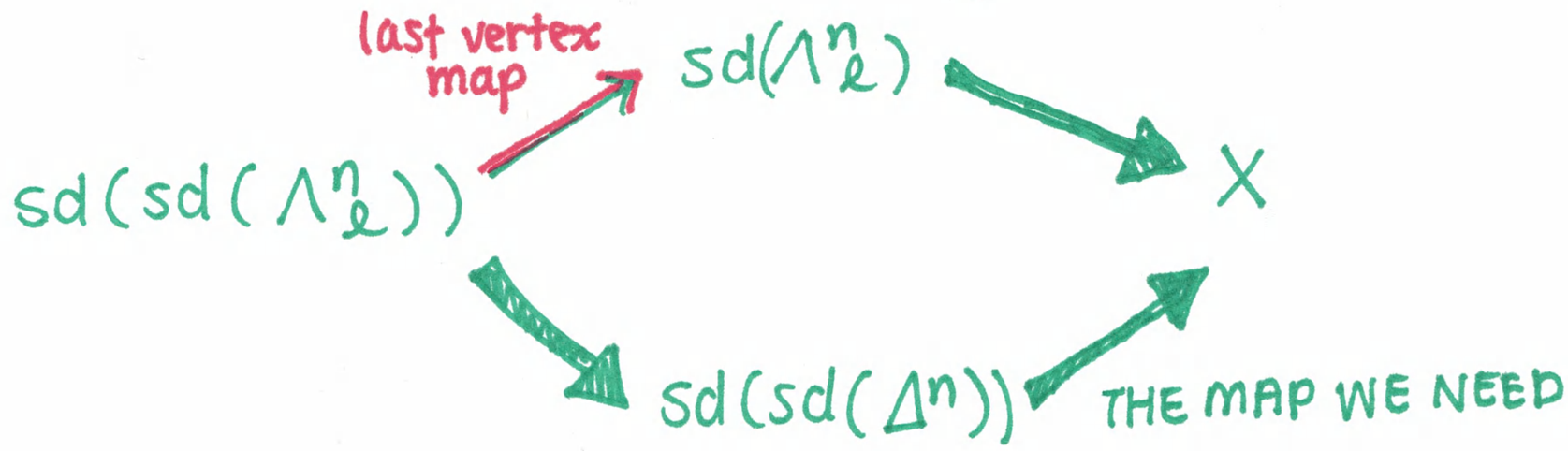
THE PLAN:

WHICH MEANS that if we can show the second diagram commutes, then we know that the first diagram commutes, which means that we can construct a map in the original diagram from which we can define the map necessary is the diagram that would then show $Ex^\infty(X)$ is Kan.



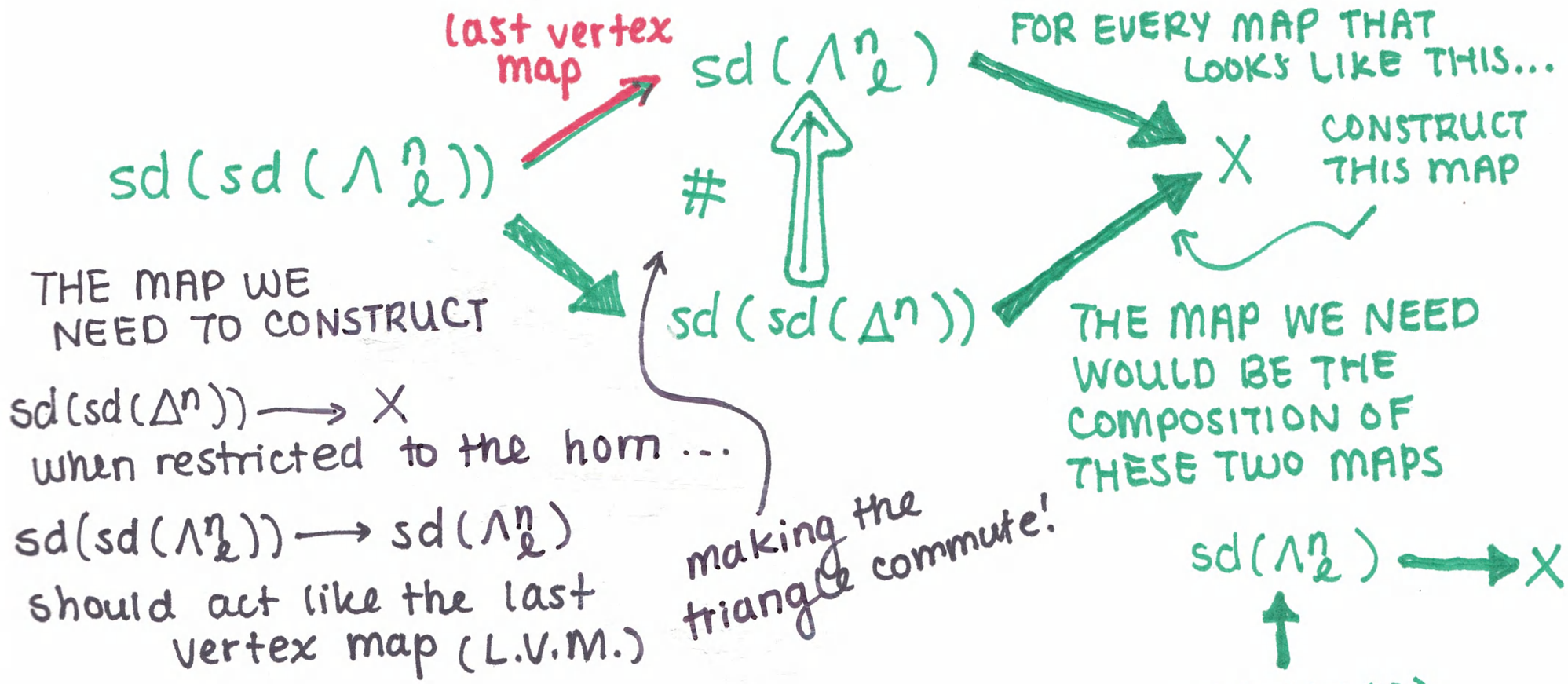


QUESTION: How do we show that the second diagram commutes?



IF we construct a map $sd(sd(\Lambda^n)) \longrightarrow sd(\Lambda^n_k)$

THEN we can define the map we need in terms of it!



SO... THE NEW PLAN:

Construct a map $sdsd\,\Delta^n \longrightarrow sd\,\Lambda^n_k$

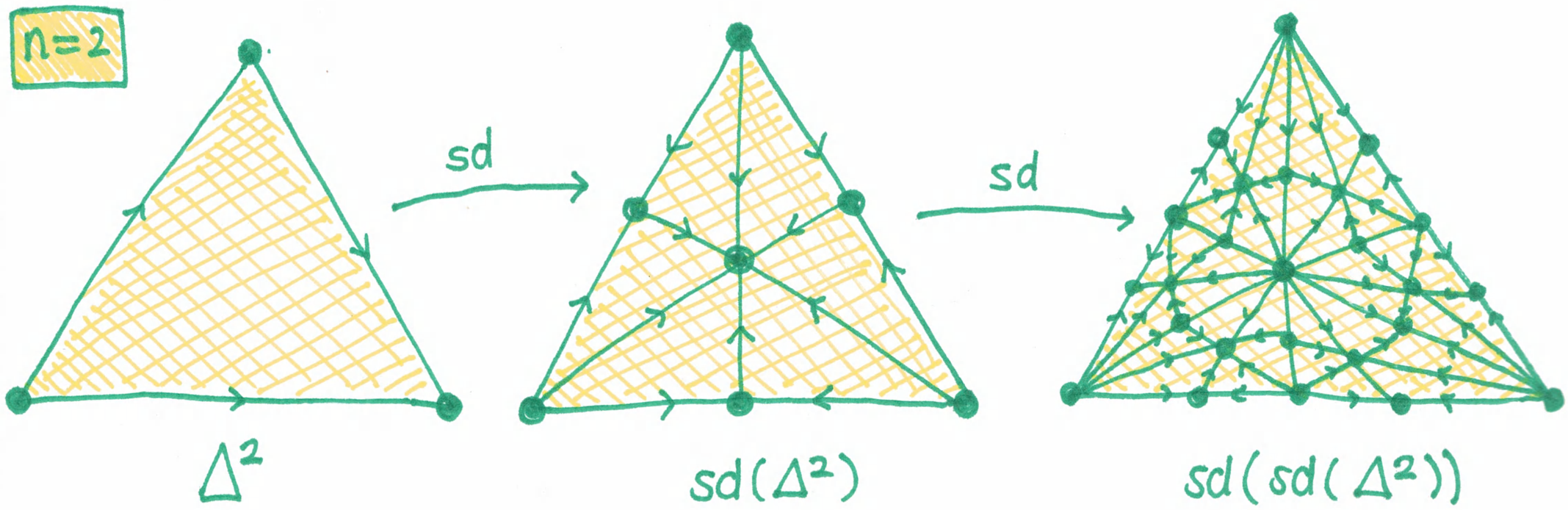
(and this will take a little work...)

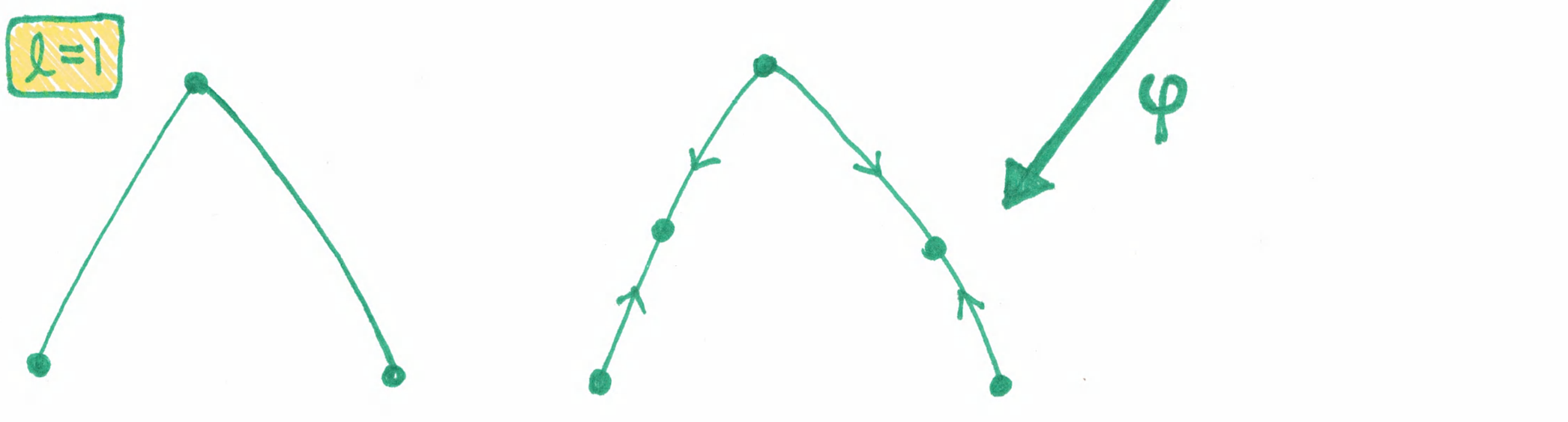
THIS MAP WILL BE USED TO DEFINE
$sd(sd(\Delta^n)) \longrightarrow X$
AND SHOULD MAKE THE LEFT TRIANGLE COMMUTE WHEN RESTRICTED TO THE BOUNDARIES.

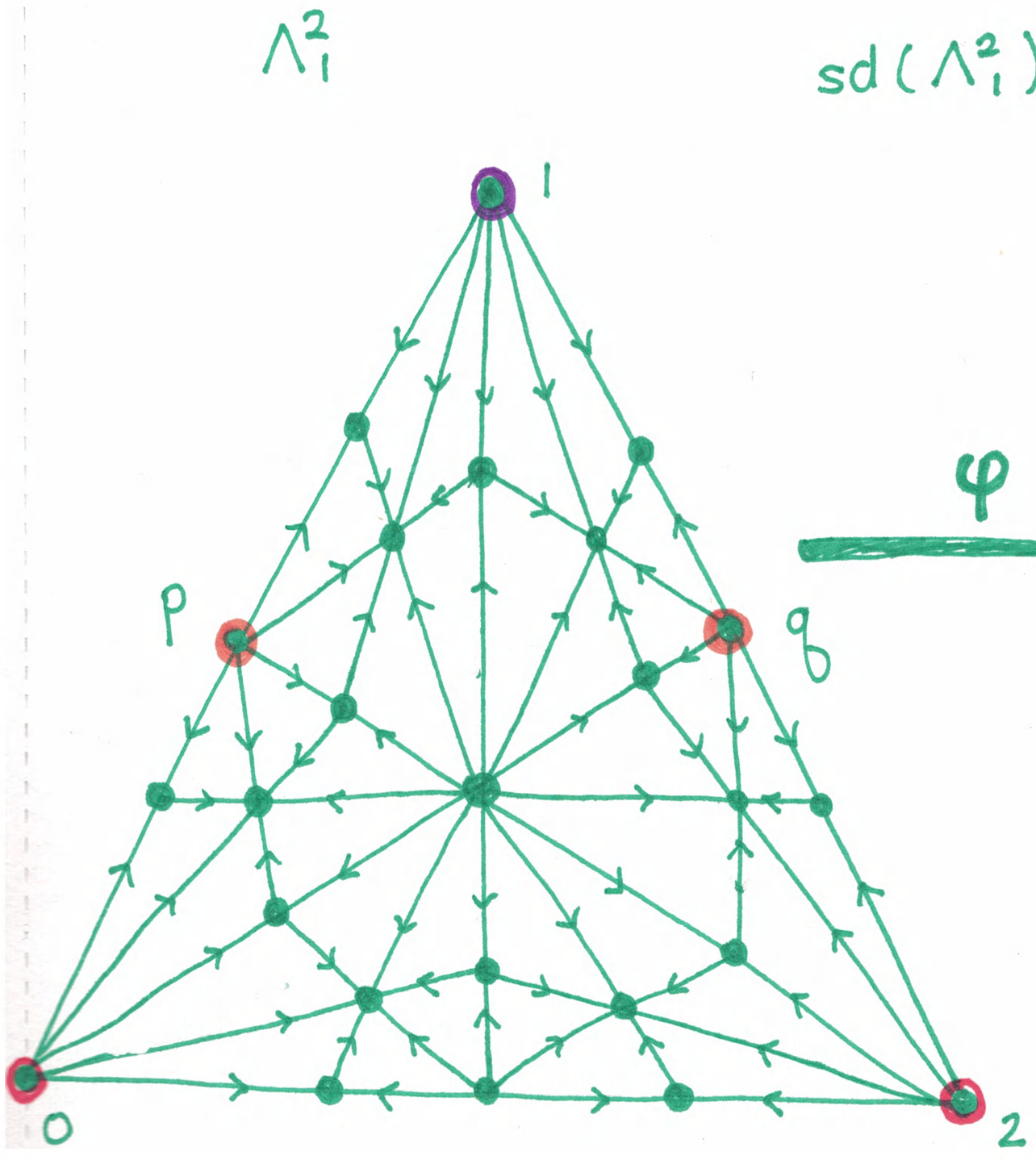
For an example of how to construct this map

$$sd(sd(\Delta^n)) \longrightarrow sd\,\Lambda^n_j$$

consider the case $\boxed{n=2}$



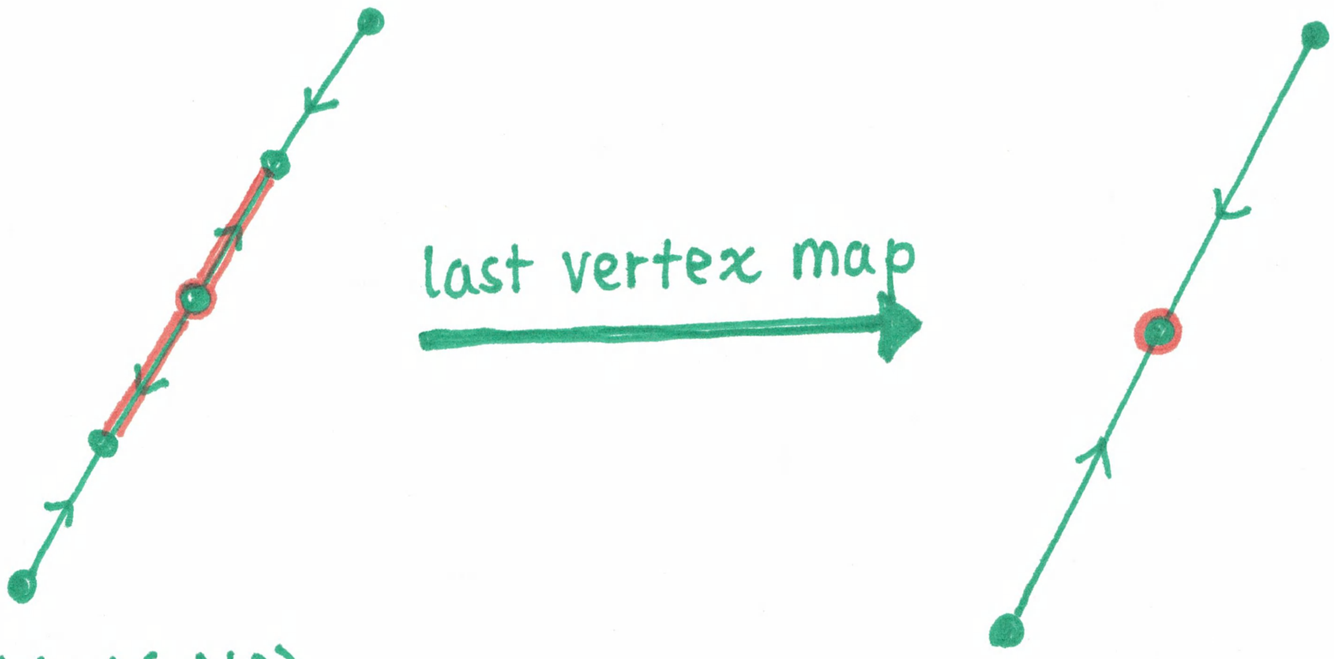




For the boundary of Δ^2,
the "last vertex map"

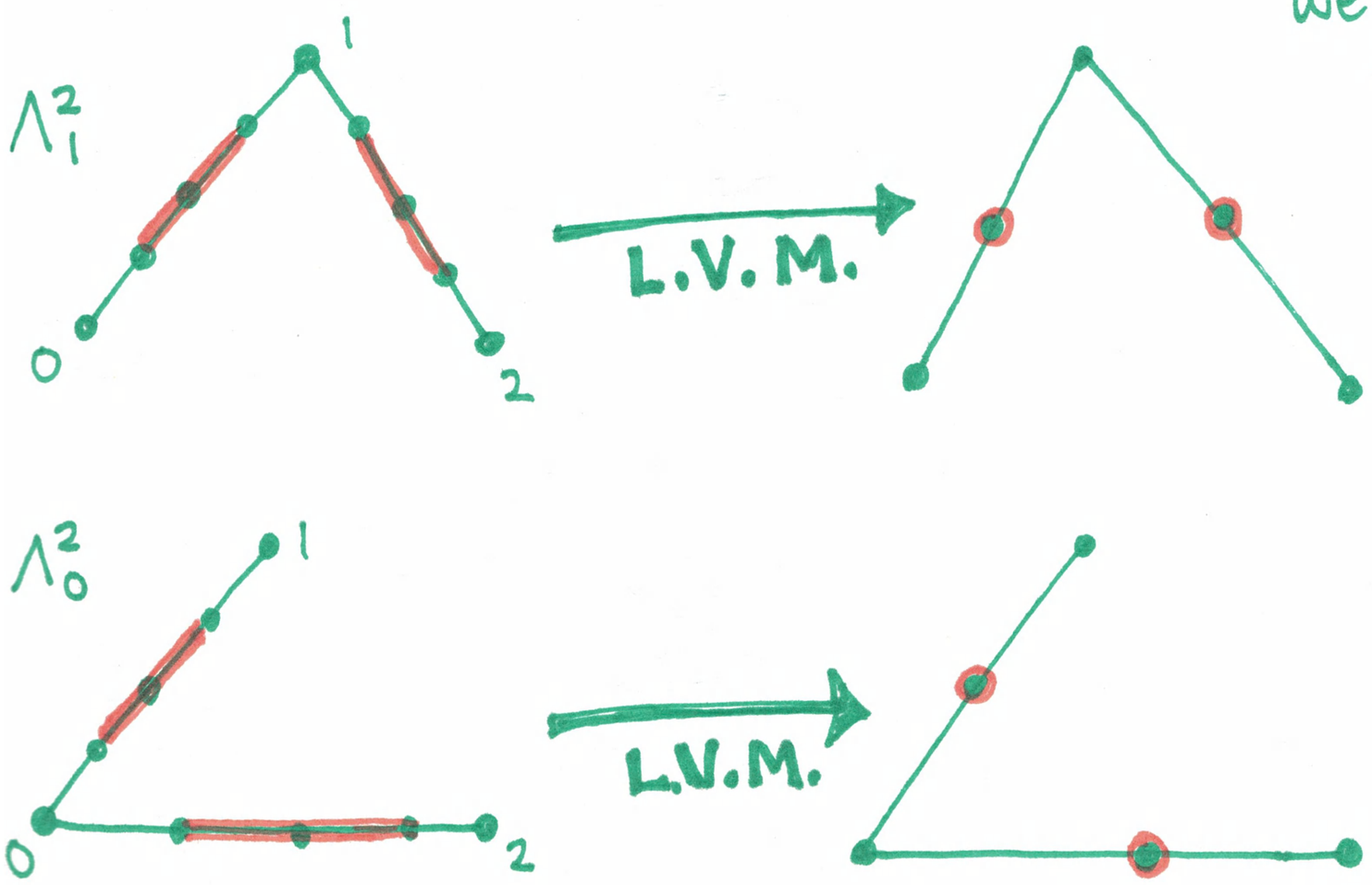
$$sd(sd(\Lambda^2_2)) \longrightarrow sd\,\Lambda^2_2$$

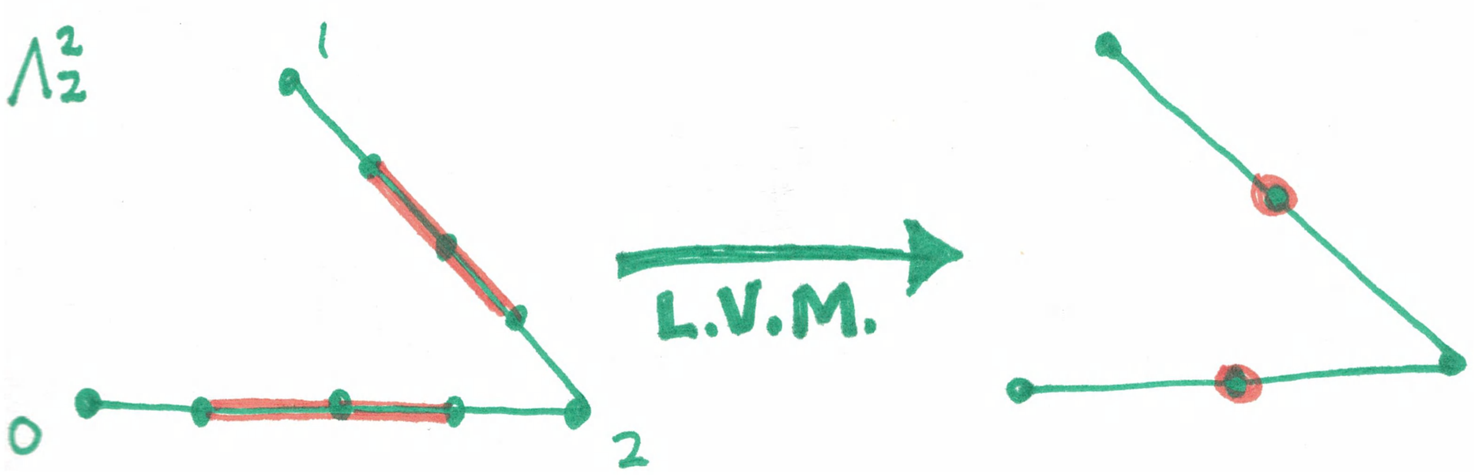
will collapse edges in the following way:

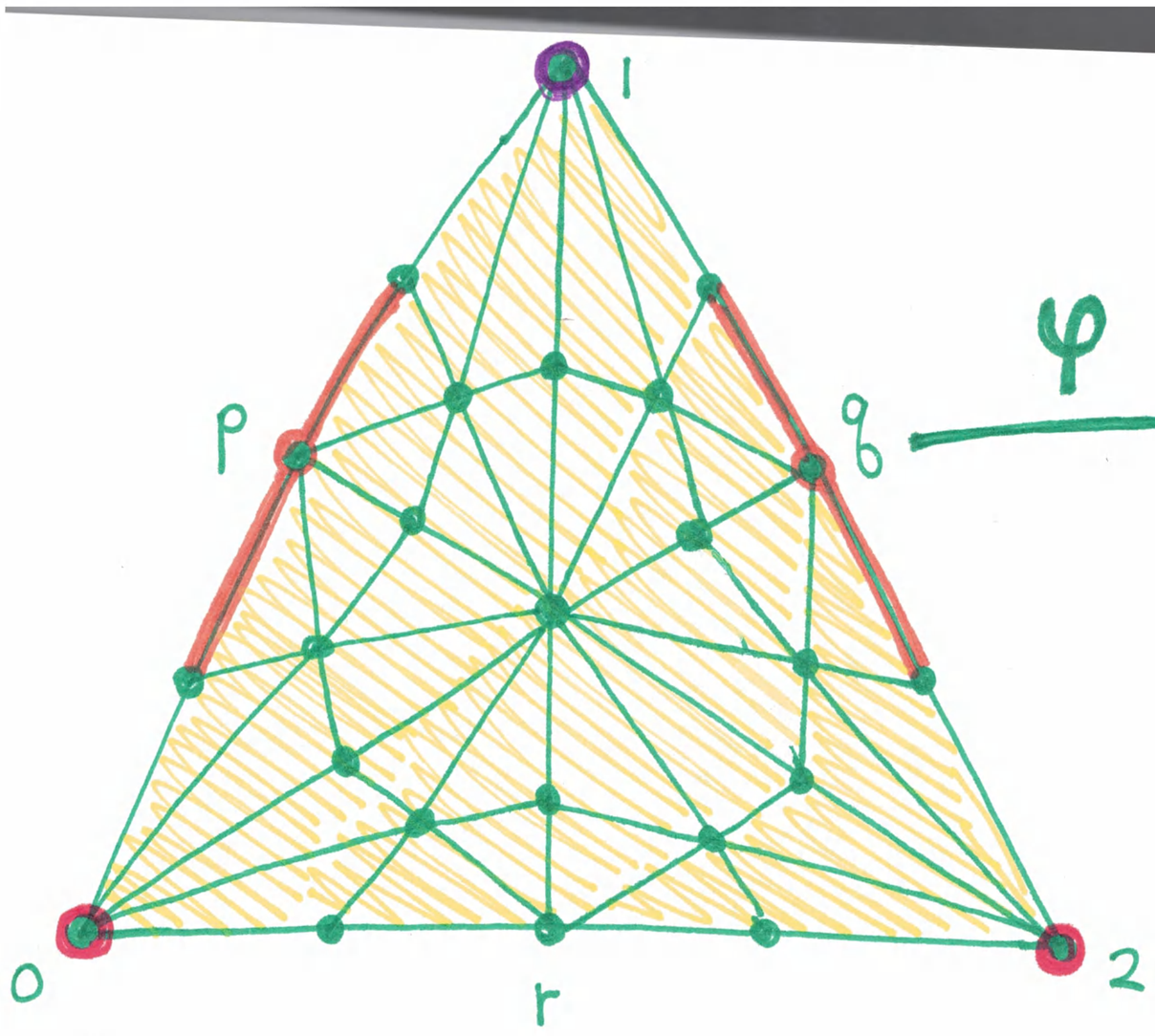


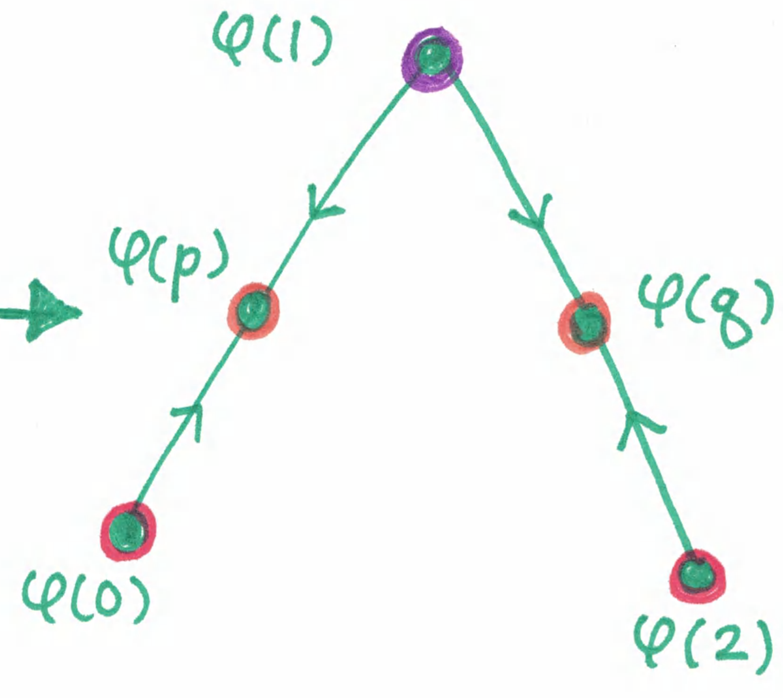
$sd(sd(\Delta^1))$

so, on ~~the boundary~~ Δ^2; the last vertex map dictates how we must collapse the boundary.

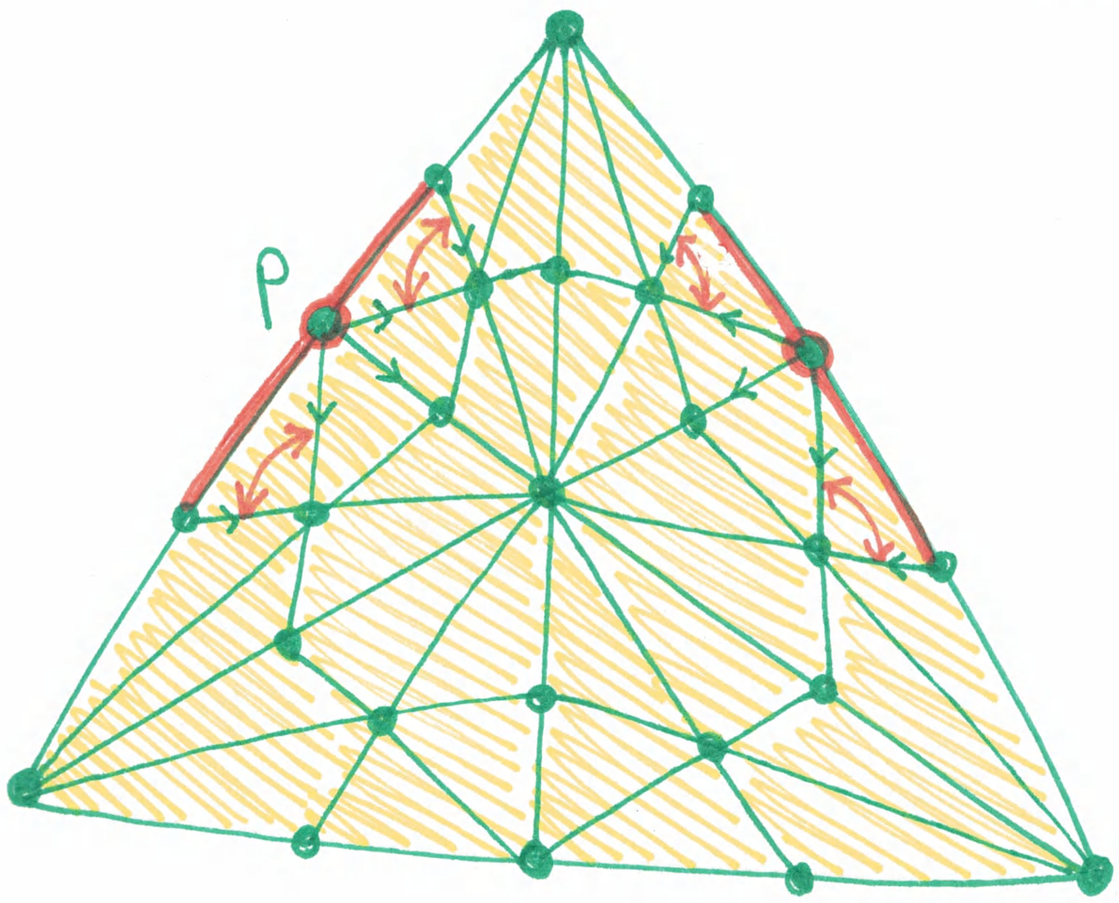




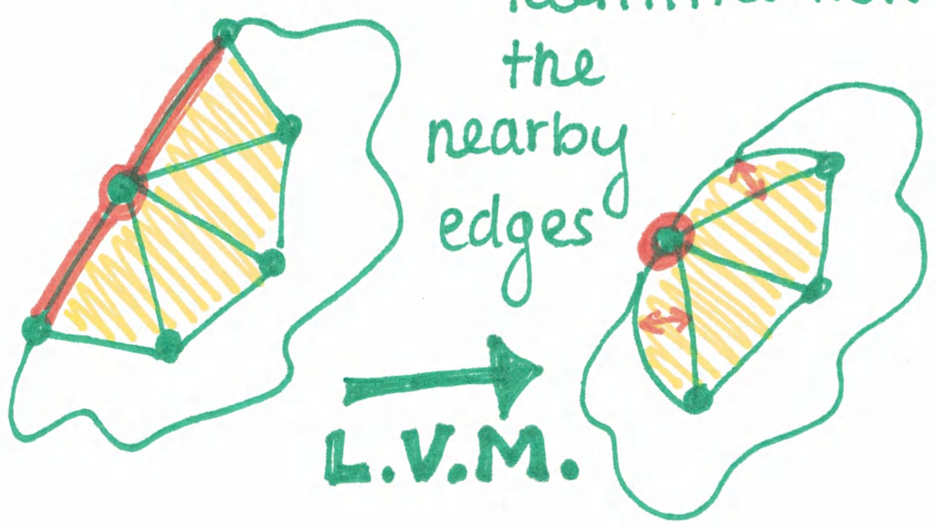




φ

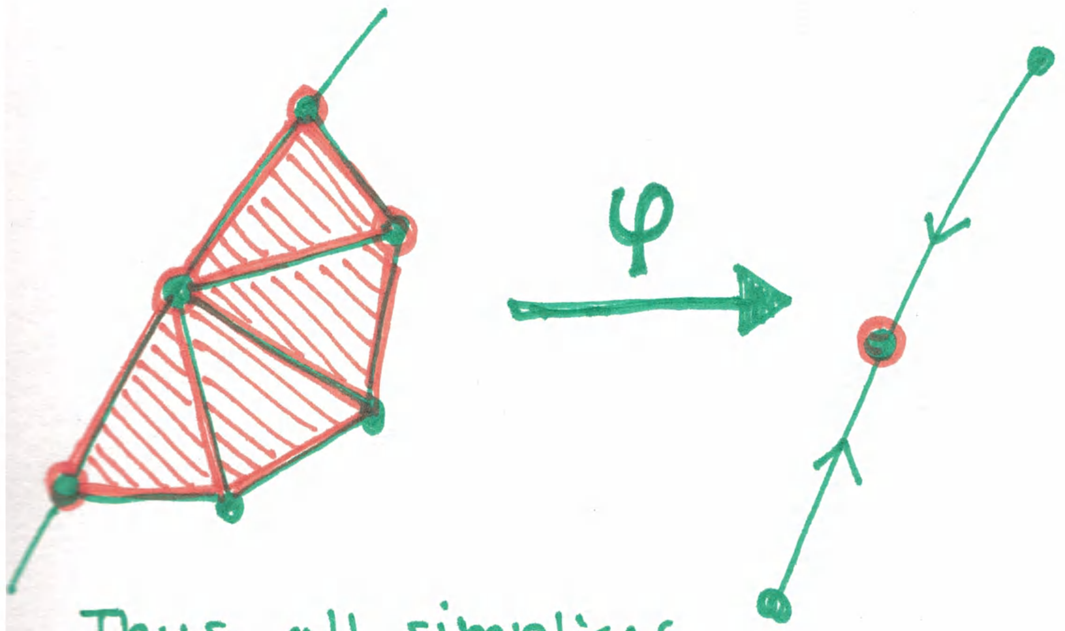


collapsing this segment forces the identification of the nearby edges

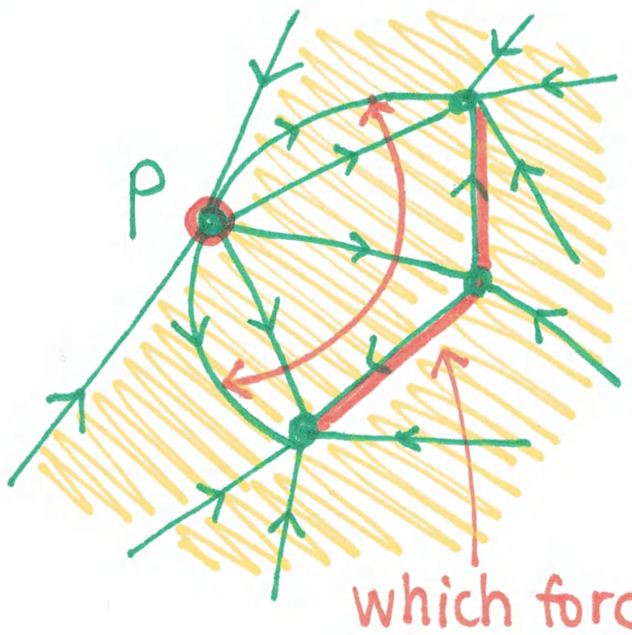


L.V.M.

and the identification of these edges forces the identification:



φ

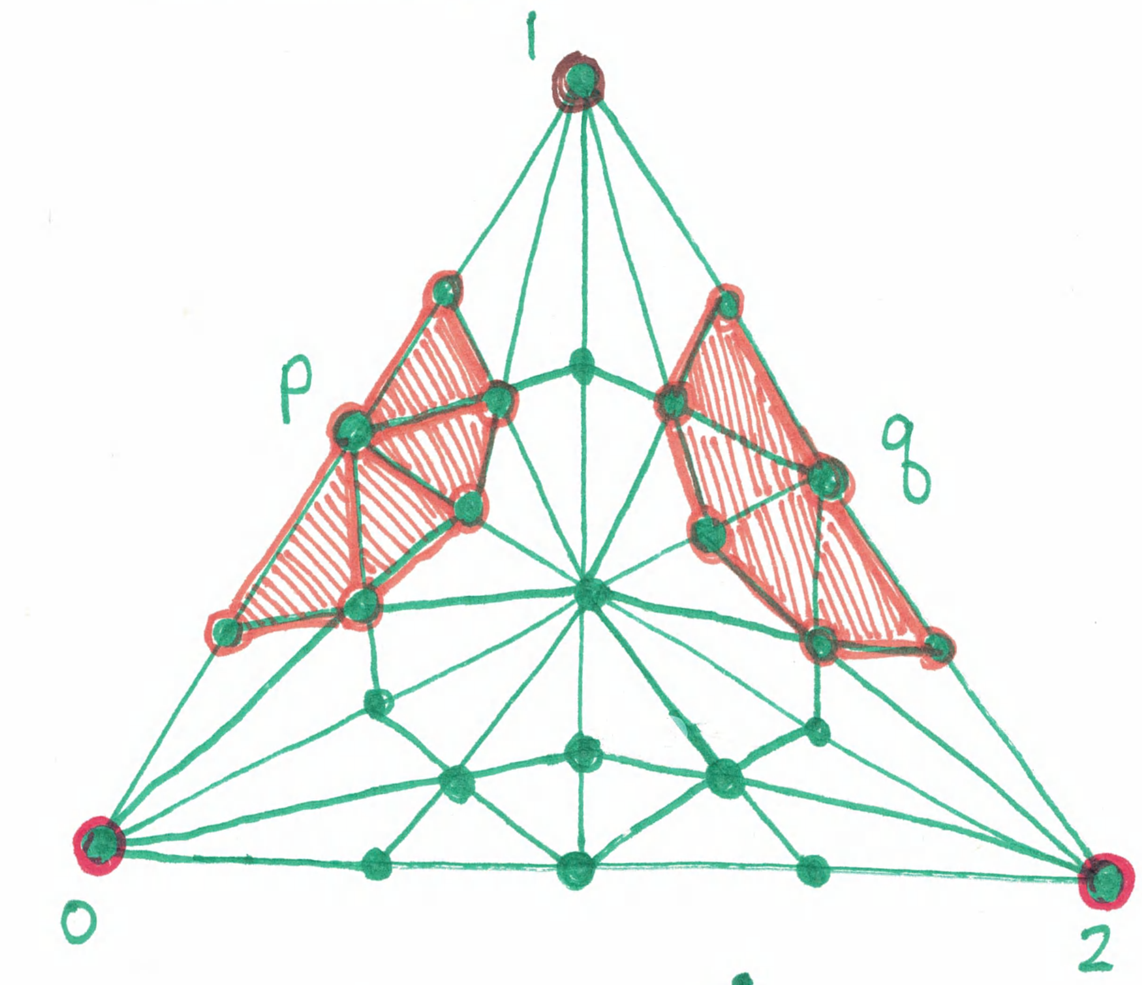


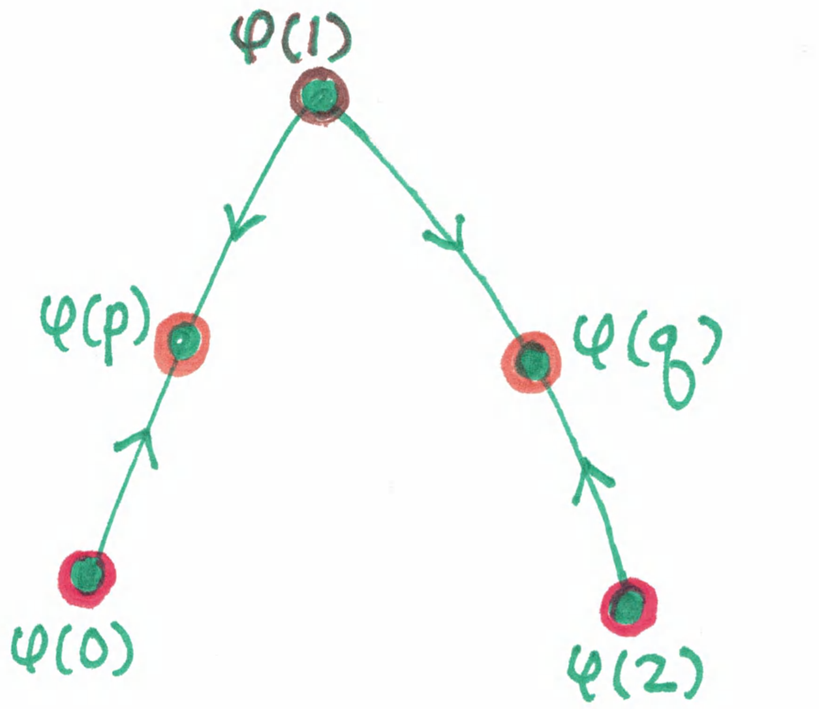
which forces the collapse of these edges to a point

Thus, all simplices surrounding $p \in sd(sd(\Delta^2))$ (as shown) must map to $\varphi(p)$

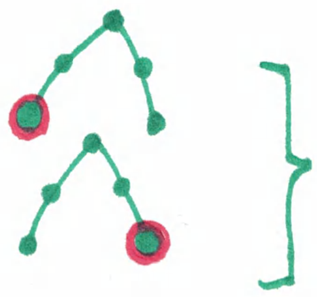
The only point in the horn that it can collapse to is the image of p $\varphi(p)$

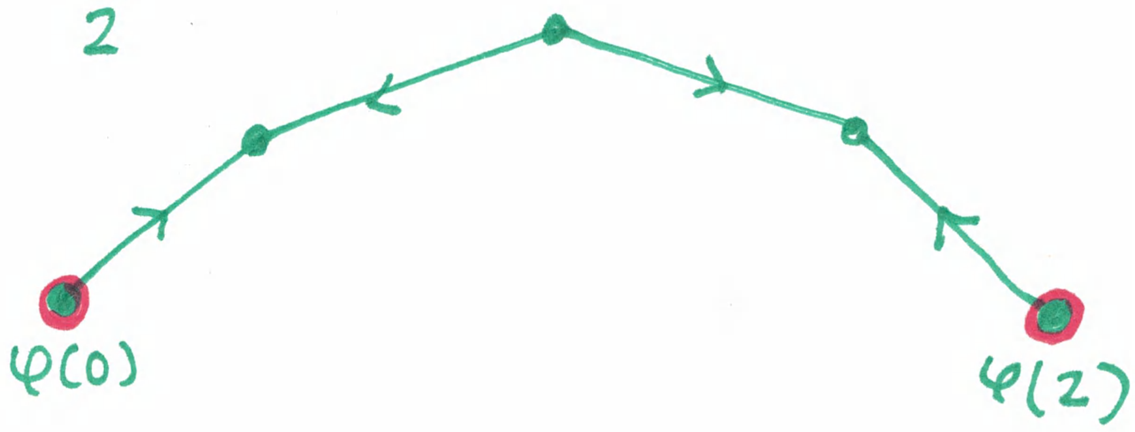
so we have,



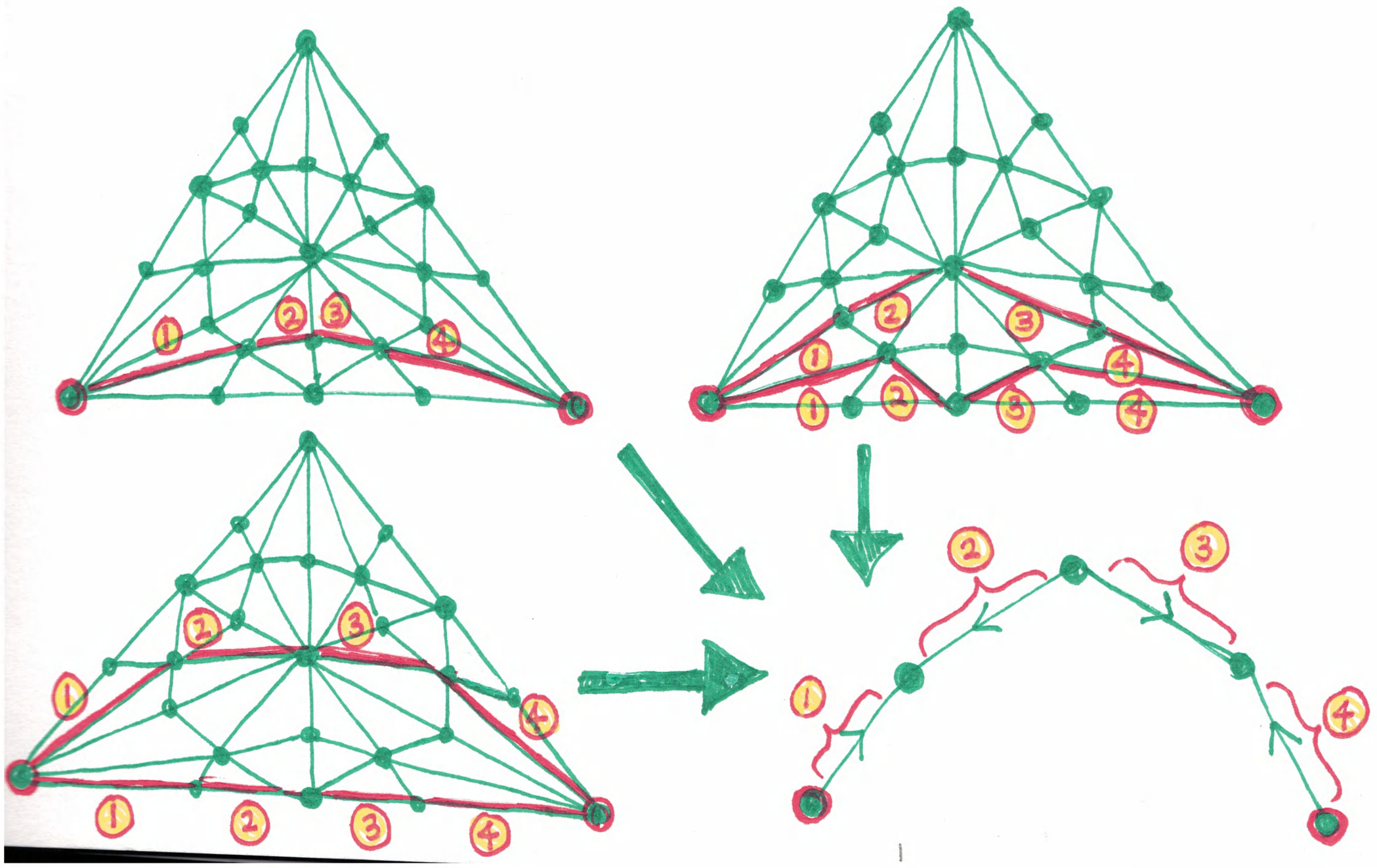


Now, $0 \xmapsto{\varphi} \varphi(0)$
$2 \longmapsto \varphi(2)$

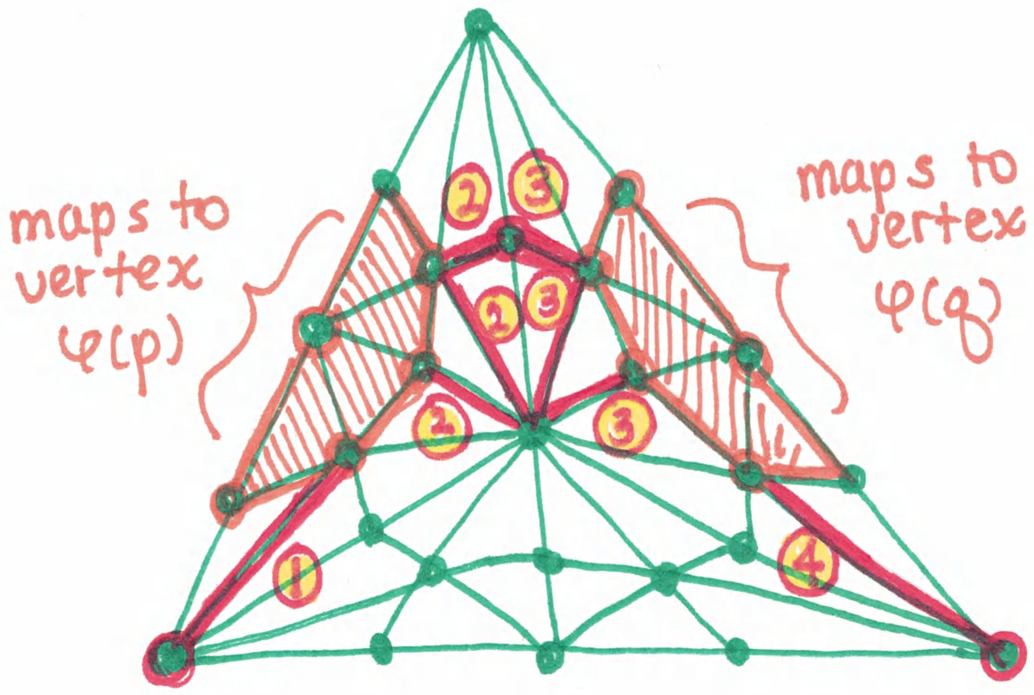


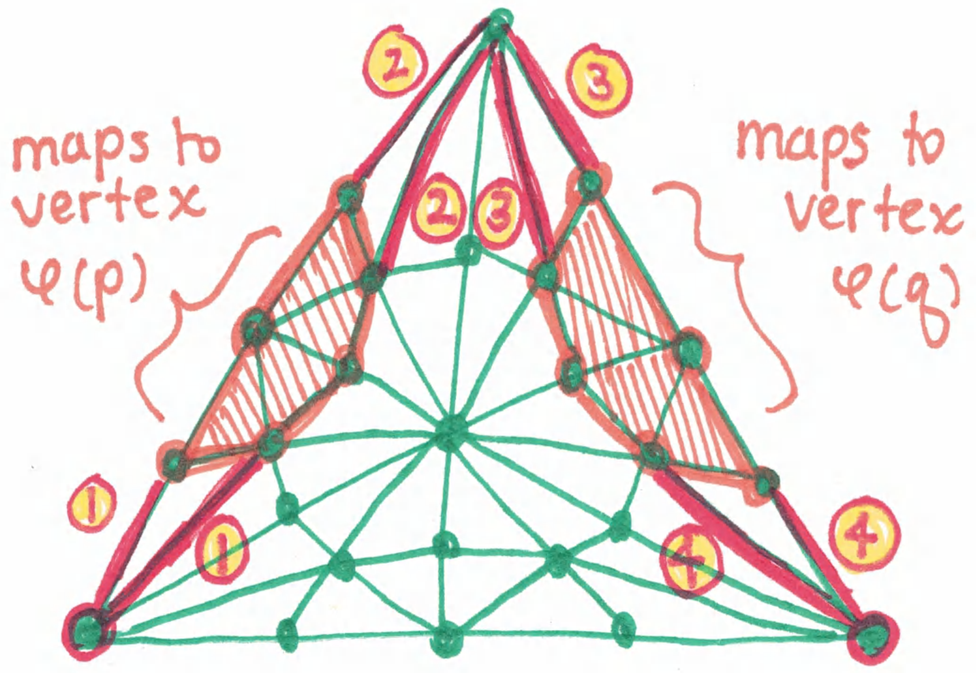


and any path of four 1-simplices with corresponding directions connecting 0 and 1 in the simplex $sd(sd(\Delta^2))$ must map to the only path of four 1-simplices connecting $\varphi(0)$ and $\varphi(1)$ in Λ^2_1... namely...

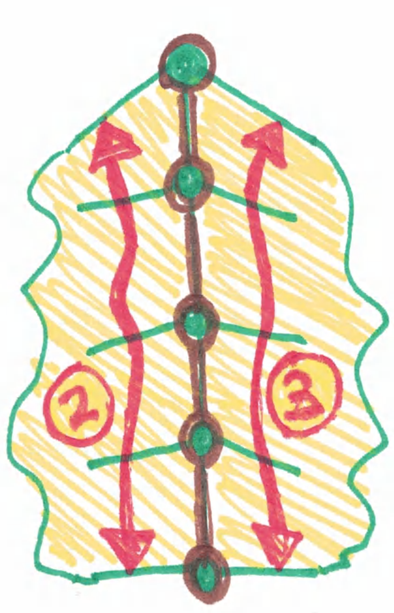


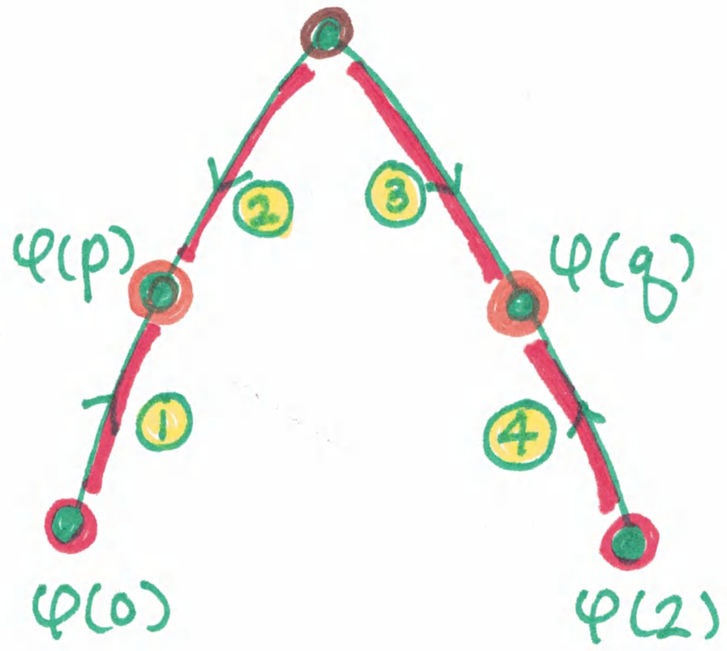
Similarly, we can consider any path of four 1-simplices connecting 0 and 1 (when we recognize those that will collapse and map to $\varphi(p)$ and $\varphi(q)$ as shown:)



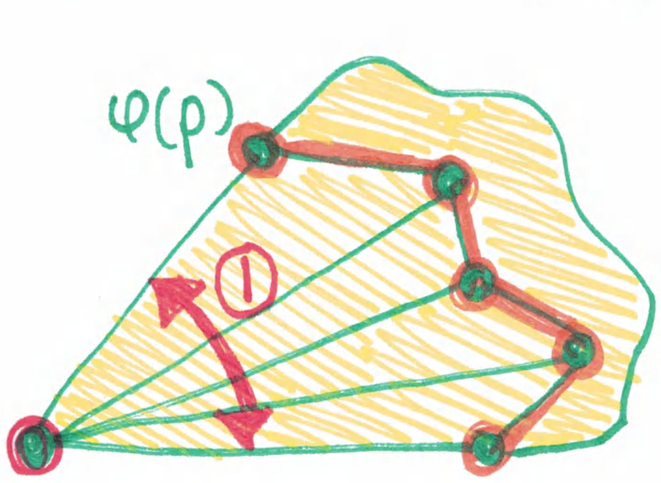


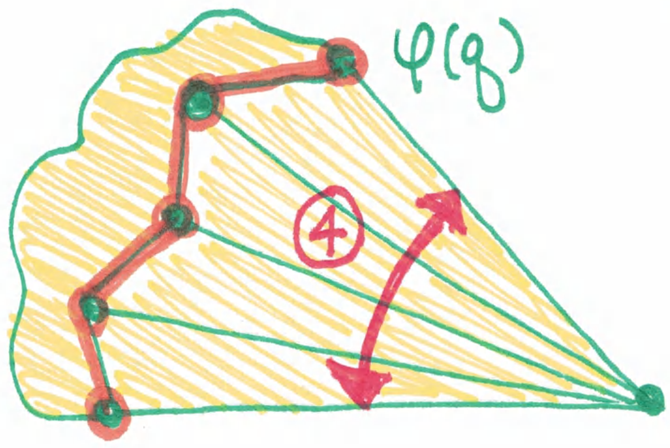
so, now we have:





The identification of these edges forces the collapse of the other edges to the vertices as shown.





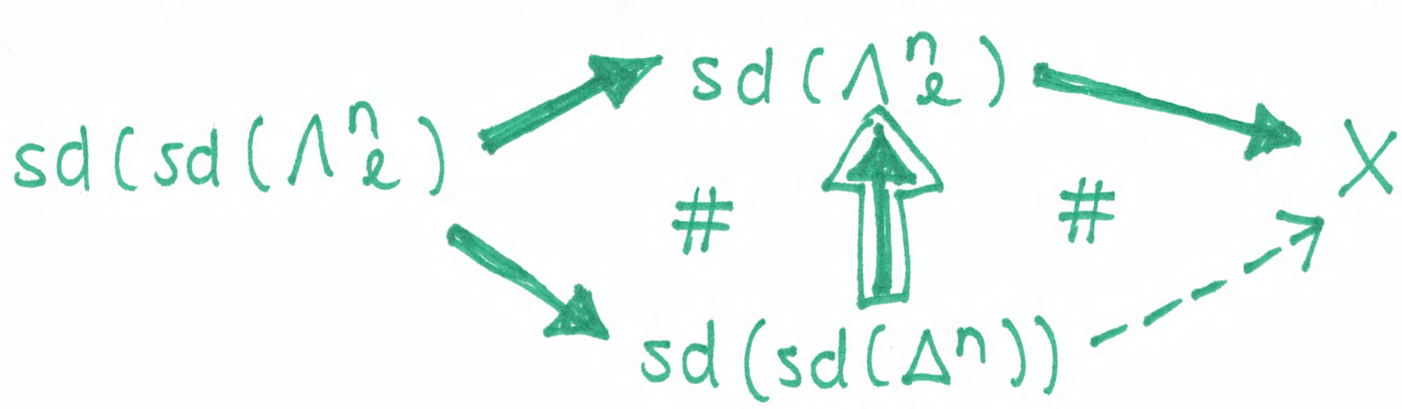
EXERCISE #5 Is the map $sd(sd(\Delta^2)) \longrightarrow sd\Lambda^2_2$ actually unique?

In this way, we ~~[redacted]~~ can construct a map

$$sd(sd(\Delta^n)) \longrightarrow sd(\Lambda^n_k)$$

for each n and each k. (we use the same argument in each dimension, specifying where to send vertices by sending them to the same vertices in the horn. The paths connecting their images force the collapse of edges; as seen in the previous example.)

This map restricts to the Last Vertex Map on the boundary, and so the left triangle commutes:



we define $sd(sd(\Delta^n)) \longrightarrow X$ as the composition $sd(sd(\Delta^n)) \longrightarrow sd(\Lambda^n_k) \longrightarrow X$

and the adjoint of this map

$sd(sd(\Delta^n)) \longrightarrow X$ ..... $\Delta^n \longrightarrow Ex(Ex(X))$

IS THE MAP WE WANTED!

So... $Ex^\infty X$ is Kan!

(as desired...) ∎

PROPOSITION:

If $X, Y \in sSet$ and $X \xrightarrow{f} Y$

ⓐ $X, Y \in Kan$

f weak equivalence $\rightarrow$ f homotopy equivalence

[**NOTE**: the converse is already true:

f weak equivalence $\leftarrow$ f homotopy equivalence]

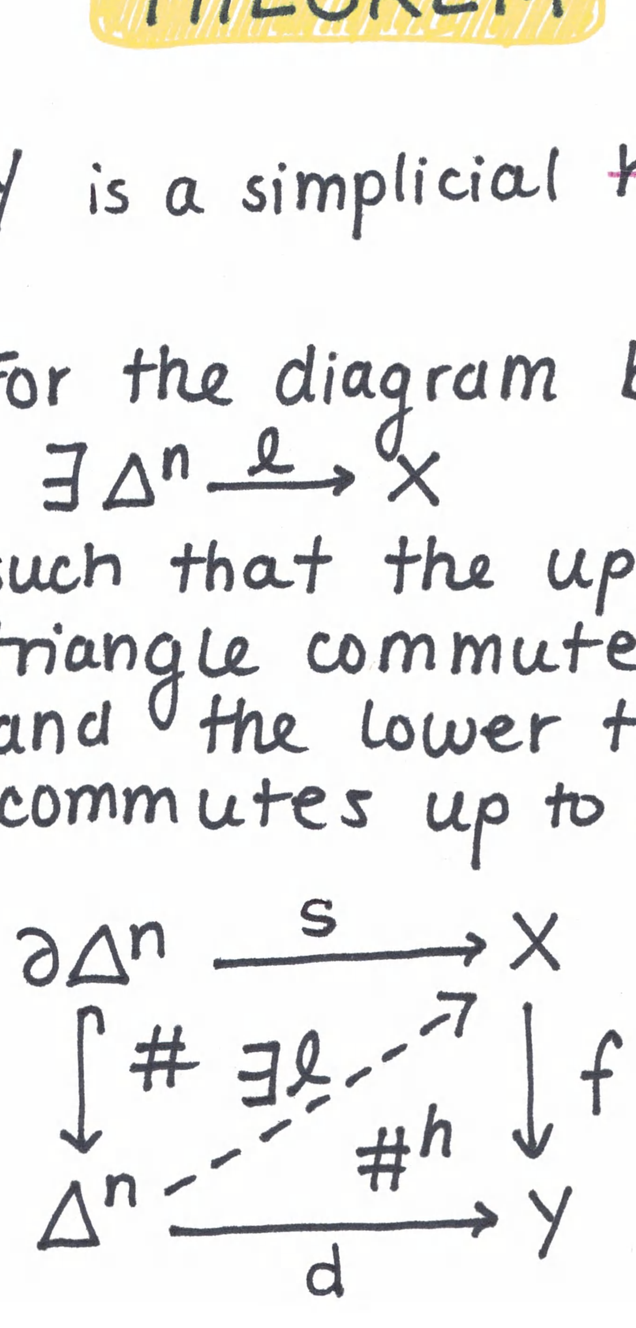
ⓑ f weak equivalence $\leftrightarrow$ $Ex^{\infty} f$ homotopy equivalence

[part ⓑ holds $\forall X, Y \in sSet$, not necessarily Kan]

PROPOSITION: SIMPLICIAL WHITEHEAD THEOREM

If $X, Y \in Kan$ and $X \xrightarrow{f} Y$ is a simplicial ~~homotopy~~

f homotopy equivalence $\leftrightarrow$ For the diagram below $\exists\, \Delta^n \xrightarrow{\ell} X$ such that the upper triangle commutes (#) and the lower triangle commutes up to homotopy ($\#^h$)



$f \circ \ell \overset{h}{\simeq} d$

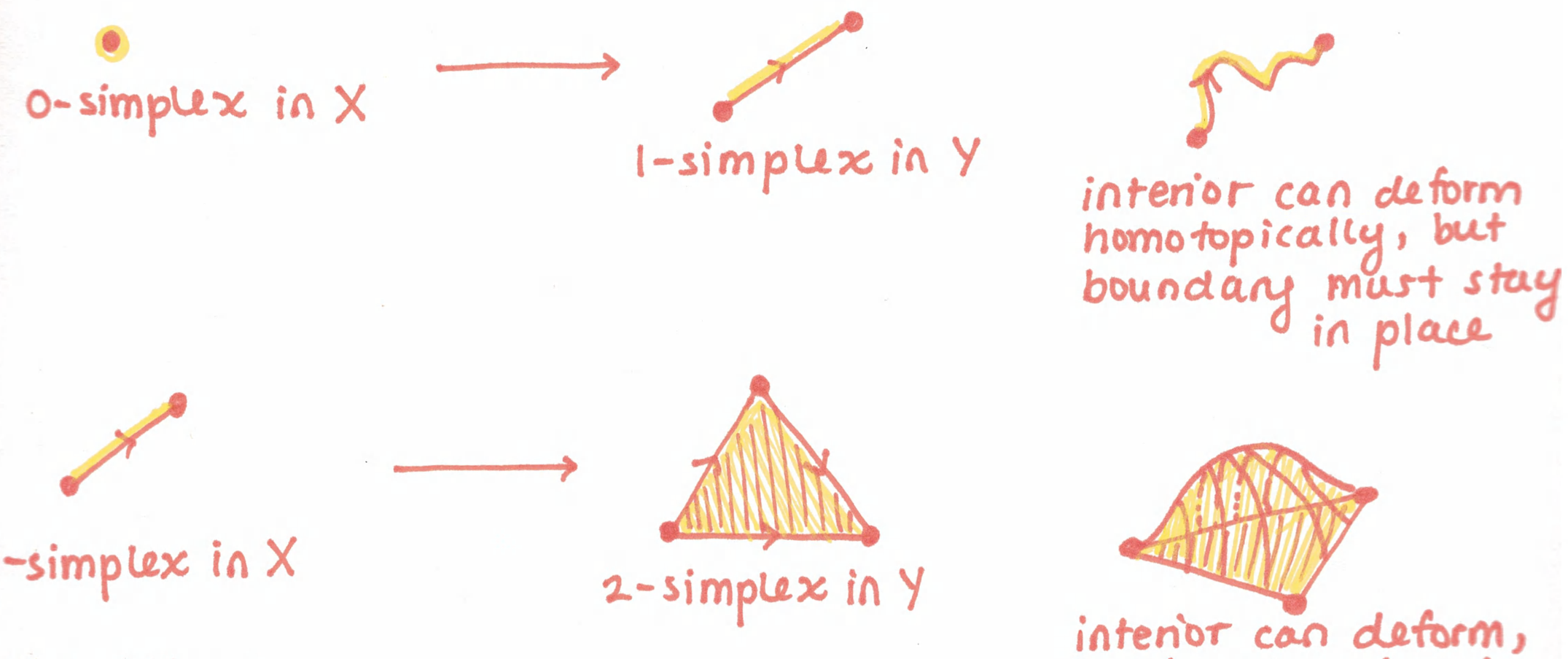
homotopic via map h

* This condition $\#^h$ means there is a map $\Delta^1 \times \Delta^n \longrightarrow Y$ whose restriction to $\Delta^1 \times \partial\Delta^n$ is a constant homotopy ...

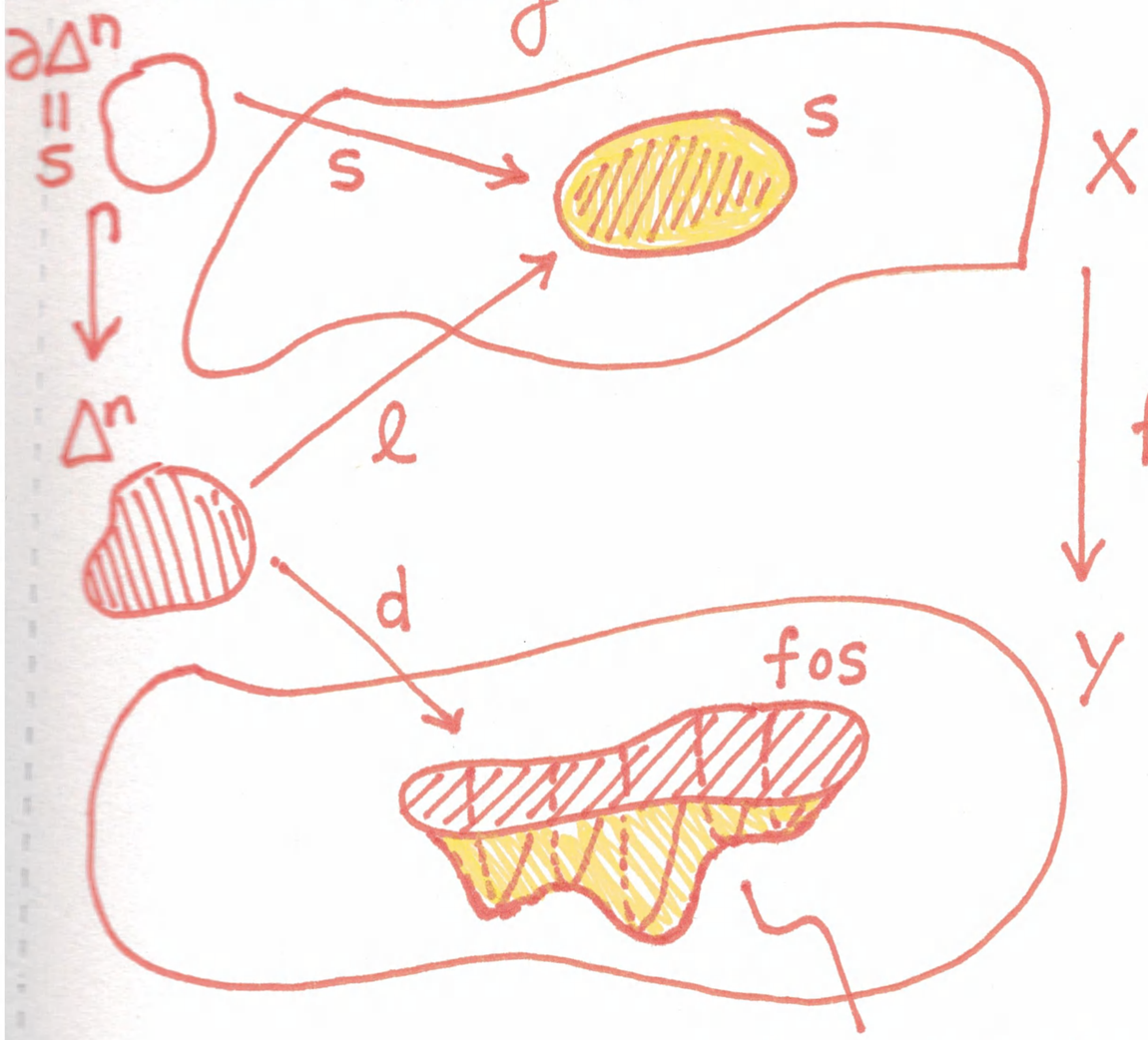
→ continued...

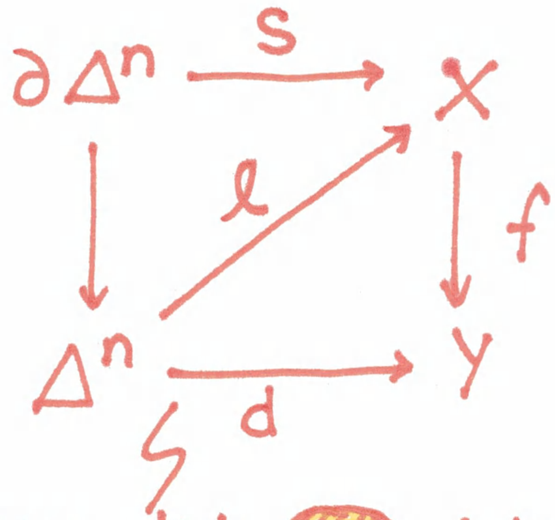
→ continued... #^{h}

There is a map: $\Delta^1 \times \Delta^n \longrightarrow Y$
(which, by the Yoneda Lemma, means that there is a collection of $(n+1)$-simplices in Y.)
whose restriction to $\Delta^1 \times \partial\Delta^n$ is a constant homotopy (which means that the interior of the $(n+1)$-simplex can be deformed homotopically, but the boundary must stay in place!)



Intuitively...



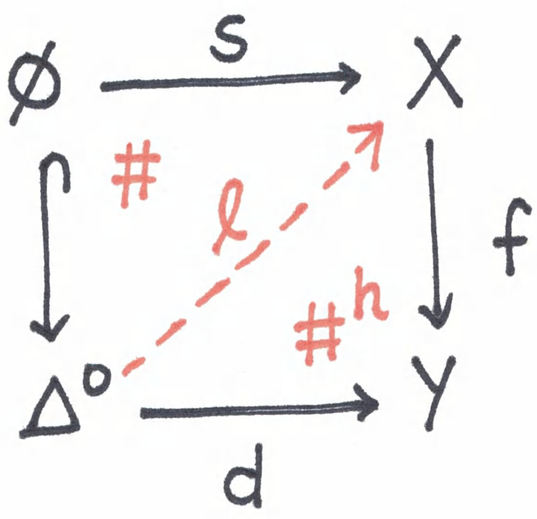


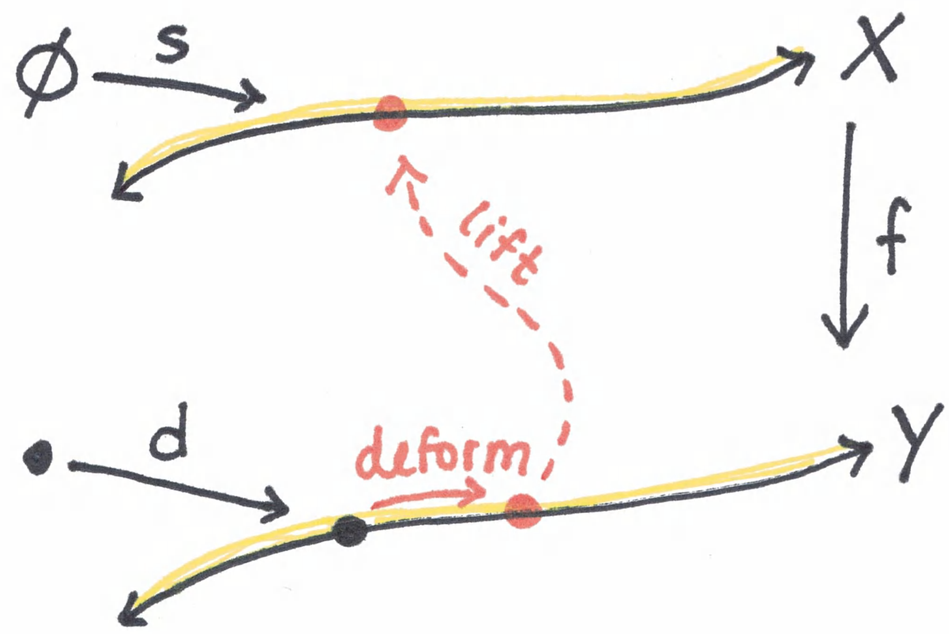
Map from disk into Y whose restriction to the boundary is the image of the sphere ○s in X

deform interior of disk in Y; the lift to X provides a "filling" of ○s ⇒ in X

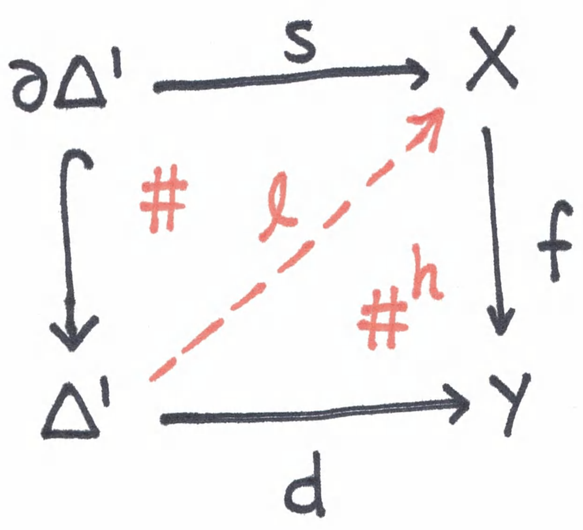
USING THE SIMPLICIAL WHITEHEAD THEOREM FOR n=0,1,2 *Examples*

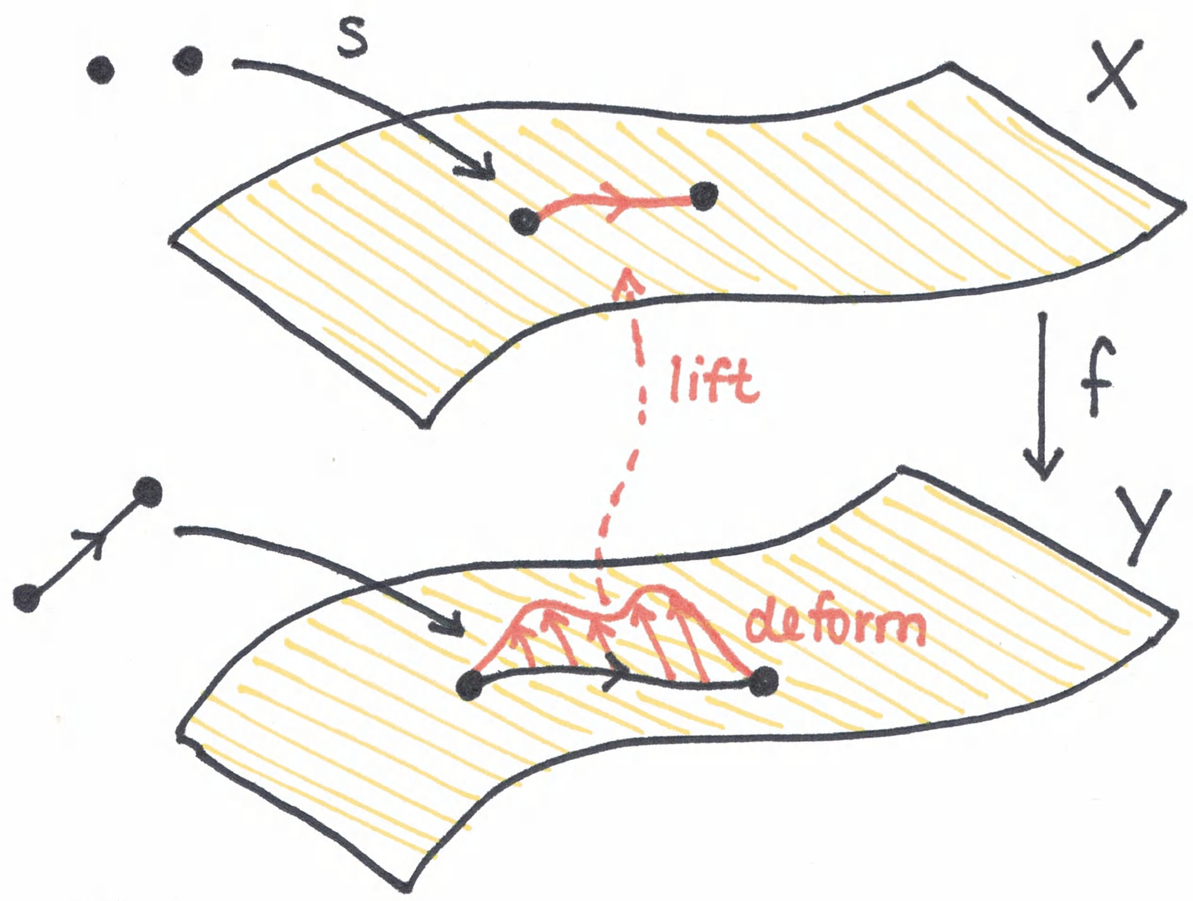
n=0 THE DIAGRAM



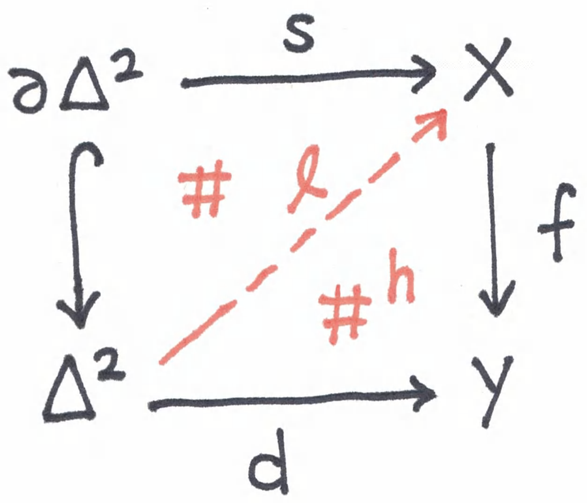


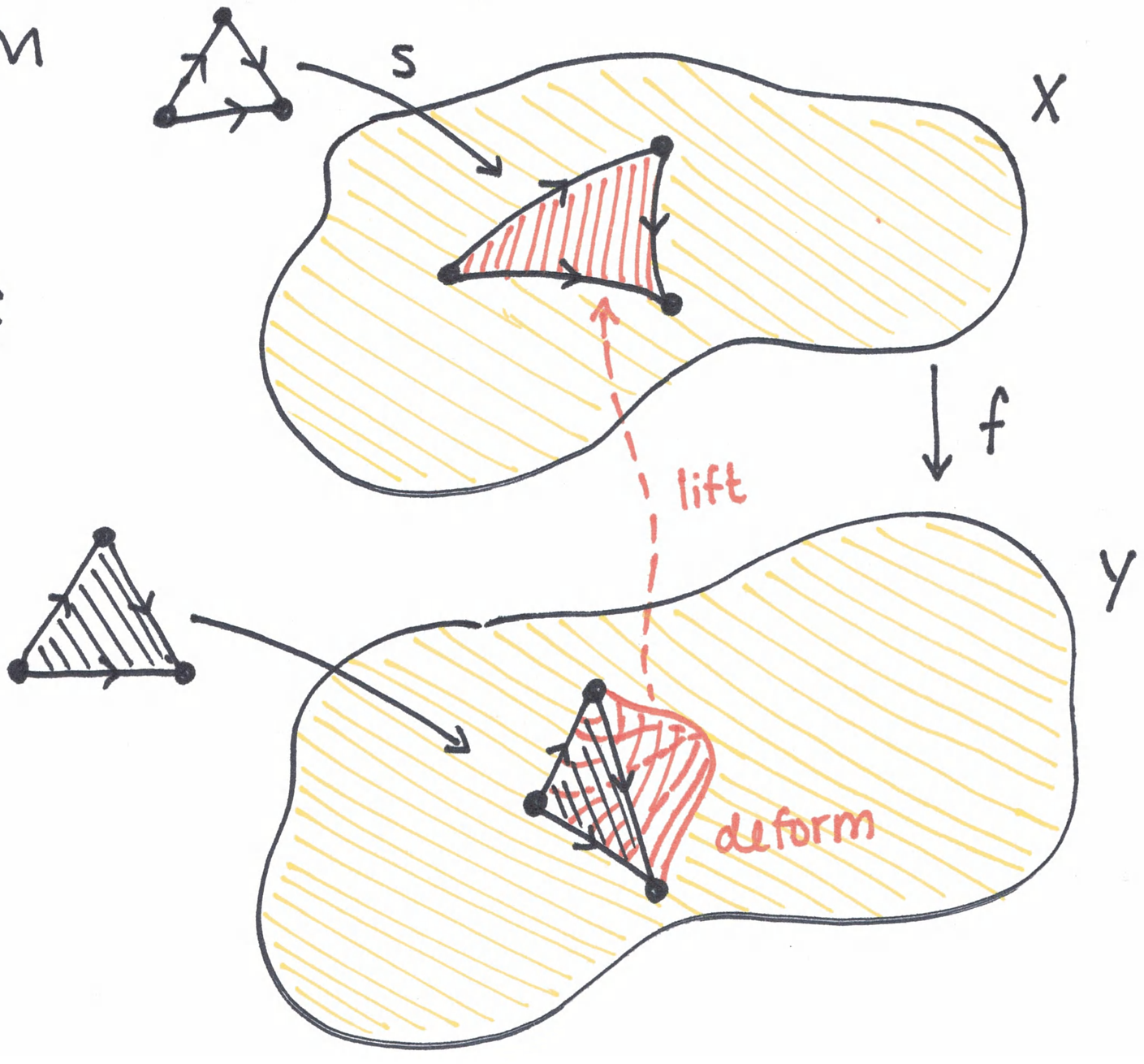
n=1 THE DIAGRAM

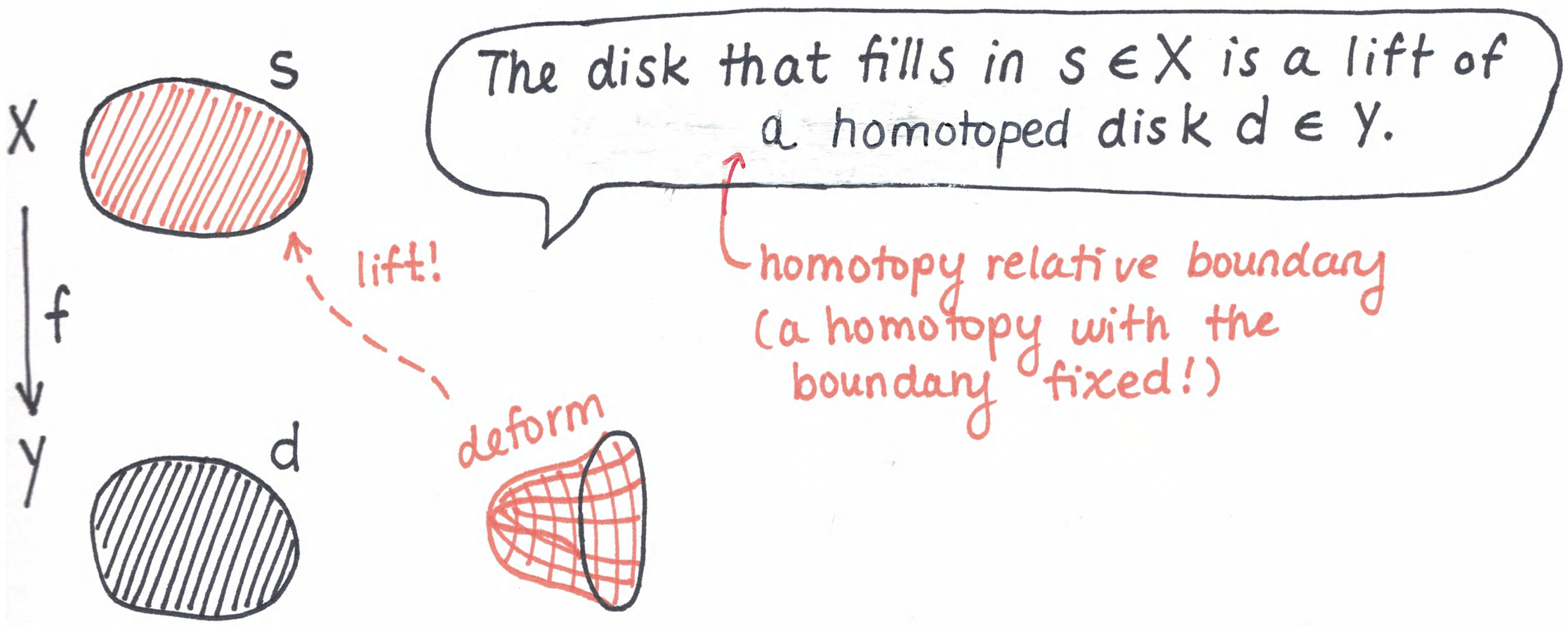




n=2 THE DIAGRAM







NOTE: Recall, $\pi_{\leq 1} f$ gives equivalence classes of homotopic paths. $\pi_{\leq 1} f$ is surjective on morphisms, as $d \in Y$ and $f(\ell) \in Y$ are identified (since they are homotopic, they are in the same equivalence class.)

$\forall s_0, s_1 \in \pi_{\leq 1} X$

$\forall d : f(s_0) \longrightarrow f(s_1)$

$\exists \ell : s_0 \longrightarrow s_1$ such that $(\pi_{\leq 1} f)(\ell) = d$

(This equality is the set of morphisms of the fundamental groupoid.)

EXERCISE #6

Show that the lifting condition in the Simplicial Whitehead Theorem is equivalent to the following two conditions:

(i) the map $\pi_0 f : \pi_0 X \longrightarrow \pi_0 Y$ is an isomorphism of sets

(ii) the map $\pi_n(f, x) : \pi_n(X, x) \longrightarrow \pi_n(Y, f(x))$ is an isomorphism of sets (or groups) for any $n \geq 1$ and any vertex $x \in X_0$.

RECALL, SKELETAL FILTRATION:

$\forall X \in$ sSet, $\exists \{X_i\}$ (with maps $f_i : X_i \to X_{i+1}$)

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots$$

where $X_n \subset X$ is generated by all simplices of dim $\leq n$.

This is thought of as:

$$X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots$$

vertices [0]-simplices; edges [1]-simplices; faces [2]-simplices

The simplicial set X is the colimit of this chain:

$$X = \operatorname*{colim}_{n \geq 0} X_n$$

Thus, to define a map out of X (a colimit), we construct a map out of each finite stage:

$$X_n \longrightarrow X_{n+1}$$

[or rather, $X_{\leq n} \longrightarrow X_{\leq (n+1)} \cdots$
since the degenerate $n+1$ simplices are already present in $X_{\leq n}$, we define the map by specifying how to add in nondegenerate $n+1$ simplices.]

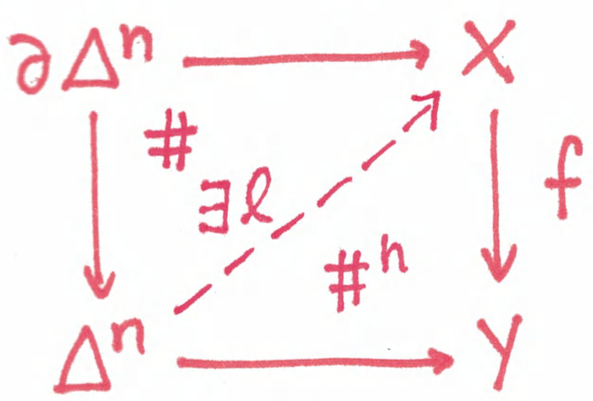
We have a pushout diagram:

$$\begin{array}{ccc} \coprod_{\sigma \in X^{nd}_{n+1}} \partial\Delta^{n+1} & \xrightarrow{\text{attaching map}} & X_{\leq n} \\ \downarrow & \# & \downarrow \\ \coprod_{\sigma \in X^{nd}_{n+1}} \Delta^{n+1} & \longrightarrow & X_{\leq n+1} \end{array}$$

∎

Proof (Simplicial Whitehead Theorem)

Given a diagram



$X, Y \in sSet_{Kan}$

such that $\exists \ell : \Delta^n \longrightarrow X$ which commutes as shown

in order to show that f is a homotopy equivalence, we will construct a map $g : Y \longrightarrow X$ and homotopies $h : \Delta^1 \times X \longrightarrow X$ and $i : \Delta^1 \times Y \longrightarrow Y$

such that $gof \overset{h}{\sim} id_X$ and $fog \overset{i}{\sim} id_Y$

$gof \overset{h}{\sim}$ $\quad$ $\overset{i}{\sim}$

We can filter X and Y by skeletal filtration

$$X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow X_3 \longrightarrow \cdots$$

$$Y_0 \longrightarrow Y_1 \longrightarrow Y_2 \longrightarrow Y_3 \longrightarrow \cdots$$

and proceed by induction on the dimension, n.

INDUCTION

[Thus, we will assume the desired result for the case $n-1$, assuming we have maps

$$g_{n-1} : Y_{n-1} \longrightarrow X_{n-1}$$
$$h_{n-1} : \Delta^1 \times X_{n-1} \longrightarrow X_n$$
$$i_{n-1} : \Delta^1 \times Y_{n-1} \longrightarrow Y_n$$

and we wish to construct maps

$$g_n : Y_n \longrightarrow X_n$$
$$h_n : \Delta^1 \times X_n \longrightarrow X_{n+1}$$
$$i_n : \Delta^1 \times Y_n \longrightarrow Y_{n+1}$$

NOTE: The base case, $n=-1$, is trivial (base step of induction)]

From the skeletal filtration of X, we have the commutative pushout square (below) for each inclusion $X_{n-1} \longrightarrow X_n$:

$$\begin{array}{ccc} \coprod_{\sigma \in Y_n^{nd}} \partial\Delta^n & \longrightarrow & Y_{n-1} \\ \downarrow & \# & \downarrow \\ \coprod_{\sigma \in Y_n^{nd}} \Delta^n & \longrightarrow & Y_n \end{array}$$

The Universal Property of Pushouts allows us to construct the relevant maps on the coproducts of simplices instead, while verifying that they are compatible with existing maps on the boundary

$$\begin{array}{ccc} \coprod_{\sigma \in Y_n^{nd}} \partial\Delta^n & \longrightarrow & Y_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{\sigma \in Y_n^{nd}} \Delta^n & \longrightarrow & Y_n \\ & \searrow & \downarrow \\ & & X_n \end{array}$$

Recall, the Universal Property of Pushouts:

Given a pushout square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & Q \end{array}$$

Morphisms $B \coprod_A C \longrightarrow Q$ are in bijection to pairs of morphisms

$B \longrightarrow Q$, $C \longrightarrow Q$

(such that the square #)

The Universal Property of Coproducts allows us to construct the maps we need for $\coprod_{\sigma \in Y_n^{nd}} \Delta^n \longrightarrow X_n$ by constructing these map for the individual components $\sigma : \Delta^n \longrightarrow Y$ (for some nondegenerate n-simplex)

(and its boundary $\partial\sigma : \partial\Delta^n \longrightarrow Y$, which factors through Y_{n-1})

Now, we have assumed $\exists\, g_{n-1} : Y_{n-1} \longrightarrow X_{n-1}$

(with homotopies $h_{n-1} : \Delta^1 \times X_{n-1} \longrightarrow X_n$

$i_{n-1} : \Delta^1 \times Y_{n-1} \longrightarrow Y_n$)

consider the composition map : $g_{n-1} \circ \partial\sigma$

$\partial\sigma : \partial\Delta^n \longrightarrow Y$

$g_{n-1} : Y_{n-1} \longrightarrow X_{n-1}$ $\Big\}$ $g_{n-1} \circ \partial\sigma : \partial\Delta^n \longrightarrow X$

Then $f \circ g \circ \partial\sigma : \partial\Delta^n \longrightarrow Y$

$\simeq$ homotopic via $\Delta^1 \times \partial\Delta^n \longrightarrow Y$

$\partial\sigma : \partial\Delta^n \longrightarrow Y$

The homotopy being given by the restriction of i_{n-1} (from the previous stage)

The homotopy $\Delta^1 \times \partial\Delta^n \longrightarrow Y$

combined with $\sigma : \Delta^n \longrightarrow Y$

yields the coproduct over $1 \times \partial\Delta^n$

$$A = (\Delta^1 \times \partial\Delta^n) \bigsqcup_{1 \times \partial\Delta^n} (\Delta^n \longrightarrow Y)$$

Consider the map $A \longrightarrow Y$

We can extend this to a map

$\pi : B \longrightarrow Y$

where $B = \Delta^1 \times \Delta^n$.

define $\tau : \Delta^n \longrightarrow Y$ to be $\pi\big|_{0 \times \Delta^1}$

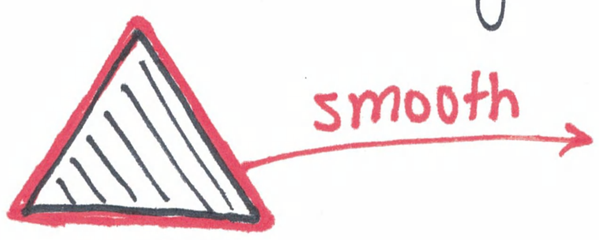
For a given simplex:

$n=2$



Δ^2

The map $\partial\Delta^n \xrightarrow{S} \underset{\text{smooth}}{\text{Sing}}(X)$ is a smooth map on the boundary:



and the map $\Delta^n \xrightarrow{d} \underset{\text{cont.}}{\text{Sing}}(X)$ is a continuous map on the interior:



we can deform the interior such that the resulting simplex is smooth

deform



We have a homotopy:

$$[0,1]\times\Delta^n \xrightarrow{h}_{\sim} X$$

continuous

such that

$h|_{0\times\Delta^n} = d : \partial\Delta^n \longrightarrow \underset{\text{smooth}}{\text{Sing}}(X)$

$h|_{0\times\partial\Delta^n}$ stays constant and smooth

$h|_{1\times\Delta^n}$ = a smooth map

stone weirstrass proves lift exists

Hence, we have a map

$\Delta^n \longrightarrow \underset{\text{smooth}}{\text{sing}}(X)$ that reduces to

$\partial\Delta^n \longrightarrow \underset{\text{smooth}}{\text{sing}}(X)$ on the boundary! ∎

5. $X \longrightarrow Ex^{\infty}(X)$ is a weak equivalence.

First, recall that we have several definitions of simplicial weak equivalence (equivalent definitions!):

see previous PROPOSITION

(1) $\forall K \in \mathrm{Kan}$, $f: X \longrightarrow Y$ is a simplicial weak equivalence if

$$\mathrm{Hom}(Y,K) \xrightarrow{\sim} \mathrm{Hom}(X,K)$$

is a simplicial homotopy equivalence

(2) $f: X \longrightarrow Y$ is a simplicial weak equivalence iff

$$Ex^{\infty}(f): Ex^{\infty}(X) \longrightarrow Ex^{\infty}(Y)$$

is a simplicial homotopy equivalence.

By the simplicial Whitehead Theorem, we can give another very concrete equivalent definition of simplicial weak equivalence ...

APPLY SIMPLICIAL WHITEHEAD THEOREM TO (2) ...

---

(3) $Ex^{\infty}(f): Ex^{\infty}(X) \longrightarrow Ex^{\infty}(Y)$

is a simplicial homotopy equivalence

[which from (2) means that $f: X \longrightarrow Y$ is a simplicial weak equivalence ...]

$\forall s, d$ (shown)

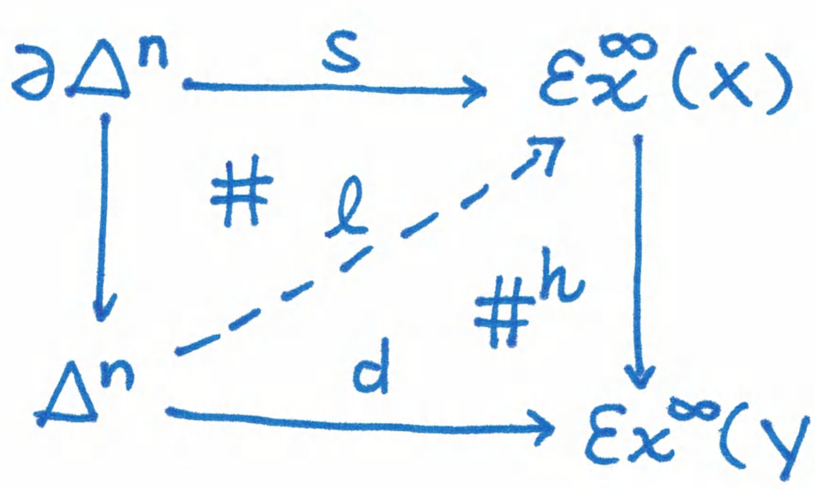
if there exists a map

$\ell: \Delta^n \longrightarrow Ex^{\infty}(X)$

such that the diagram commutes as shown:

upper triangle : #

lower triangle : $\#^h$



→

Recall, Y has a skeletal filtration

$$Y_0 \longrightarrow Y_1 \longrightarrow Y_2 \longrightarrow Y_3 \longrightarrow \cdots.$$

and $\mathcal{E}x^\infty(Y) = \operatorname{colim}_{n\geq 0} \mathcal{E}x^n(Y)$

$$Y \longrightarrow \mathcal{E}x\,Y \longrightarrow \mathcal{E}x(\mathcal{E}x(Y)) \longrightarrow \mathcal{E}x^3(Y) \longrightarrow \cdots$$

since $\partial\Delta^n$ and Δ^n are compact, the maps

$\partial\Delta^n \xrightarrow{s} \mathcal{E}x^\infty(X)$ and $\Delta^n \xrightarrow{d} \mathcal{E}x^\infty(Y)$

factor through some finite stage, k:

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & \mathcal{E}x^k(X) \\ \downarrow & & \downarrow \\ \Delta^n & \longrightarrow & \mathcal{E}x^k(Y) \end{array}$$

Recall, $sd \dashv \mathcal{E}x$ (adjoint functors), so we can rewrite the diagram above as:

$$\begin{array}{ccc} sd^k\partial\Delta^n & \longrightarrow & X \\ \downarrow & & \downarrow \\ sd^k\Delta^n & \longrightarrow & Y \end{array}$$

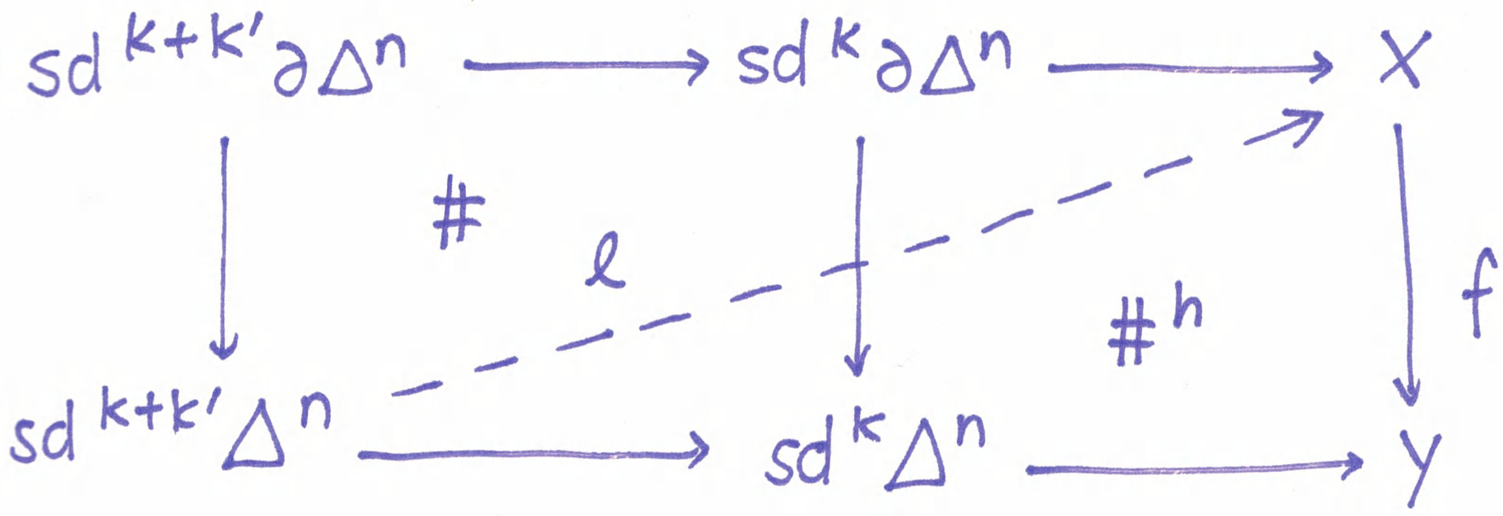
Now, to say that there exists a map $l: \Delta^n \longrightarrow \mathcal{E}x^\infty(X)$ such that the original diagram commutes means that this map also factors through some finite stage,

HOWEVER

it may NOT be the same stage as the maps s and d.

If l factors through some stage less than k, then we are done (the lift map exists at stage k). But, if l factors through some stage greater than k...

... we can find a k' such that ℓ factors through the stage $k+k'$. Then we have the diagram below:



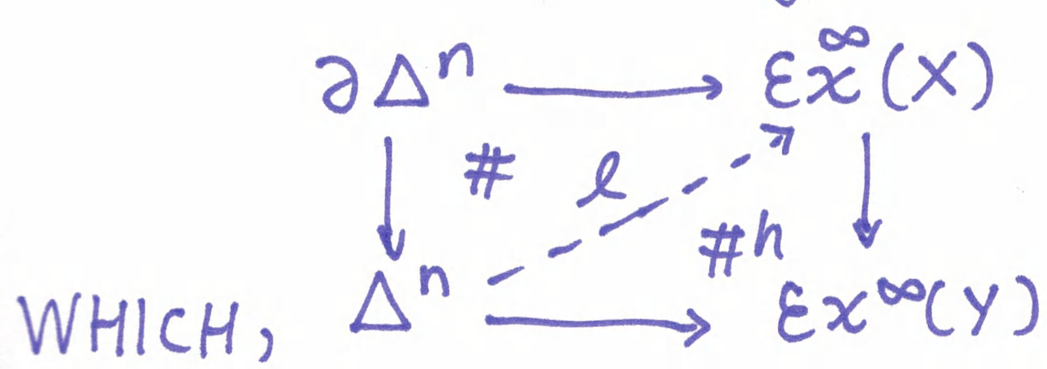
The upper triangle commutes, and the lower triangle commutes up to a homotopy h, where h is induced by some map

$$\Delta^1 \times \Delta^n \longrightarrow Ex^\infty(Y)$$

adjoint to the map

$$sd^{k+k'}(\Delta^1 \times \Delta^n) \longrightarrow Y$$

In practice, we can prove the existance of such maps h and ℓ in the diagram above, and this proves the ~~original~~ existance of h and ℓ in the original diagram:

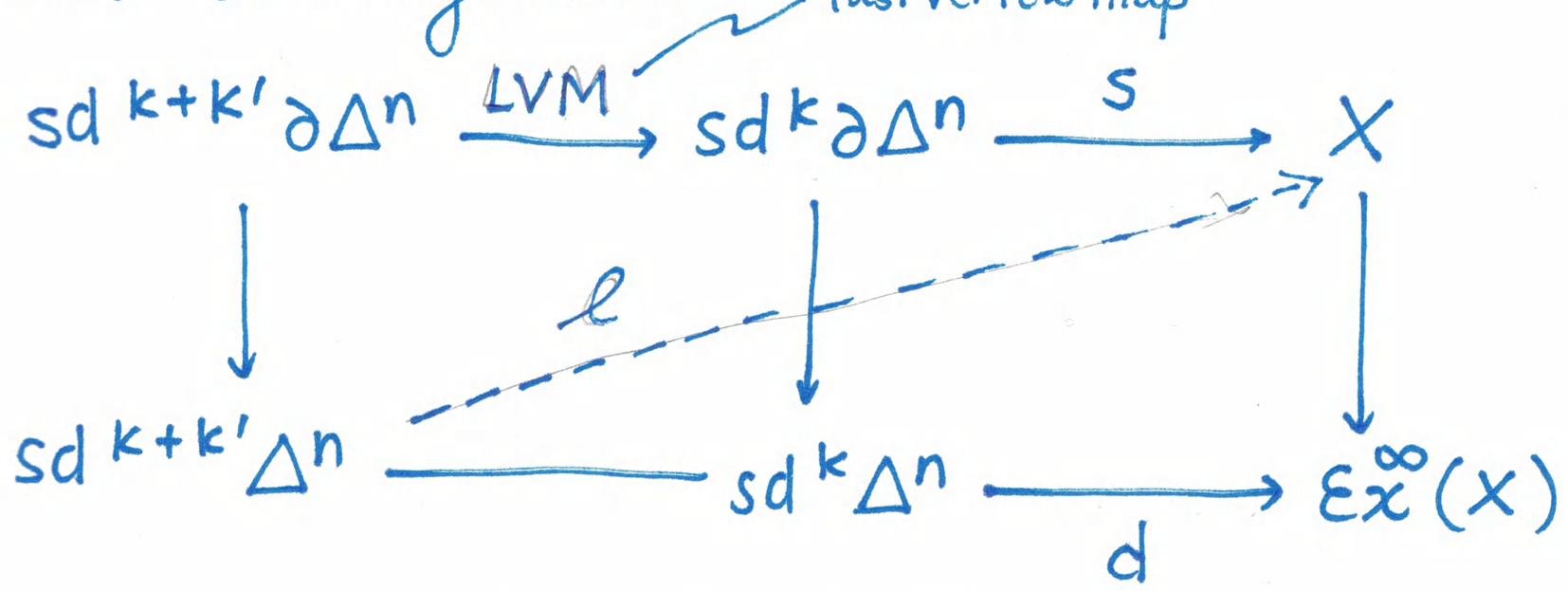


WHICH, By simplicial whitehead Theorem, means that the map $Ex^\infty(f): Ex^\infty(X) \longrightarrow Ex^\infty(Y)$ is a simplicial homotopy equivalence, WHICH, by (2), means that $f: X \longrightarrow Y$ is a simplicial weak equivalence.

THIS WILL BE A USEFUL WAY TO PROVE THAT $f: X \longrightarrow Y$ IS A SIMPLICIAL WEAK EQUIVALENCE!

← We can immediately employ this method to prove that $X \longrightarrow \mathcal{E}x^{\infty}(X)$ is a simplicial weak equivalence.

Given a diagram: (last vertex map)



we need to show that $\forall s, d$, we can construct maps h and ℓ such that the upper triangle commutes and the lower triangle commutes up to a homotopy induced by h.

$(\ell : sd^{k+k'}\Delta^n \longrightarrow X)$

construct ℓ

ℓ is easy to determine!

We already know how to define the map

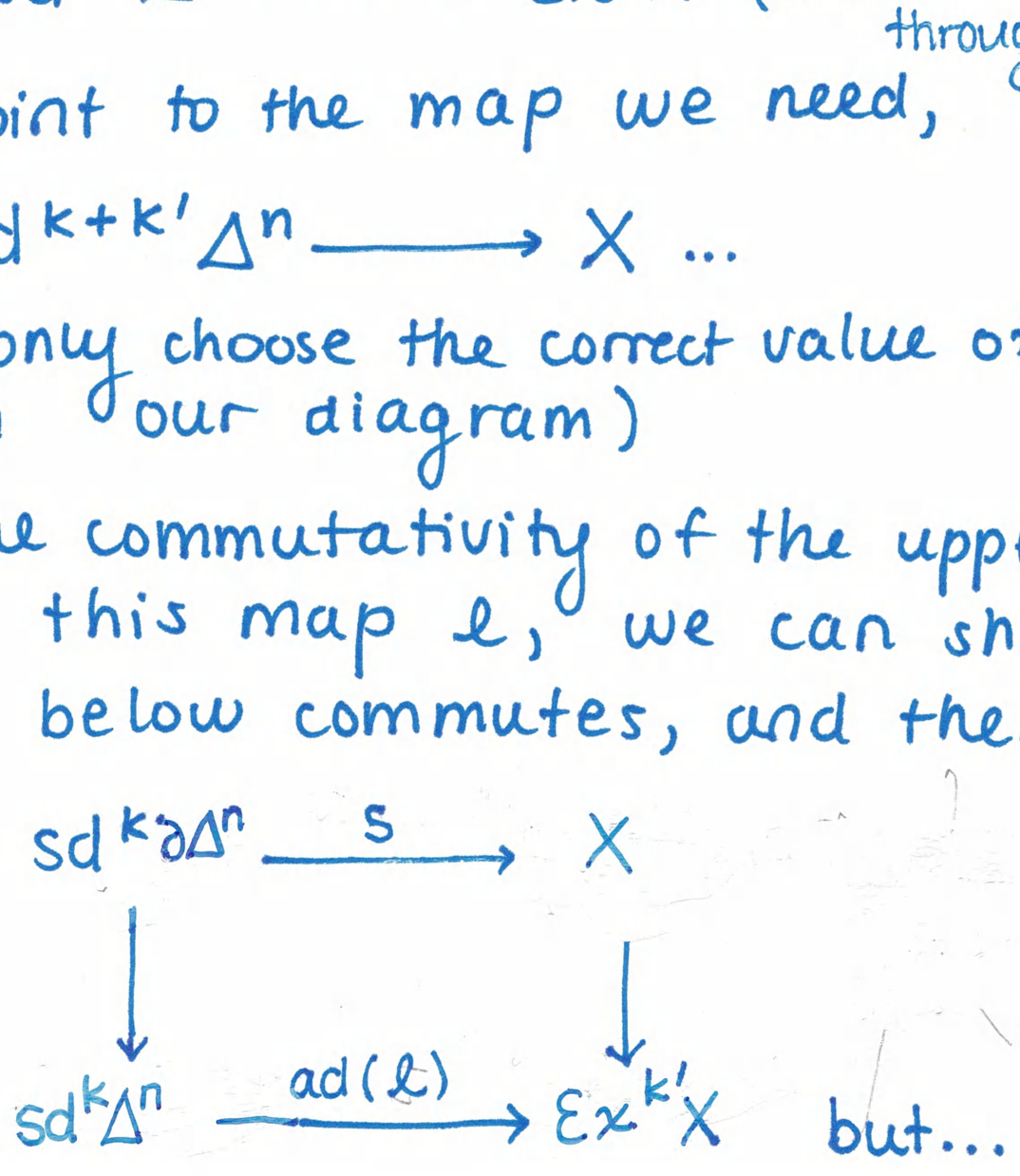
$$sd^k\Delta^n \longrightarrow \mathcal{E}x^{k'}X$$ (because it can be factored through some finite stage!)

(which is adjoint to the map we need,

$$sd^{k+k'}\Delta^n \longrightarrow X \ldots$$

we need only choose the correct value of k' to match our diagram)

and to show the commutativity of the upper triangle with this map ℓ, we can show the diagram below commutes, and then pass to the adjoint:



but...

$$\begin{array}{ccc} sd^k \partial\Delta^n & \longrightarrow & X \\ \downarrow & \# & \downarrow \\ sd^k \Delta^n & \longrightarrow & \mathcal{E}x^k X \end{array}$$

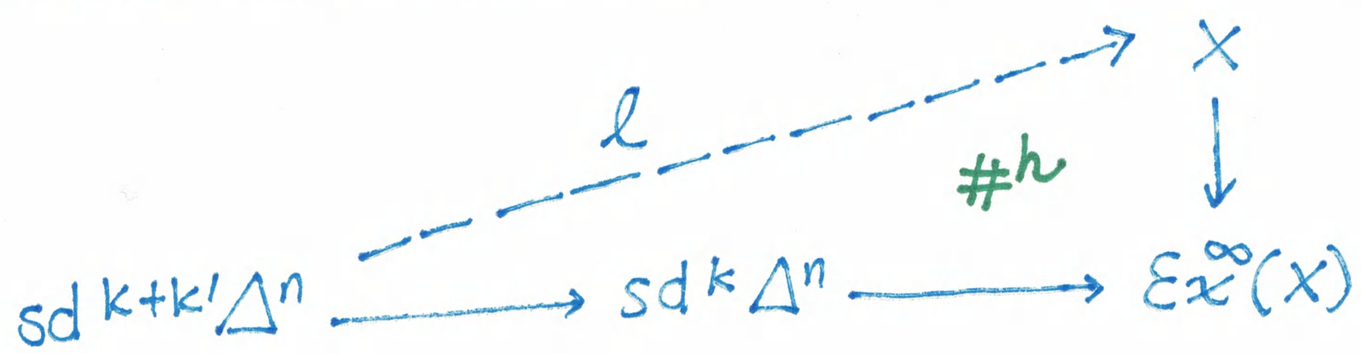
THIS DIAGRAM COMMUTES BECAUSE

$$\begin{array}{ccc} sd^k \partial\Delta^n & \longrightarrow & X \\ \downarrow & \# & \downarrow \\ sd^k \Delta^n & \longrightarrow & \mathcal{E}x^\infty X \end{array}$$

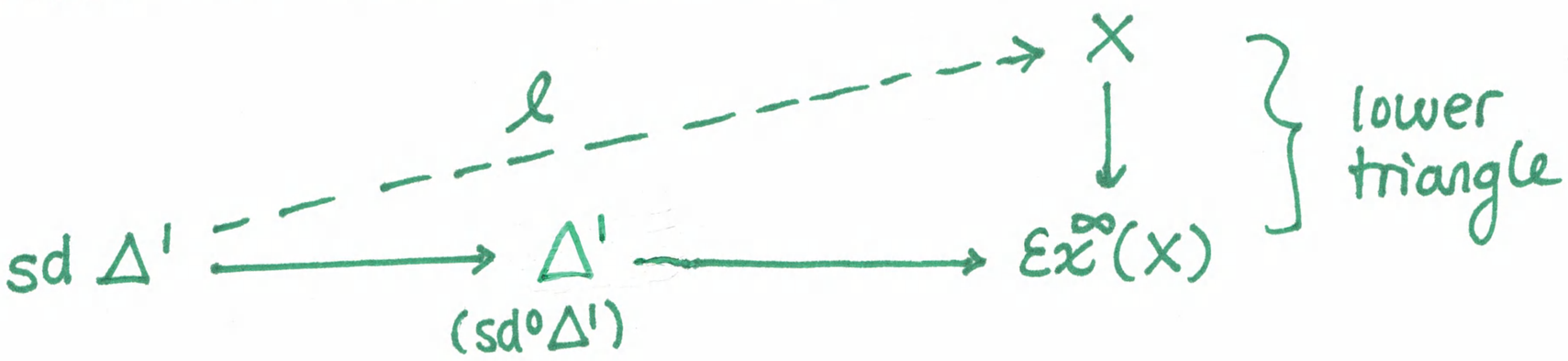
THIS DIAGRAM COMMUTES!

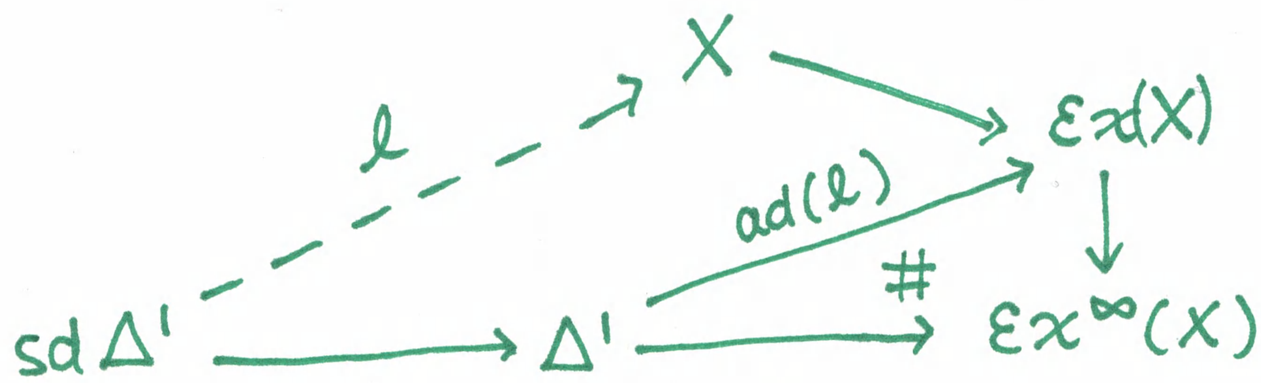
SO THE UPPER TRIANGLE COMMUTES!

TO SHOW THE LOWER TRIANGLE COMMUTES UP TO A HOMOTOPY:

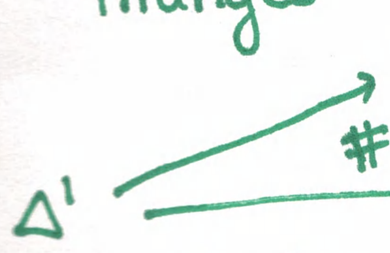


CONSIDER THE CASE $k=0$, $k'=1$, $n=1$





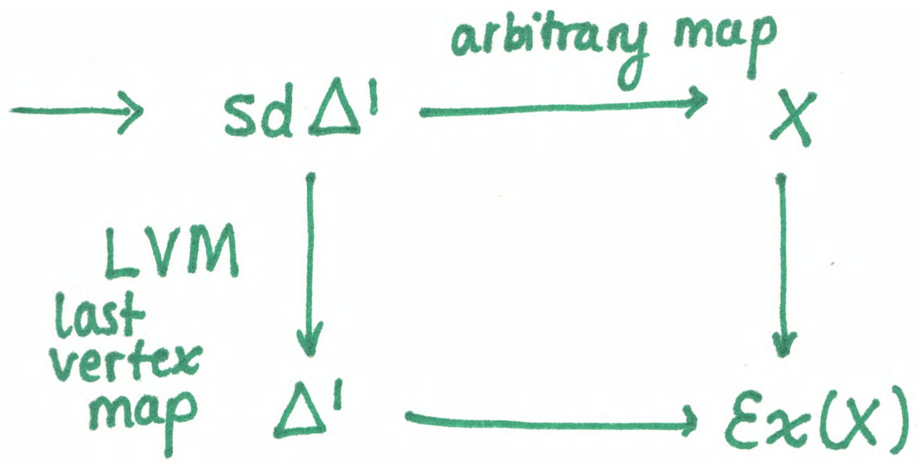
We claim that $sd\Delta^1 \longrightarrow \Delta^1 \longrightarrow \mathcal{E}x^\infty(X)$ is nullhomotopic, which, since the triangle



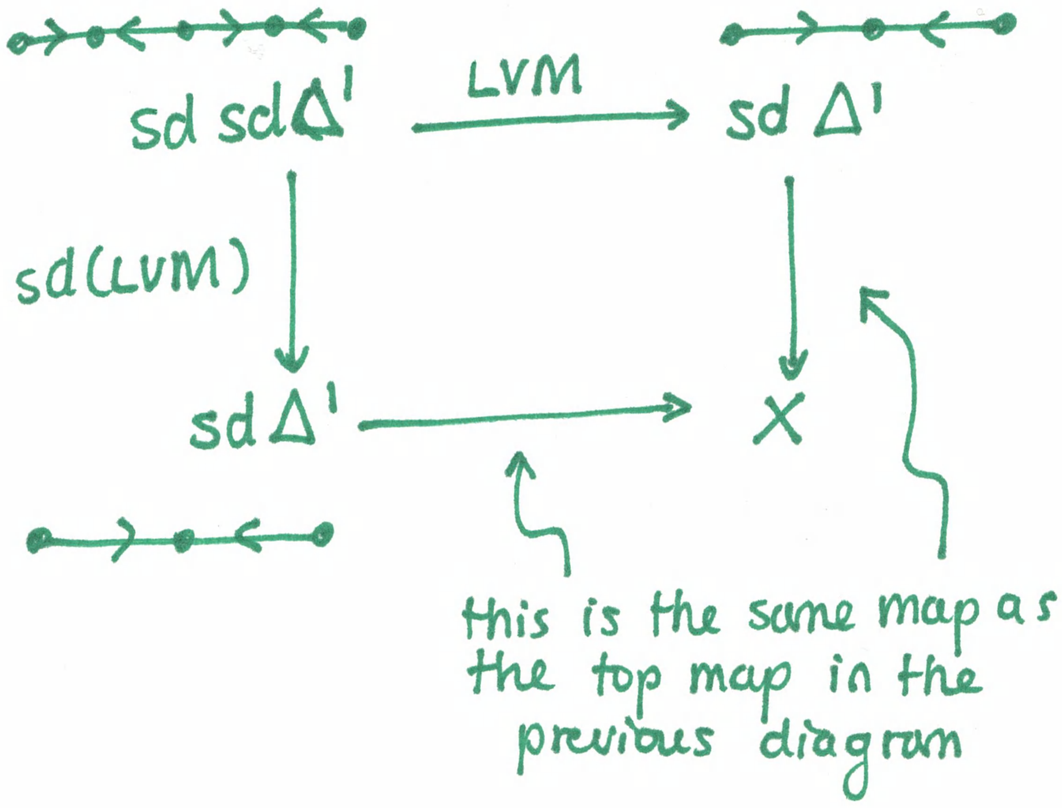
commutes by definition, this is equivalent to saying the diagram

$$\begin{array}{ccc} sd\Delta^1 & \longrightarrow & X \\ \downarrow & & \downarrow \\ \Delta^1 & \longrightarrow & \mathcal{E}x(X) \end{array}$$

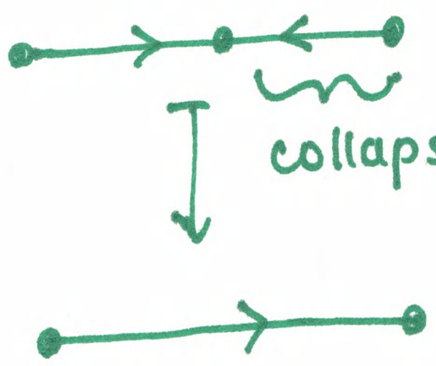
commutes. →



AND THIS DIAGRAM IS EQUIVALENT TO THE ADJOINT DIAGRAM:

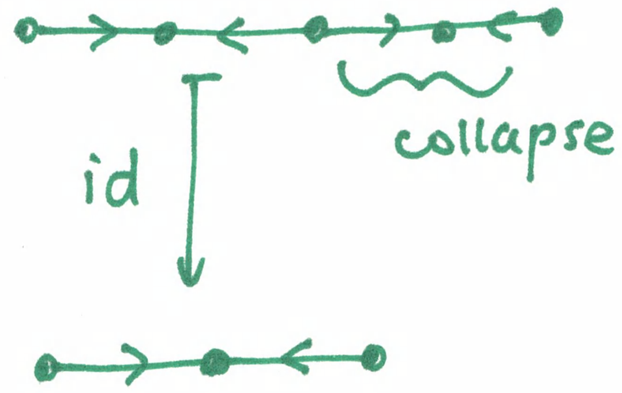


The last vertex map (LVM) maps



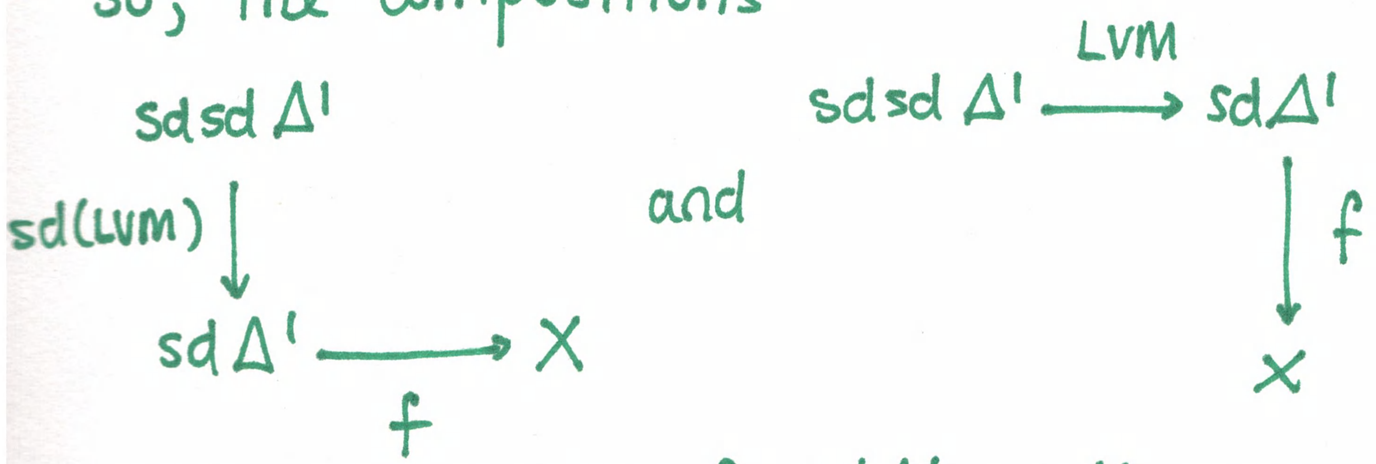
by collapsing the second simplex to the last vertex

The map sd(LVM) maps



by mapping the first half to itself (identity) and collapsing the second half.

So, the compositions

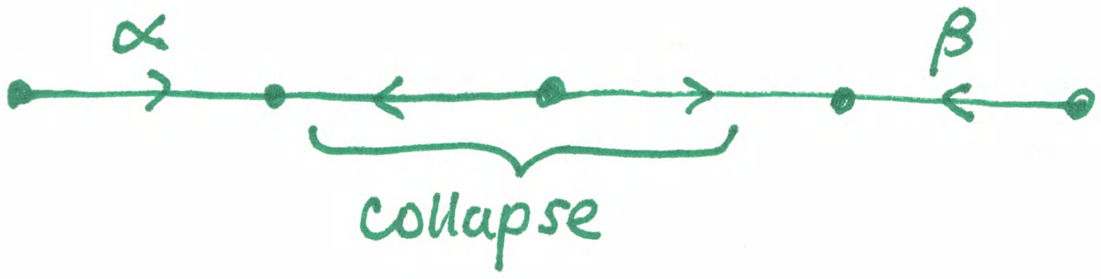


are the same map $f: sd\Delta^1 \longrightarrow X$ precomposed with two different maps (sd(LVM) and LVM ...) $\longrightarrow$

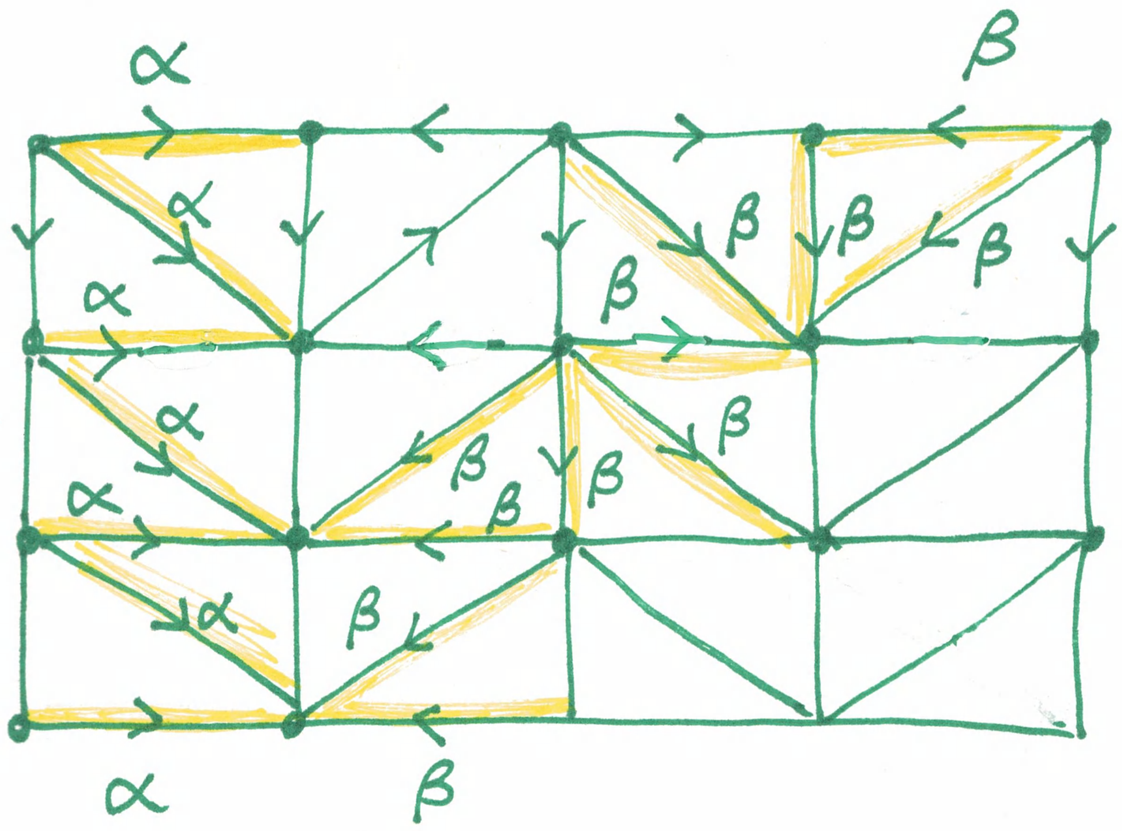
We can show that these compositions are the same up to a homotopy:



Are these homotopic?



We can construct a homotopy:



Hence

$$\begin{array}{ccc} \Delta^1 & \longrightarrow & Sd\,\Delta^1 \\ \downarrow & \# & \downarrow \\ \Delta^1 & \longrightarrow & Ex(X) \end{array}$$

, and the lower triangle commutes up to a homotopy.

A SIMILAR ARGUMENT CAN BE MADE IN HIGHER DIMENSIONS. ▨

So $X \longrightarrow Ex^{\infty}(X)$ is a weak equivalence!

We have defined the functor Ex^∞ and showed how this functor improves a simplicial set to a Kan complex:

$$X \in sSet$$

$$\mathrm{Ex}^\infty X \in sSet_{Kan}$$

and $X \xrightarrow{\sim} \mathrm{Ex}^\infty X$ is a weak equivalence.

We can thus replace a simplicial set X with something weakly equivalent but better behaved, namely, $\mathrm{Ex}^\infty X$.

The fact that $\mathrm{Ex}^\infty X$ is Kan carries with it the important fact that restricting to Kan complexes preserves weak equivalences, and in $sSet_{Kan}$

$$\begin{matrix}\text{simplicial}\\ \text{weak equivalences}\end{matrix} \equiv \begin{matrix}\text{simplicial}\\ \text{homotopy equivalences}\end{matrix}$$

DEFINITION **DERIVED INTERNAL HOM**

$$RHom : sSet^{op} \times sSet \longrightarrow sSet$$

$$(X, Y) \longmapsto Hom(\mathrm{Ex}^\infty X, \mathrm{Ex}^\infty Y)$$

This derived internal hom maps a pair of simplicial sets to the internal hom of these simplicial sets, improved to Kan complexes ($\mathrm{Ex}^\infty X$ and $\mathrm{Ex}^\infty Y$)

To say that RHom preserves weak equivalences is to say that:

$$\begin{array}{ccc} X & & Y \\ \downarrow\sim & \text{and} & \downarrow\sim \\ X' & & Y' \end{array} \Rightarrow \begin{array}{c} RHom(X,Y) = Hom(\mathrm{Ex}^\infty X, \mathrm{Ex}^\infty Y) \\ \downarrow\sim \qquad\qquad\qquad \downarrow\sim \\ RHom(X',Y') = Hom(\mathrm{Ex}^\infty X, \mathrm{Ex}^\infty Y) \end{array}$$

For sSet, simplicial homotopy equivalences $\subset$ simplicial weak equivalences

So, to say RHom preserves weak equivalences will follow from the fact that internal hom (Hom) preserves homotopy equivalences.

RECALL (... from Topology ...)

PROPOSITION Hom preserves homotopy equivalences

Suppose we have a homotopy equivalence between simplicial sets X and X'.

$$X \underset{g}{\overset{f}{\rightleftarrows}} X'$$

$$\Delta^1 \times X \xrightarrow{h} X, \quad id_X \overset{h}{\sim} g \circ f$$

$$\Delta^1 \times X' \xrightarrow{i} X', \quad id_{X'} \overset{i}{\sim} f \circ g$$

Applying the internal hom functor

$$Hom(-, Y)$$

(a contravariant functor)

$$X \underset{g}{\overset{f}{\rightleftarrows}} X' \quad \xrightarrow{\text{apply } Hom(-,Y)} \quad Hom(X,Y) \underset{Hom(g,Y)}{\overset{Hom(f,Y)}{\rightleftarrows}} Hom(X',Y)$$

and the maps:

$$\Delta^1 \times X \xrightarrow{h} X \quad \Rightarrow \quad Hom(\Delta^1 \times X, Y) \xleftarrow{Hom(h,Y)} Hom(X,Y)$$

$$\Delta^1 \times X' \xrightarrow{i} X' \quad \Rightarrow \quad Hom(\Delta^1 \times X', Y) \xleftarrow{Hom(i,Y)} Hom(X',Y)$$

By the Hom-product-adjunction property ...

$$Hom(\Delta^1 \times X, Y) = Hom(X, Hom(\Delta^1, Y))$$

we also have a composition property on these maps:

$Hom(f, Y): Hom(X', Y) \longrightarrow Hom(X, Y)$

$Hom(g, Y): Hom(X, Y) \longrightarrow Hom(X', Y)$

Then

Recall, contravariance!

$$Hom(f,Y) \circ Hom(g, Y) = Hom(g \circ f, Y)$$

where $Hom(g \circ f, Y): Hom(X, Y) \longrightarrow Hom(X, Y)$

This map has the same domain and codomain as:

$$Hom(id_X, Y): Hom(X, Y) \longrightarrow Hom(X, Y)$$

We want to show that:

$$Hom(f, Y) \circ Hom(g, Y) \simeq id_{Hom(X,Y)}$$

and

$$Hom(g, Y) \circ Hom(f, Y) \simeq id_{Hom(X', Y)}$$

Consider the diagram:

$$\begin{array}{ccc} \Delta' \times Hom(X,Y) & \xrightarrow{\beta} & Hom(X,Y) \\ \downarrow \alpha & \nearrow Hom(h,Y) & \\ Hom(\Delta' \times X, Y) & & \end{array}$$

We want to construct a map α, for then the top map β

$\Delta' \times Hom(X,Y) \longrightarrow Hom(X,Y)$

can be written in terms of α and $Hom(h,Y)$

Take the adjoint of the diagram, and construct α':

$$\begin{array}{ccc} (\Delta' \times X) \times (\Delta' \times Hom(X,Y)) & \xrightarrow{\beta_1'} & \Delta' \times \Delta' \times Y \\ \downarrow \alpha' & \swarrow \beta_2' & \\ Y & & \end{array}$$

$\Delta' \times \Delta' \times Y$

$=$

$(\Delta' \times X) \times \Delta' \times Hom(X,Y)$

$(\Delta' \times \Delta') \times X \times Hom(X,Y)$

commutativity of product

(continued) Then $\alpha' = \beta_2' \circ \Delta'$, and α is the adjoint m

The adjoint of the identity map is the evaluation map

$$\begin{array}{ccc} Hom(X,Y) & \xrightarrow{id} & Hom(X,Y) \\ X \times Hom(X,Y) & \xrightarrow{eval} & Y \end{array}$$

DEFINITION $(\mathcal{C}, W_{\mathcal{C}})$ and $(\mathcal{D}, W_{\mathcal{D}})$ are relative categories.

and $F: \mathcal{C} \longrightarrow \mathcal{D}$ is a functor that need not preserve weak equivalences.

The functor F is called "**RIGHT-DERIVABLE**" if

1. $\exists\, \mathcal{C}' \subset \mathcal{C}$ (a full subcategory)
   & with fully faithful inclusion functor $\iota: \mathcal{C}' \longrightarrow \mathcal{C}$
2. $\exists\, R: \mathcal{C} \longrightarrow \mathcal{C}'$ that preserves weak equivalences
   with a natural weak equivalence $r: id_{\mathcal{C}} \xrightarrow{\sim} \iota \circ R$

such that ...

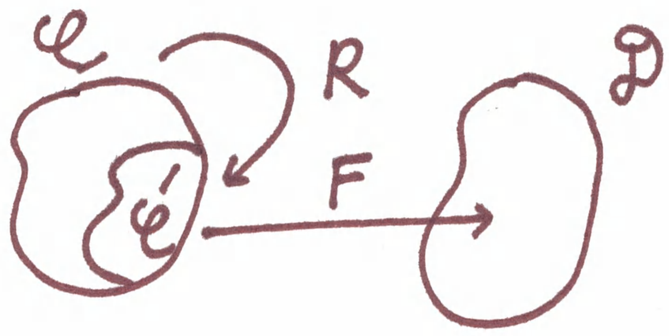
$F|_{\mathcal{C}'}$ preserves weak equivalences

$\forall X \in \mathcal{C}$
$X \xrightarrow{\sim} RX$
is a weak equivalence in $\mathcal{C}$

In this case, we define the "right-derived functor of F"

$$RF := F \circ R : (\mathcal{C}, W_{\mathcal{C}}) \longrightarrow (\mathcal{D}, W_{\mathcal{D}})$$

This composition then preserve's weak equivalences



F only preserves weak equivalence on the subset $\mathcal{C}' \subset C$, the codomain of R...

The "left-derived functor of F" is defined similarly: **LEFT-DERIVABLE**

- replace R by L
- replace r by $\ell: \iota \circ L \xrightarrow{\sim} id_{\mathcal{C}}$

$\forall X \in \mathcal{D}$
$LX \xrightarrow{\sim} X$
is a weak equivalence in $\mathcal{D}$

$$LF := L \circ F : (\mathcal{C}, W_{\mathcal{C}}) \longrightarrow (\mathcal{D}, W_{\mathcal{D}})$$

EXAMPLES

Example 1 Take $(\mathscr{C}, W_{\mathscr{C}}) = (sSet, \text{simplicial weak equivalences})$

and $\mathscr{C}' = \text{simplicial kan complexes} = sSet_{kan}$

Then $R = Ex^{\infty} : sSet \longrightarrow sSet_{kan}$
$(\mathscr{C}) \qquad (\mathscr{C}')$

and $r : id_{sSet} \longrightarrow U \circ R = Ex^{\infty}$
is a natural weak equivalence,
that is, a natural transformation
whose components $X \longrightarrow RX$
here, $X \longrightarrow Ex^{\infty}X$
are weak equivalences.

(Recall, we already showed $X \xrightarrow{\sim} Ex^{\infty}X$)

Example 2 Fix $X \subset sSet$

and consider the functor

$$Hom(X, -) : sSet \longrightarrow sSet$$
$$(\mathscr{C}) \qquad (\mathscr{C}')$$

The functor $F = Hom(X, -)$ is right derivable!

$R = Ex^{\infty} : sSet \longrightarrow sSet_{kan} \subset sSet$

Then $RF = \underbrace{RHom(X, -)}_{\text{derived internal hom}} = F \circ Ex^{\infty} = Hom(X, Ex^{\infty}(-))$

Example 3 If F already preserves weak equivalences, then F is always right/left derivable

take: $RF = F = LF$

and
$\mathscr{C}' = \mathscr{C}$
$R = id_{\mathscr{C}}$
$L = id_{\mathscr{C}'}$
$r = id_{id_{\mathscr{C}}}$

Example 4 Given a finite set I,

define $\prod : sSet^I \longrightarrow sSet$

where $sSet^I$ is an I-indexed family of simplicial sets

Finite products "$\prod$" are automatically derived (right-derived):

$$R\prod = \prod$$

This is because $\prod$ preserves weak equivalences:

Suppose we have weak equivalences in $sSet^I$ given by

$$f_i : X_i \xrightarrow{\sim} Y_i$$

we need to show that: $\prod_i f_i : \prod_i X_i \longrightarrow \prod_i Y_i$ is a weak equivalence

METHOD #1:

[use simplicial Whitehead Theorem]

We have:

$\prod_i X_i$

$\downarrow$

$\prod_i Y_i$

} subdivide :

$$\begin{array}{ccc} sd^k \partial\Delta^n & \longrightarrow & \prod_i X_i \\ \downarrow & & \downarrow \\ sd^k \Delta^n & \longrightarrow & \prod_i Y_i \end{array}$$

By theorem, $\forall i$, $\exists k'_i$, l_i, h_i, such that

$$\begin{array}{ccccc} sd^{k+k'}\partial\Delta^n & \longrightarrow & sd^k\partial\Delta^n & \longrightarrow & X_i \\ \downarrow & & \downarrow & \nearrow_{l_i} \# & \downarrow \\ sd^{k+k'}\Delta^n & \longrightarrow & sd^k\Delta^n & \longrightarrow & Y_i \end{array} \qquad \#\, h_i$$

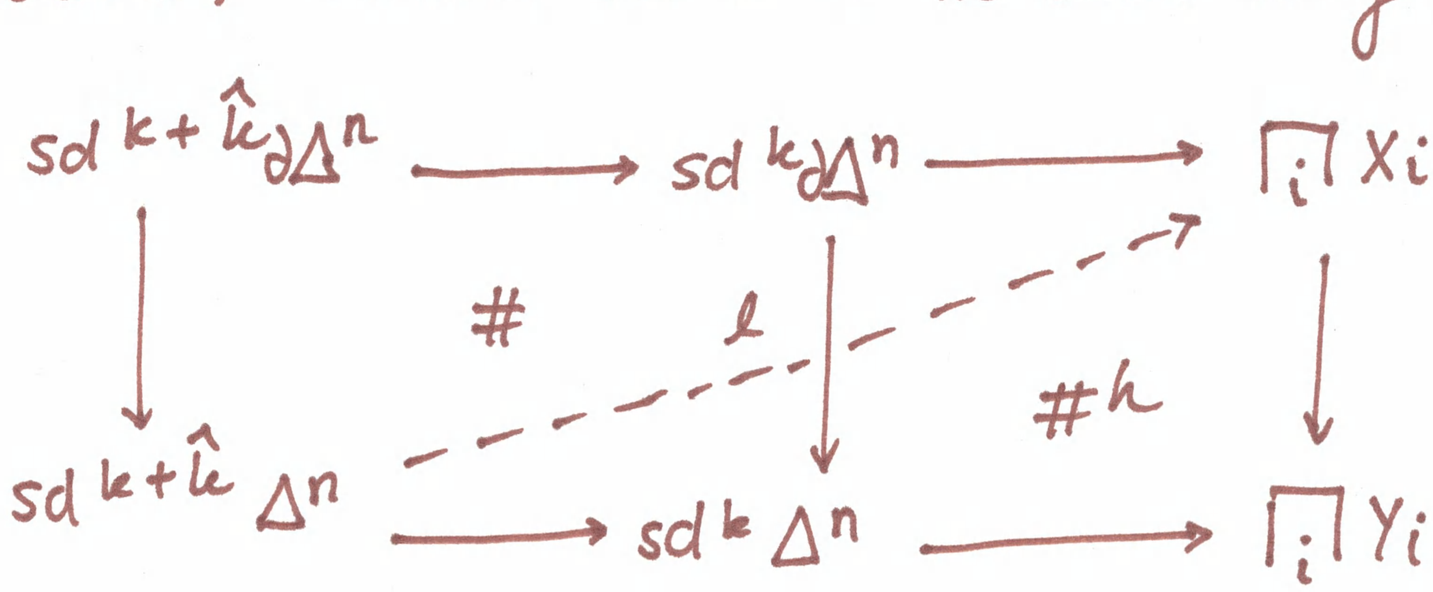
But we can always increase the value of k' (subdividing further) without changing the $\longrightarrow$

→ commutativity of the diagram, so, ...

Let $\hat{k} = \max_{i \in I} k'_i$

(since I is finite, maximum exists)

WLOG, assume $k'_i = \hat{k}$ in each diagram; then we have,



where ℓ and h are constructed individually for each i (universal property of products)

[Alternate proof w/ Ex^∞ functor]

METHOD #2: Ex^∞ preserves finite products

[first, right adjoint functors preserve small limits;
second, taking the sequential colimit of X_n,
note that sequential colimits commute
with finite products]

$\prod_i f_i \in$ s.w.e $\iff$ $Ex^\infty \prod_{i \in I} f_i = \prod_{i \in I} Ex^\infty f_i \in$ s.h.e

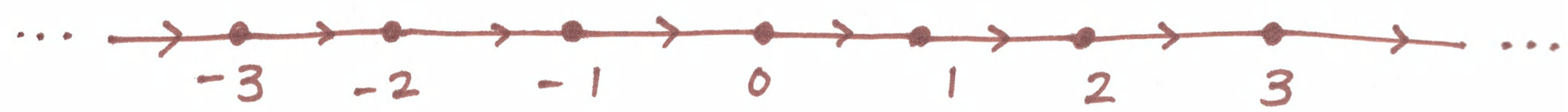
simplicial weak equivalence

simplicial homotopy equivalence

By the Universal property of products, consider each component, and then any product of shes is a she. ▨

Exercise

Consider $X :=$ infinite line
(set of infinite points,
numbered by $\mathbb{Z}$, with 1-simplices
joining n to $n+1$)



Consider $f: X \longrightarrow \Delta^0$

ⓐ show f is a simplicial weak equivalence

ⓑ Consider $\prod_{n\geq 0} f : \prod_{n\geq 0} X \longrightarrow \prod_{n\geq 0} \Delta^0 \cong \prod_{n\geq 1} \Delta^0$

$\prod_{n\geq 0} f$: indentical copies of f

$\prod_{n\geq 0} X$: infinite identical copies of X

$\prod_{n\geq 1} \Delta^0$: infinite identical copies of Δ^0

Show this is NOT
a simplicial weak equivalence.
[that is, compute π_0]

Hint: X is contractible,
but the infinite product is not contractible
[π_0 detects this; compute π_0]

Since each edge can only contract along
$n \longrightarrow n+1$,
each vertex can only move up or down.
(to $n+1$) (to n)

This divides the
result into two
collections; cannot
move between collections
in a finite number
of stages.

Proposition Given an infinite set I,

$$\prod : sset^{I} \longrightarrow sset$$

$\prod$ is right derivable.

$$R\prod^{I} = \prod^{I} \circ (\mathcal{E}x^{\infty})^{I}$$

proof Take $\mathcal{C}' \doteq (sSet_{Kan})^{I}$

$$R' = (\mathcal{E}x^{\infty} I)$$

then r is a map in each component

$$X \longrightarrow \mathcal{E}x^{\infty} X$$

Exercise Show that $\prod$ preserves simplicial homotopy equivalences.

(where I could be finite or infinite)

(continued) HOMOTOPY LIMITS & COLIMITS

DEFINITION

(indexing category) $I \in Cat$

(target category) $(\mathcal{C}, W) \in RelCat$

relative category of I-indexed diagrams

define $(\mathcal{C}, W)^I := (\mathcal{C}^I, W^I)$

where $\mathcal{C}^I$ is the category of I-indexed diagrams in $\mathcal{C}$ $\quad \mathcal{C}^I = \{ I \longrightarrow \mathcal{C} \}$

and W^I is the set of natural transformations

~~category of I-indexed~~

~~maps $I \to \mathcal{C}(P)$~~

$t : F \longrightarrow G$

$\downarrow \qquad \downarrow$

$F : I \to \mathcal{C} \qquad G : I \to \mathcal{C}$

$F \in \mathcal{C}^I \qquad G \in \mathcal{C}^I$

such that

$t(i) : F(i) \longrightarrow G(i)$

belongs to W, $\forall i \in I$.

DEFINITION

Recall, the limit functor

$$\lim : \mathcal{C}^I \longrightarrow \mathcal{C}$$

sends I-indexed diagrams to their limit

DEFINITION

The homotopy limit functor

$$\text{holim} : (\mathcal{C}, W)^I \longrightarrow (\mathcal{C}, W)$$

is the "right derived functor" of the limit functor above.

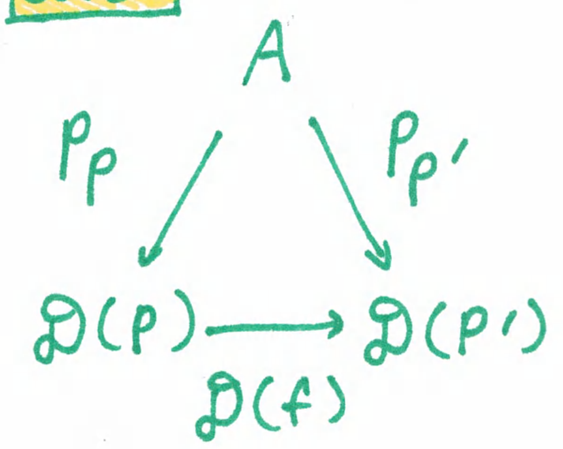
$$\text{holim} := R\lim$$

diagram $\mathcal{D} : I \longrightarrow \mathcal{C}$

object $A \in \mathcal{C}$

projection maps $p_p : A \longrightarrow \mathcal{D}(P)$, $p \in I$

cone



for any $f : p \to p'$ (morphism in I)

limit the terminal object (A) in the category of cones over $\mathcal{D}$

From nLab:

OLD SCHOOL DEFINITION WITH HOMOTOPY CATEGORIES

Given a relative category $(\mathcal{C}, W)$,
the "homotopy category" (if such a category exists)
is the result of forcing weak equivalences to
become isomorphisms.

Forcing the invertibility of these weak equivalences
is tantamount to localizing $\mathcal{C}$ at W:

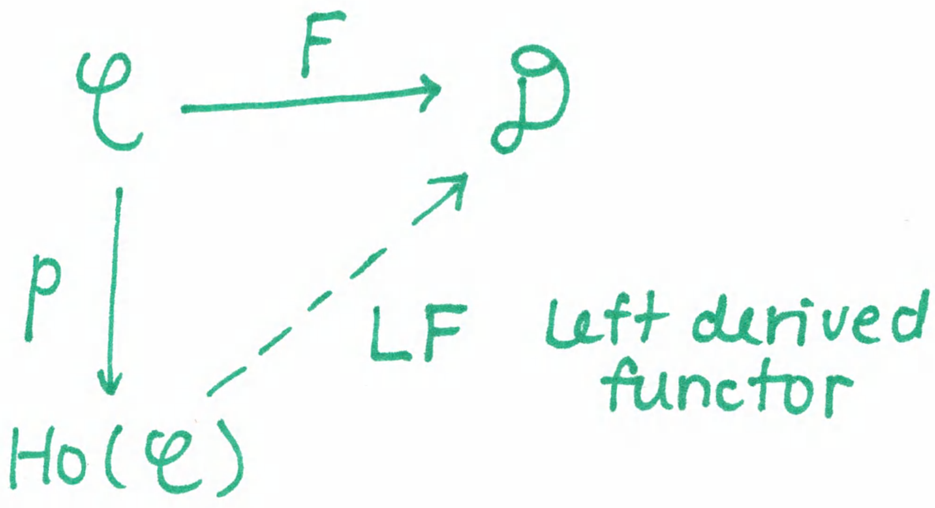
$$\begin{matrix}\text{homotopy}\\ \text{category}\end{matrix} \equiv \begin{matrix}\text{localizing } \mathcal{C}\\ \text{at } W\end{matrix}$$

$$Ho(\mathcal{C}) \equiv \mathcal{C}[W^{-1}]$$

The map from $\mathcal{C}$ to this new homotopy category $Ho(\mathcal{C})$
is called the projection map

$$p: \mathcal{C} \longrightarrow Ho(\mathcal{C})$$

Now, given a functor $F: \mathcal{C} \longrightarrow \mathcal{D}$, we can
define the "left-derived functor" as
the RIGHT Kan extension along the
projection map:



Similarly, the "right derived functor" is the
LEFT Kan extension along the projection map.

NOTE: Such extensions do not always exist!

(NOTE)^2: This definition is antiquated, and we will use the "modern" definition!

DEFINITION

Recall, the colimit functor

$$\text{colim} : \mathcal{C}^{I} \longrightarrow \mathcal{C}$$

sends I-indexed diagrams to their colimit

DEFINITION

The homotopy colimit functor

$$\text{hocolim} : (\mathcal{C}, W)^{I} \longrightarrow (\mathcal{C}, W)$$

is the "left derived functor" of the colimit functor above.

$$\text{hocolim} := \mathbb{L}\,\text{colim}$$

diagram $\mathcal{D} : I \to \mathcal{C}$

objects $A \in \mathcal{C}$

injection maps $c_p : \mathcal{D}(p) \to A$

cocone

$$\begin{array}{ccc} \mathcal{D}(p) & \xrightarrow{\mathcal{D}(f)} & \mathcal{D}(p') \\ & {\scriptstyle c_p} \searrow \quad \swarrow {\scriptstyle c_{p'}} & \\ & A & \end{array}$$

for any $f : p \to p'$ (morphisms in I)

colimit the initial object in the category of cocones under $\mathcal{D}$.

EXAMPLE I: discrete category

Recall... [objects in the discrete category are elements of I and morphisms in the discrete category are all identity morphisms] [NOTE: The discrete category is a groupoid]

$\lim = \prod_I$ (limit / ordinary product)

$\text{colim} = \coprod_I$ (colimit / ordinary coproduct)

Then, I-indexed diagrams (relative category) $(\mathcal{C}, W)^{I}$ are just families of objects of $\mathcal{C}$ indexed by elements of I [W^I is only natural transformations $t : F \longrightarrow$ such that $t(i) : F(i) \longrightarrow (i)$ belongs to W, $\forall i \in I$, that is, only identity morphisms.]

In this case...

$\mathbb{R}\lim_I := R\prod_I$ homotopy product

and

$\mathbb{L}\,\text{colim}_I := L\coprod_I$ homotopy coproduct

EXAMPLE

CASE 1

$\mathcal{C}$: simplicial sets
W : simplicial weak equivalences

I : discrete & finite $\Rightarrow \lim_I = \prod_I$

$R\lim = \lim$

finite products of sSet already preserve weak equivalences

CASE 2

I : discrete & infinite
$\mathcal{C}$: simplicial sets
W : simplicial weak equivalences

$\Rightarrow \lim_I = \prod_I$

$R\lim = \lim \circ (Ex^\infty)^I$

infinite products of sSet, derived functor sends simplicial weak equivalences to simplicial homotopy equivalences.

CASE 3

I : discrete & finite/infinite
$\mathcal{C}$: simplicial sets
W : simplicial weak equivalences

$\Rightarrow \operatorname{colim}_I = \coprod_I$

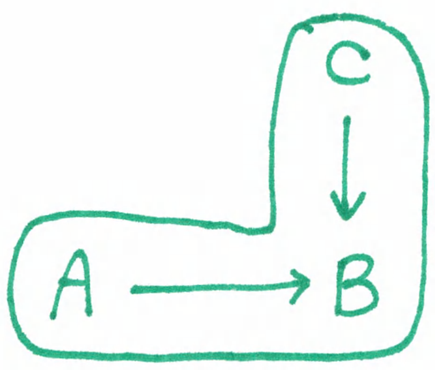
$L\operatorname{colim} = \operatorname{colim}$

coproducts of sSet already preserve weak equivalences

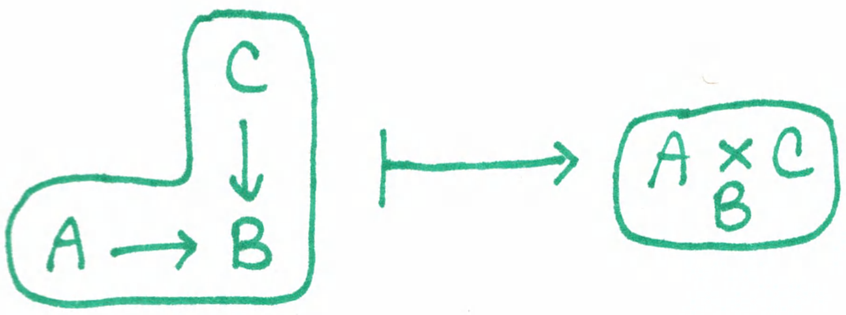
EXAMPLE Recall, the "pullback" of a diagram

PULLBACK $I = \{1 \longrightarrow 0 \longleftarrow 2\}$

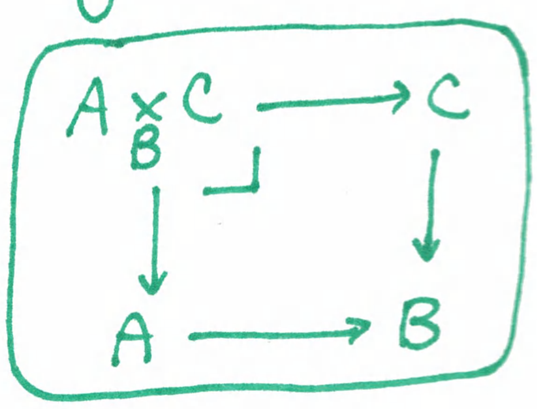
A functor from I is a diagram



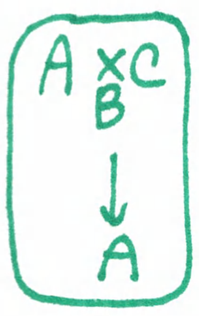
Here, the limit functor sends this diagram



such that



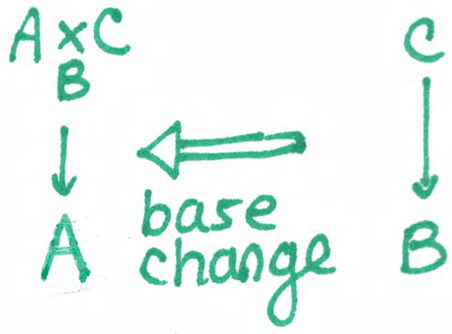
Recall, this was called the
"fiber product" or
"base change", where the map



is a base change of



along the map $A \longrightarrow B$

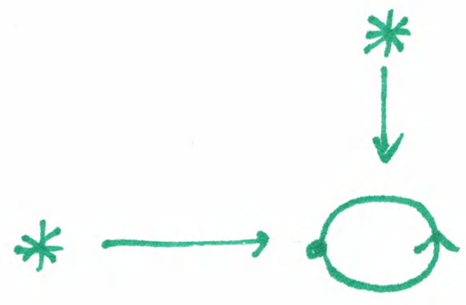


QUESTION: Does this pullback preserve weak equivalences?

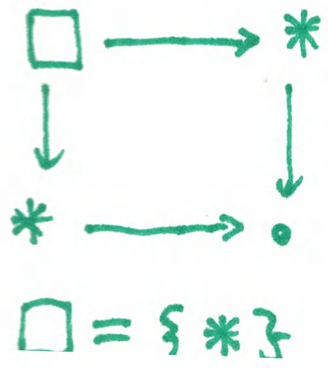
CASE #1

When $A = *$, $B =$ , $C = *$

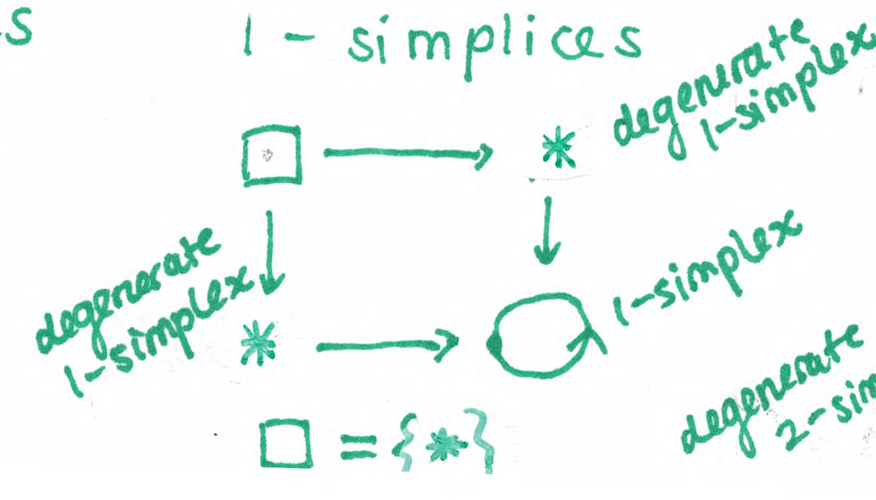
The pullback of the diagram is computed by computing in each dimension...



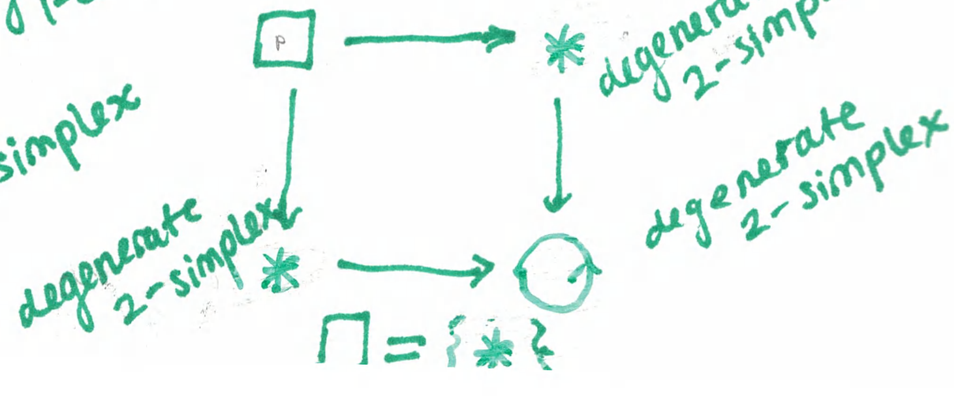
0-simplices



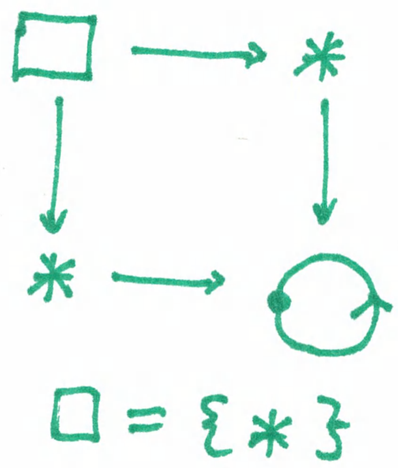
1-simplices



2-simplices ...

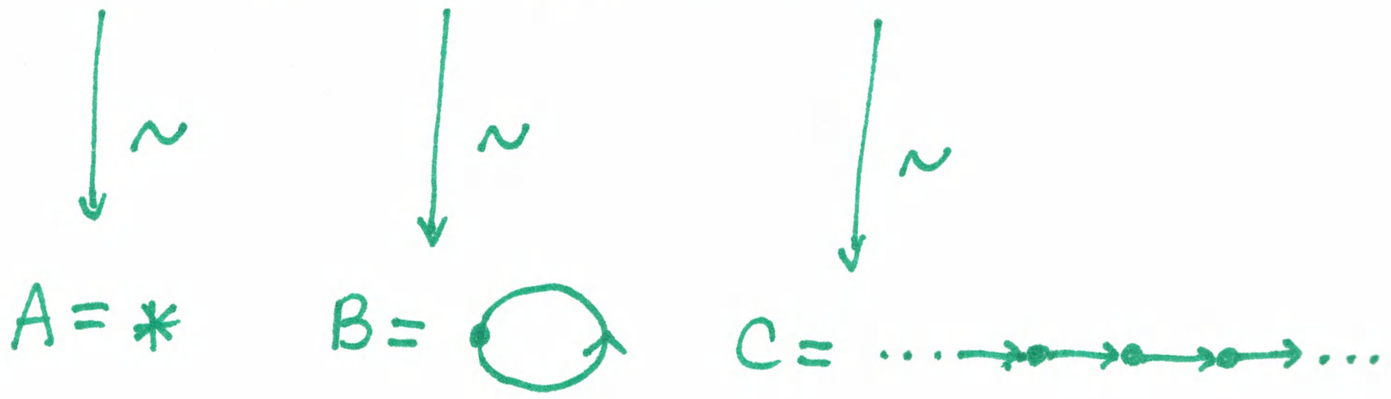


So the pullback is



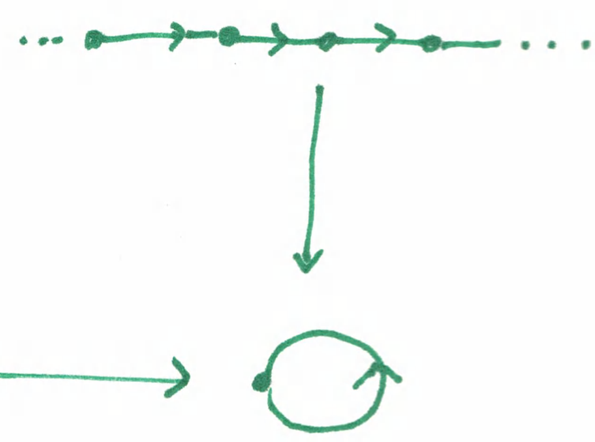
$\square = \{*\}$

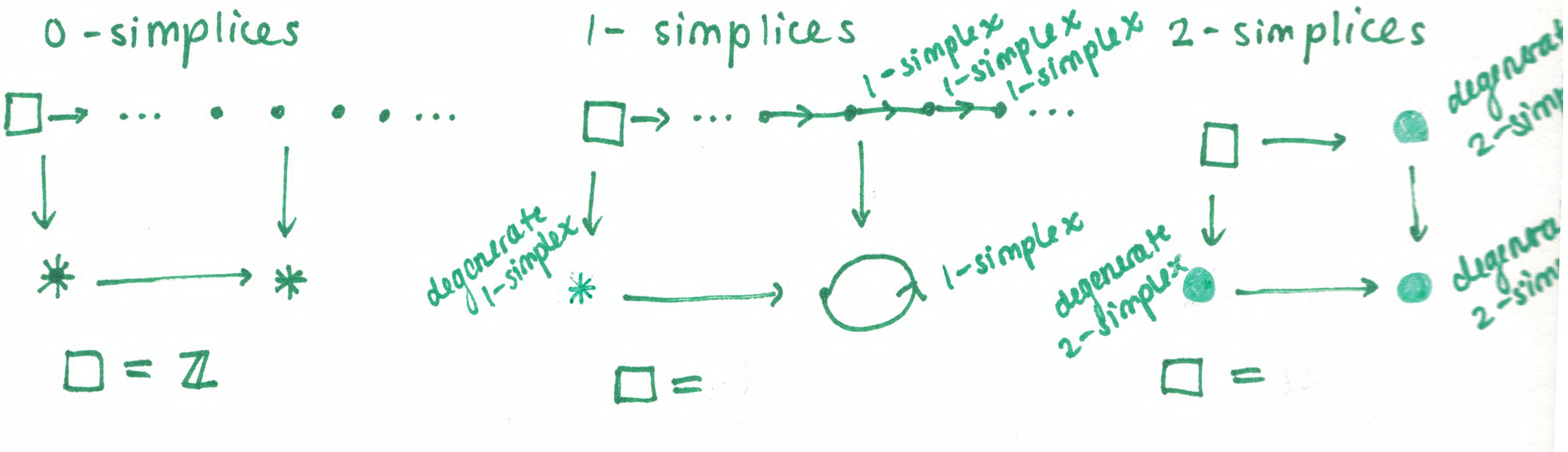
Now, replace $A = *$, $B =$ , $C = *$ with something weakly equivalent ...



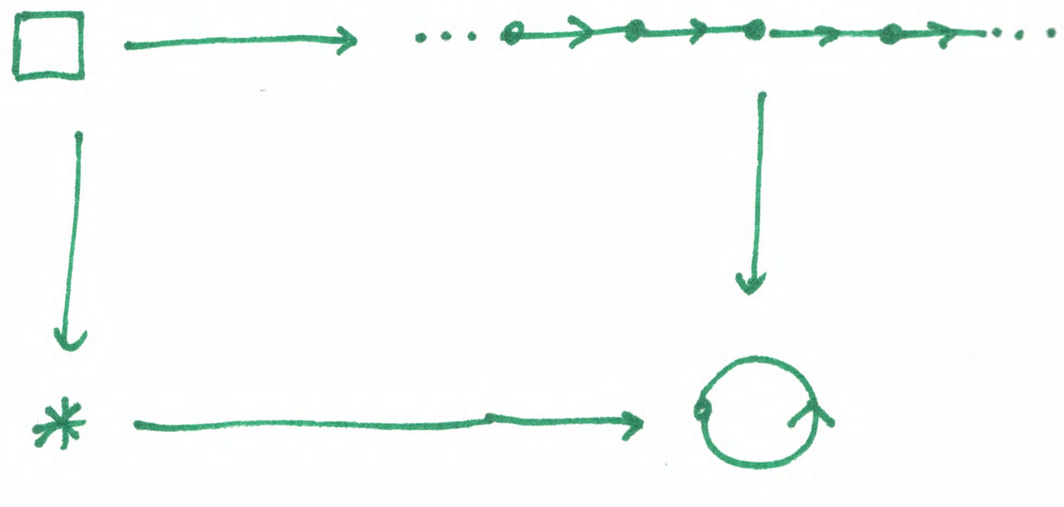
CASE #2

and compute the pullback:
(compute in each dimension)





so the pullback is



MORAL: the pullbacks of two weakly equivalent diagrams are NOT weakly equivalent!

So, pullbacks do not preserve weak equivalences! :"

$\square = \mathbb{Z}$

different from case #1

since ordinary pullbacks do not preserve w.e.'s,
We must construct a "derived pullback"
that WILL preserve weak equivalences!

THE DERIVED PULLBACK FUNCTOR!

QUESTION: What data must be specified to preserve the pullback?
(that is, to preserve weak equivalences after pullback?)

Recall, homotopy limits are like limits except with equality ($=$) replaced by homotopy ($\sim$).
Hence, homotopies will need to be specified.
Similarly, in the case of fiber products (pullbacks in sSet)
The fiber product of $A, C \in$ set is the pullback of the diagram

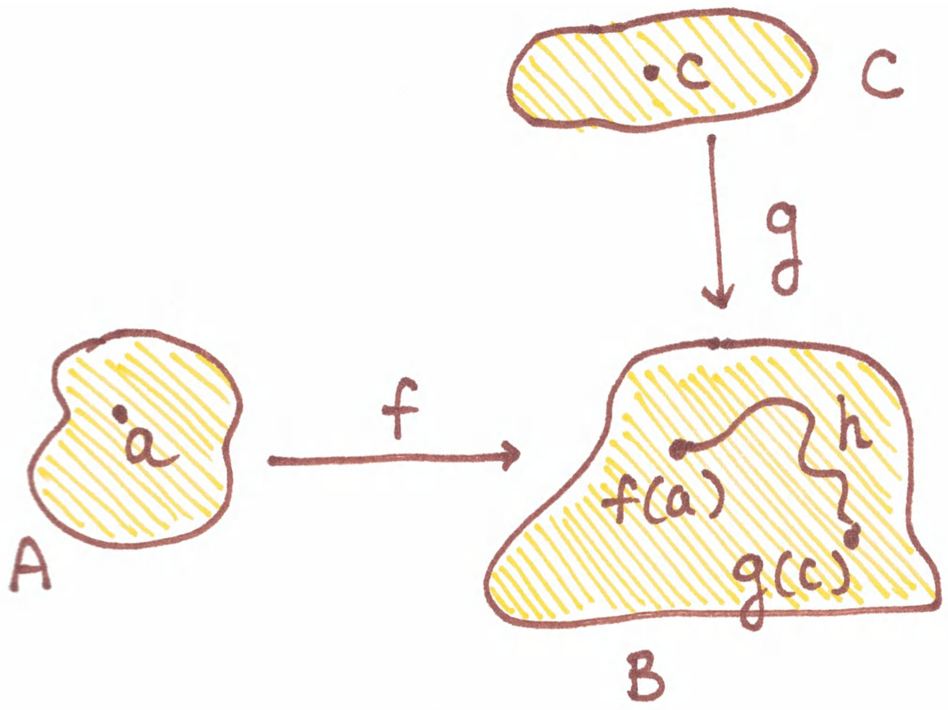
$$\begin{array}{ccc} & & C \\ & & \downarrow g \\ A & \xrightarrow[f]{} & B \end{array} \quad \Rightarrow \text{pullback!} \quad \begin{array}{ccc} A \underset{B}{\times} C & \longrightarrow & C \\ \downarrow & & \downarrow g \\ A & \xrightarrow[f]{} & B \end{array}$$

$$A \underset{B}{\times} C = \{ (a,c) \mid a \in A,\ c \in C,\ f(a) = g(c) \}$$

When we change to homotopy fiber products (homotopy pullbacks in sSet)

$A \overset{h}{\underset{B}{\times}} C$ we will change $f(a) = g(c)$
to $f(a) \overset{h}{\sim} g(c)$,
and this homotopy h must be specified.

QUESTION: How are $f(a)$ and $g(c)$ related in B? $\longrightarrow$



f(a) and g(c) are related in B by a path, where f(a) is one endpoint, and g(c) is the other endpoint.

So, when we specify this new "homotopy fiber product", we must also specify this path "h",

h: from $f(a)$ to $g(c)$

so $A \underset{B}{\times}^{h} C = \{ (a, c, h) \}$

$a \in A \quad c \in C \quad h : f(a) \longrightarrow g(c)$

This can be equivalently expressed by saying:

$$A \underset{B}{\times}^{h} C = \{ (a, c, h : f(a) \longrightarrow g(c) \}$$

$$\Downarrow$$

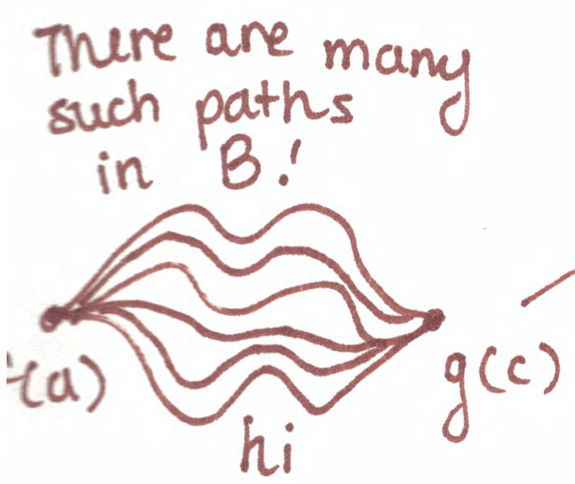
$$\{ (a, c, h : s \longrightarrow g(c)) \mid f(a) = s \}$$

$$\Downarrow$$

$$\{ (a, [\, c, h : s \longrightarrow g(c) \,]) \mid f(a) = s \}$$

and this looks like

$$A \underset{B}{\times} [\, (\text{space of paths in } B \text{ from } f(a) \text{ to } g(c)) \underset{B}{\times} C \,]$$



where maps for the product over B are specified by

$A \underset{B}{\times} (\text{Path}(B) \underset{B}{\times} C)$

$A \quad (\text{Path}(B) \underset{B}{\times} C)$

f and source of path in B $\longrightarrow B$

...and...

$\text{Path}(B) \underset{B}{\times} C$

$\text{Path}(B) \quad C$

target of path in B, g $\longrightarrow B$

Path(B): The path space in sSet has a very precise definition:

$$\mathrm{Path}(B) = \mathrm{RHom}(\Delta^1, B)$$
$$= \mathrm{Hom}(\Delta^1, \mathcal{E}x^\infty B)$$
$$= (\mathcal{E}x^\infty B)^{\Delta^1}$$ (notation for internal hom from Δ^1 to $\mathcal{E}x^\infty B$)

There are two maps associated to a path in $\mathcal{E}x^\infty B$ that we can use in the homotopy pullback:

the source map: $s \nearrow \longrightarrow \mathcal{E}x^\infty(B)$ — s is a vertex in $\mathcal{E}x^\infty(B)$, $s \in (\mathcal{E}x^\infty(B))_0$

and

the target map: $\nearrow t \longrightarrow \mathcal{E}x^\infty(B)$ — t is a vertex in $\mathcal{E}x^\infty(B)$, $t \in (\mathcal{E}x^\infty(B))_0$

This path space $(\mathcal{E}x^\infty B)^{\Delta^1}$ maps into $\mathcal{E}x^\infty(B)$...

Replacing $B \longrightarrow \mathcal{E}x^\infty(B)$ in our notion of the homotopy pullback, we obtain a new candidate for the correct notion of pullbacks (the pullback that WILL preserve weak equivalences...)

THE DERIVED PULLBACK FUNCTOR:

DEFINITION The derived homotopy pullback is given by:

$$A \times^h_B C := A \times_{\mathcal{E}x^\infty(B)} \left((\mathcal{E}x^\infty(B))^{\Delta^1} \times_{\mathcal{E}x^\infty(B)} C \right)$$

It remains to verify that this correctly defines a homotopy pullback by showing it preserves weak equivalenc

pullback diagram

$$\begin{array}{ccc} & & C \\ & & \downarrow \\ A & \longrightarrow & B \end{array}$$

fiber product

homotopy pullback diagram

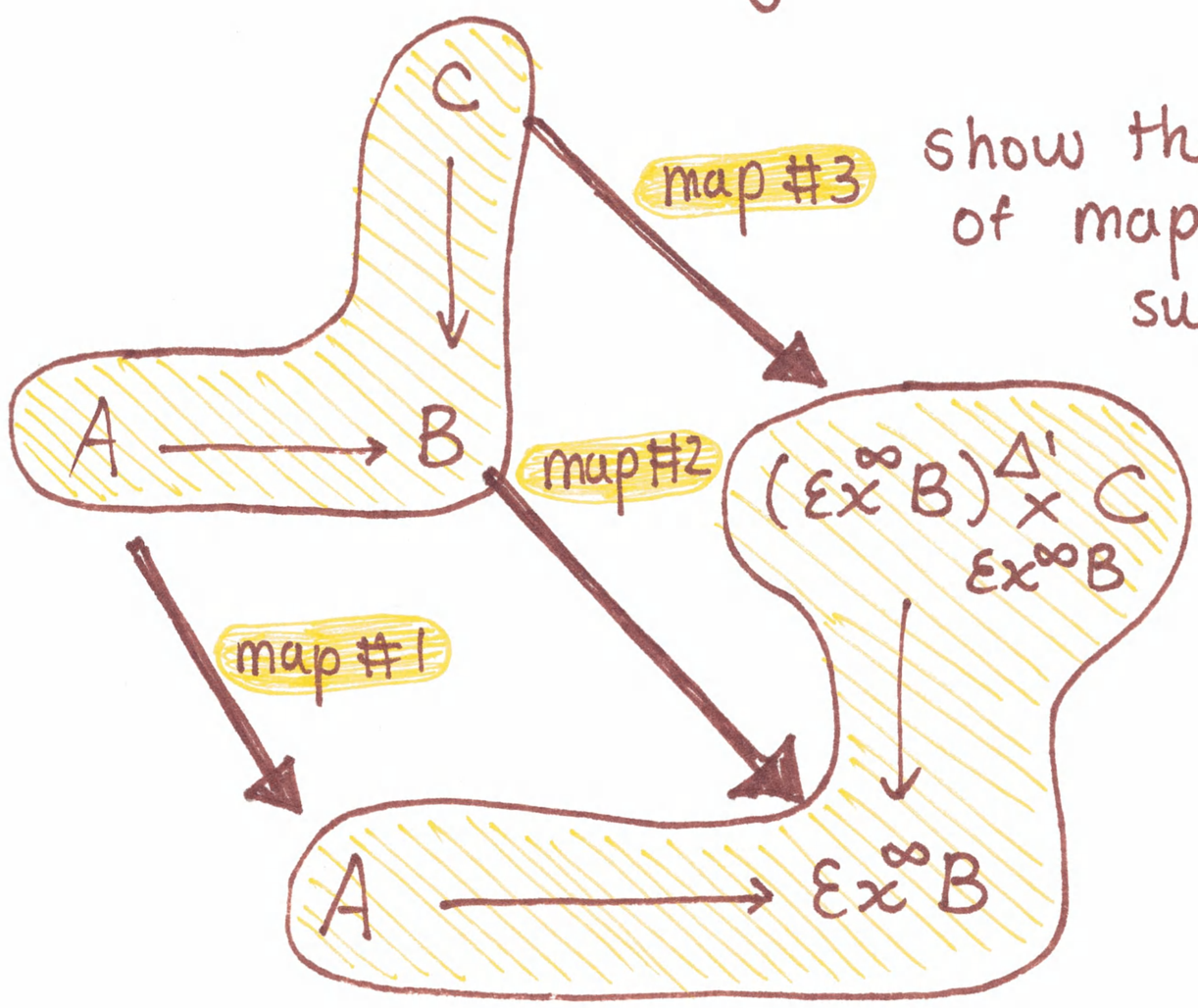
$$\begin{array}{ccc} & & (\mathcal{E}x^{\infty}B)^{\Delta'} \underset{\mathcal{E}x^{\infty}B}{\times} C \\ & & \downarrow \\ A & \longrightarrow & \mathcal{E}x^{\infty}B \end{array}$$

homotopy fiber product

$$\begin{array}{ccc} A \underset{B}{\times} C & \longrightarrow & C \\ \downarrow & \lrcorner & \downarrow \\ A & \longrightarrow & B \end{array} \quad \Longleftrightarrow \quad \begin{array}{ccc} A \underset{\mathcal{E}x^{\infty}B}{\times} ((\mathcal{E}x^{\infty}B)^{\Delta'} \underset{\mathcal{E}x^{\infty}B}{\times} C) & \longrightarrow & (\mathcal{E}x^{\infty}B)^{\Delta'} \underset{\mathcal{E}x^{\infty}B}{\times} C \\ \downarrow & \lrcorner & \downarrow \\ A & \longrightarrow & \mathcal{E}x^{\infty}B \end{array}$$

... This new homotopy pullback diagram must be weakly equivalent to the normal pullback diagram

How to show weak equivalence of diagrams:



show there is a triple of maps (#1, #2, #3) such that the internal squares commute, & each map is a weak equivalence between the corresponding parts of the diagram.

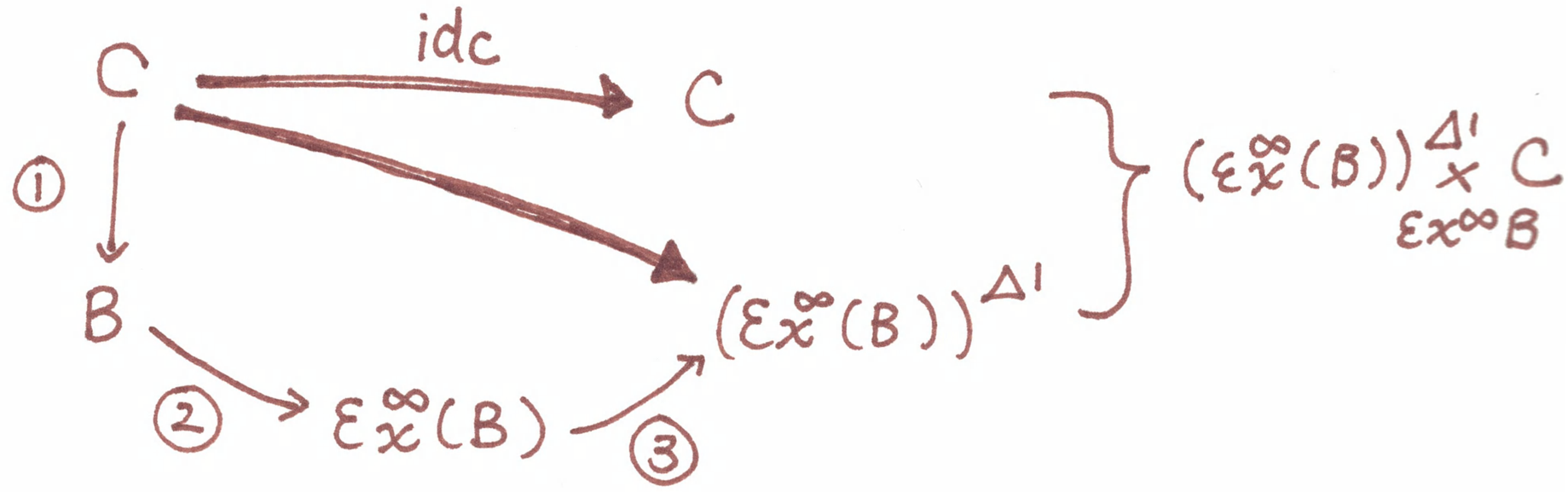
map #1 : $A \longrightarrow A$ is the identity map, idA

map #2 : $B \hookrightarrow \mathcal{E}x^{\infty}B$ is the canonical inclusion

and the lower square commutes (trivially). $\longrightarrow$

map # 3: $C \longrightarrow (\mathcal{E}x^{\infty}B)^{\Delta^1} \underset{\mathcal{E}x^{\infty}B}{\times} C$

We can define this map as follows



① Map from C to B via q in the original pullback diagram

② Map from B to $\mathcal{E}x^{\infty}B$ via the canonical inclusion

③ Map from $\mathcal{E}x^{\infty}(B)$ to $(\mathcal{E}x^{\infty}(B))^{\Delta^1}$ via the "constant path map"

$$\mathcal{E}x^{\infty}(-) \xrightarrow{\mathrm{Hom}(\varphi, \mathcal{E}x^{\infty}(-))} \mathrm{Hom}(\Delta^1, \mathcal{E}x^{\infty}(-))$$

where $\varphi: \Delta^1 \longrightarrow \Delta^0$

collapses a path to one of its endpoints.

The composition

① $\longrightarrow$ ② $\longrightarrow$ ③ defines the map from C to $(\mathcal{E}x^{\infty}(B))^{\Delta^1}$.

So we have a map

$$C \longrightarrow (\mathcal{E}x^{\infty}(B))^{\Delta^1} \underset{\mathcal{E}x^{\infty}B}{\times} C$$

} we need to show that this map is a weak equivalence!

To show that $C \xrightarrow{f} (\mathcal{E}\mathrm{x}^\infty B)^{\Delta^1} \underset{\mathcal{E}\mathrm{x}^\infty B}{\times} C$

is a weak equivalence it suffices to show that f is a simplicial homotopy equivalence.

DEFINE THIS MAP AS THE PROJECTION ONTO THE SECOND COORDINATE!

Need to show: $\exists g : (\mathcal{E}\mathrm{x}^\infty B)^{\Delta^1} \underset{\mathcal{E}\mathrm{x}^\infty B}{\times} C \longrightarrow C$

DEFINE $\tilde{\alpha}$ as =

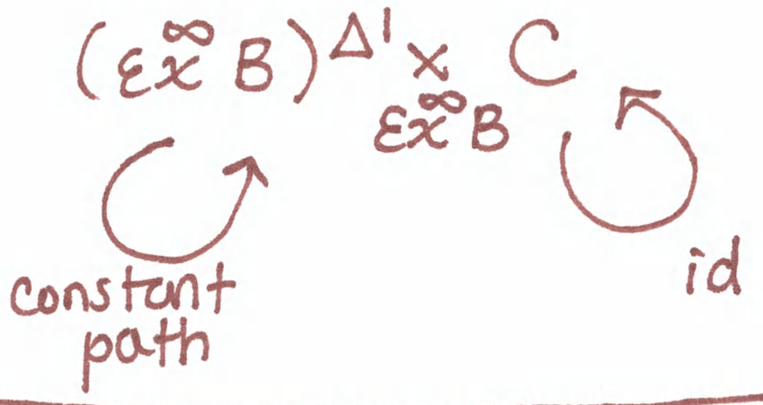
such that $gof \underset{\alpha}{\sim} id_C$ [α is a homotopy]

$fog \underset{\beta}{\sim} id_{(\mathcal{E}\mathrm{x}^\infty B)^{\Delta^1} \underset{\mathcal{E}\mathrm{x}^\infty B}{\times} C}$ [β is a homotopy]

How do we define the homotopy β?

What is the identity map on $(\mathcal{E}\mathrm{x}^\infty B)^{\Delta^1} \underset{\mathcal{E}\mathrm{x}^\infty B}{\times} C$?

we define a map that acts as the constant path in $\mathcal{E}\mathrm{x}^\infty B$ on $(\mathcal{E}\mathrm{x}^\infty B)^{\Delta^1}$ and id_C on C.



[MAP #1] We can just define this map by projection onto C

[MAP #2]

$\Delta^1 \times ((\mathcal{E}\mathrm{x}^\infty B)^{\Delta^1} \underset{\mathcal{E}\mathrm{x}^\infty B}{\times} C)$

↓ projection

$\Delta^1 \times (\mathcal{E}\mathrm{x}^\infty B)^{\Delta^1} \longrightarrow (\mathcal{E}\mathrm{x}^\infty B)^{\Delta}$

construct this map

To construct β, we need to construct a map

$\Delta^1 \times ((\mathcal{E}\mathrm{x}^\infty B)^{\Delta^1} \underset{\mathcal{E}\mathrm{x}^\infty B}{\times} C) \longrightarrow (\mathcal{E}\mathrm{x}^\infty B)^{\Delta^1} \underset{\mathcal{E}\mathrm{x}^\infty B}{\times} C$

We can accomplish this by constructing a map into each factor of the product:

$\Delta^1 \times ((\mathcal{E}\mathrm{x}^\infty B)^{\Delta^1} \underset{\mathcal{E}\mathrm{x}^\infty B}{\times} C) \longrightarrow C$ [MAP #1]

$\Delta^1 \times ((\mathcal{E}\mathrm{x}^\infty B)^{\Delta^1} \underset{\mathcal{E}\mathrm{x}^\infty B}{\times} C) \longrightarrow (\mathcal{E}\mathrm{x}^\infty B)^{\Delta^1}$ [MAP #2]

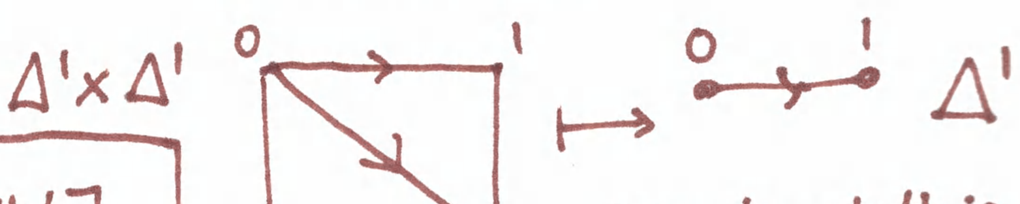
equivalent to constructing the adjoint map

$\Delta^1 \times \Delta^1 \times (\mathcal{E}\mathrm{x}^\infty B)^{\Delta^1} \longrightarrow \mathcal{E}\mathrm{x}^\infty B$

fix point in Δ^1

evaluation map

$\Delta^1 \times (\mathcal{E}\mathrm{x}^\infty B)^{\Delta^1}$



construct this homotopy ...

So, $C \xrightarrow{\sim} (\mathcal{E}\mathrm{x}^\infty B)^{\Delta^1} \underset{\mathcal{E}\mathrm{x}^\infty B}{\times} C$ is a weak equivalence. ▨

KAN FIBRATIONS

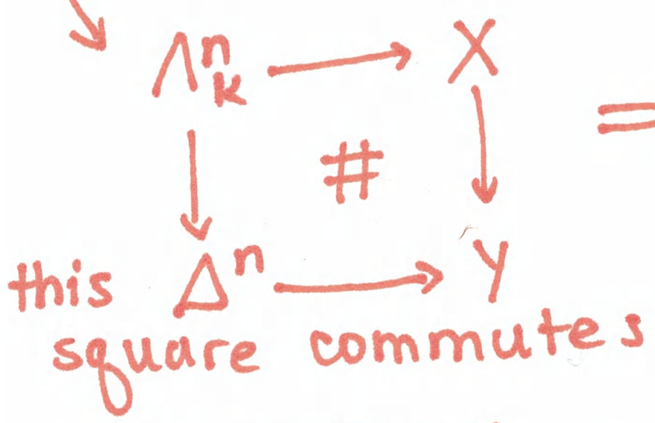
DEFINITION $f: X \longrightarrow Y$ is called a "Kan fibration"

if and only if

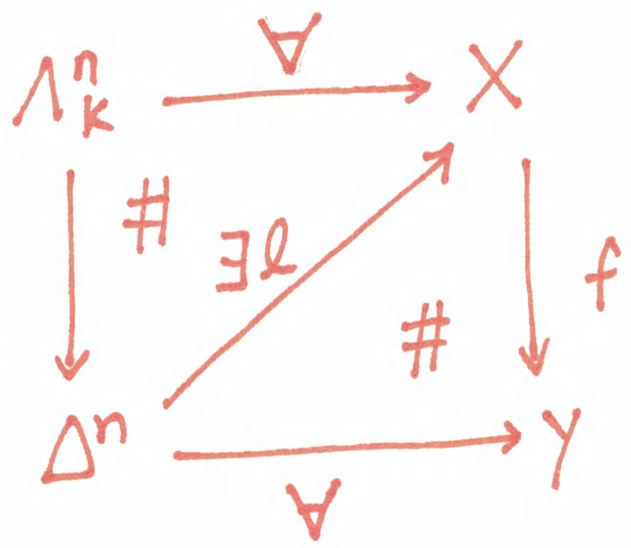
$\forall \Lambda^n_k \longrightarrow X$
$\forall \Delta^n \longrightarrow Y$
such that

$\exists\, l: \Delta^n \longrightarrow X$ such that the diagram commutes:

This map is called a "lift"



$\Rightarrow$



> A Kan fibration is a morphism of simplicial sets with the lifting property for all horn inclusions.
> — nLab

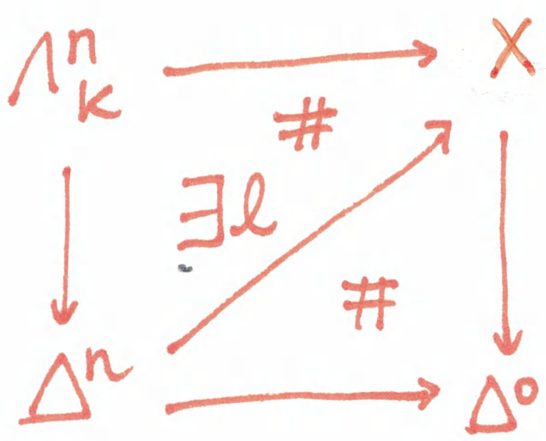
PROPOSITION $X \in sSet$ is Kan

if and only if

$X \longrightarrow \Delta^0$ is a Kan fibration

(denoted, for now, by the double headed arrow, $\twoheadrightarrow$, so, $X \twoheadrightarrow \Delta^0$)

Proof: If $X \in sSet$ is Kan,
by definition of Kan complexes, $\exists\, l: \Delta^n \longrightarrow K$
such that the
upper triangle of the
diagram commutes.



The lower triangle commutes
trivially, so

$X \twoheadrightarrow \Delta^0$.

The converse holds, since the definition of $X \twoheadrightarrow \Delta^0$ provides the lift map $l: \Delta^n \longrightarrow X$ such that X is Kan ∎

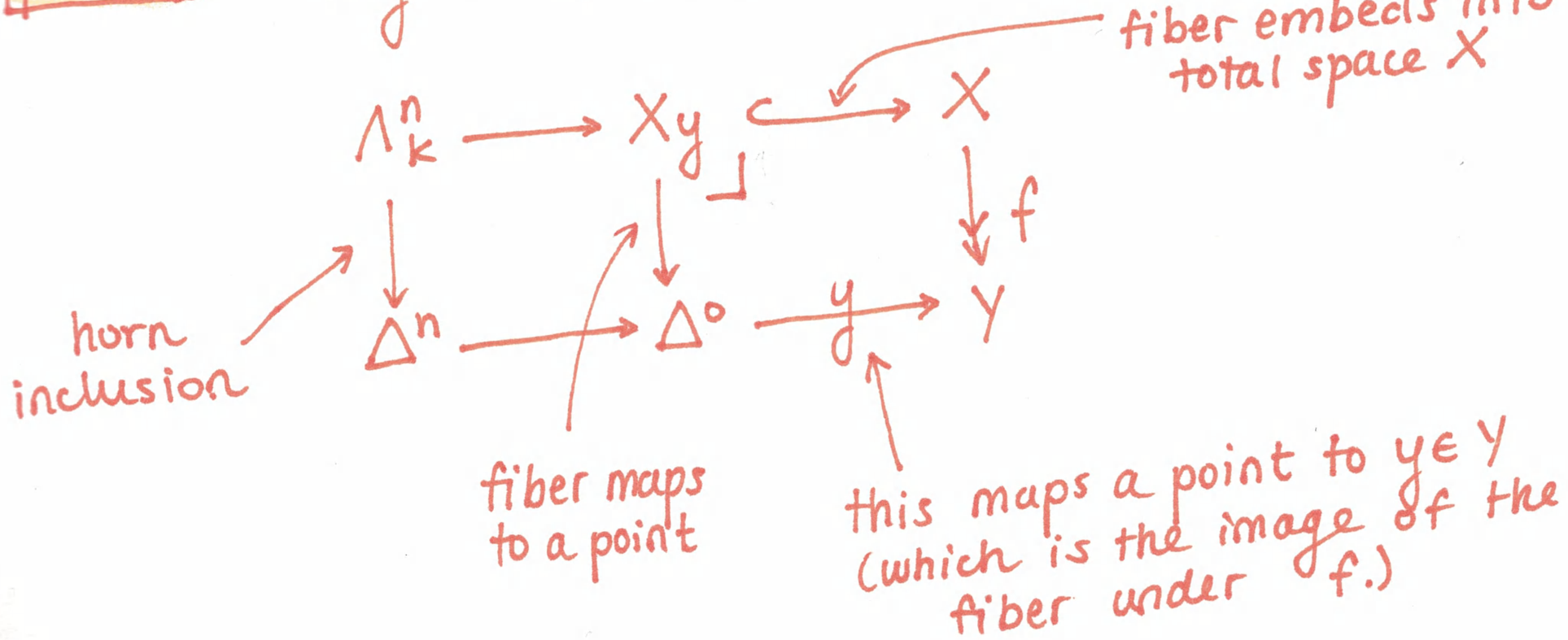
Properties of ~~the~~ Kan fibrations

(a) $\forall y \in Y_0$ define the fiber over y by:

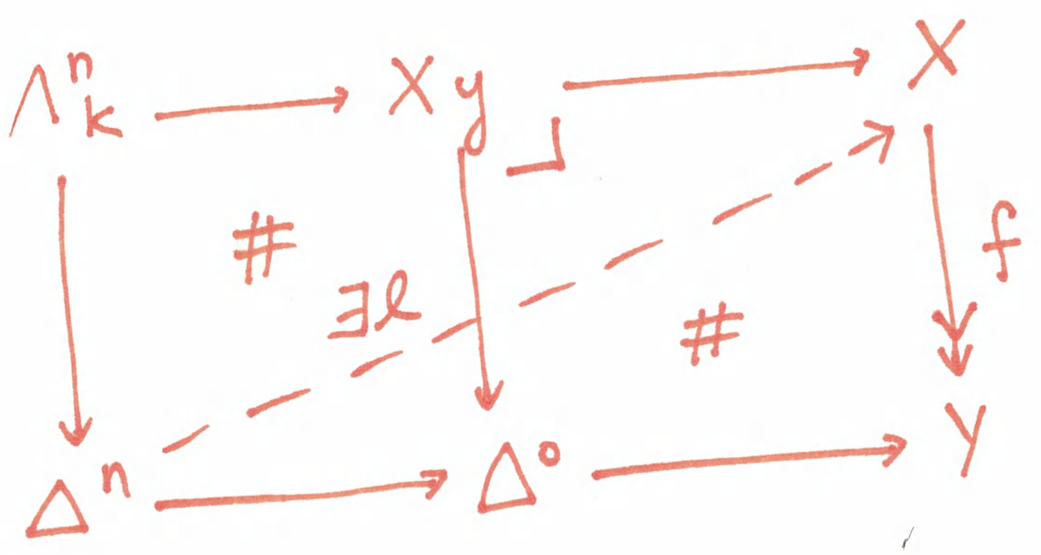
$$X_y := \Delta^0 \underset{Y}{\times} X$$

The fiber X_y is a Kan complex!

proof: $X_y \subset X$, so we have a diagram



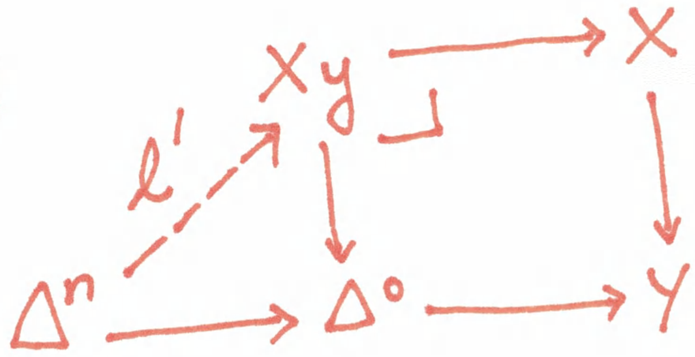
We have assumed that the map $f: X \twoheadrightarrow Y$ is a Kan fibration, so $\exists l: \Delta^n \longrightarrow X$ such that the triangles formed by l commute:



To show X_y is Kan, we need to show

$\exists l': \Delta^n \longrightarrow X_y$ such that the triangles formed by l' commute!

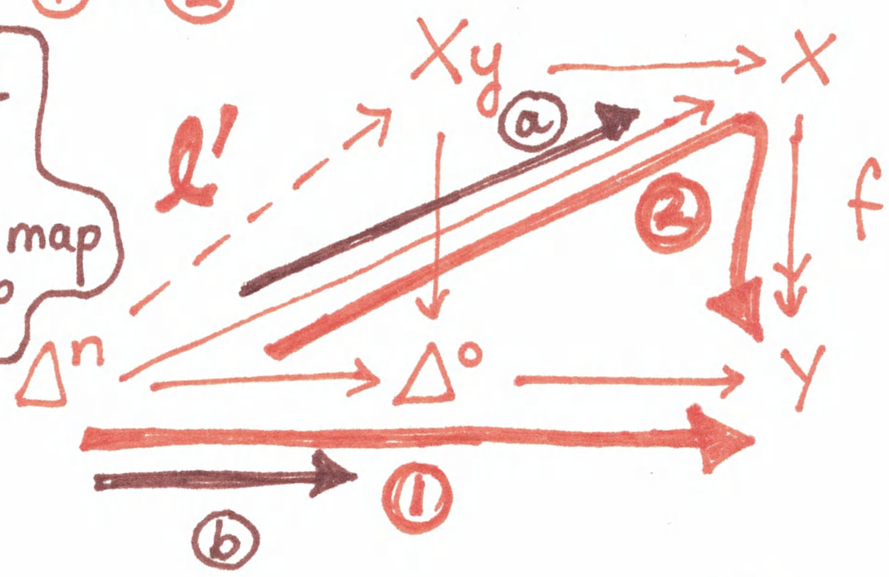
NOTE: The fiber $X_y := \Delta^0 \underset{Y}{\times} X$ is a pullback over Y.
The map we want is a map into a pullback!
[NOTE: we want $l': \Delta^n \longrightarrow X_y$]



By the UNIVERSAL PROPERTY OF PULLBACKS, a map INTO a pullback can be defined by defining maps ⓐ and ⓑ below; such that compositions are equal: ① = ②

ⓐ := l

ⓑ: the original map $\Delta^n \to \Delta^0$

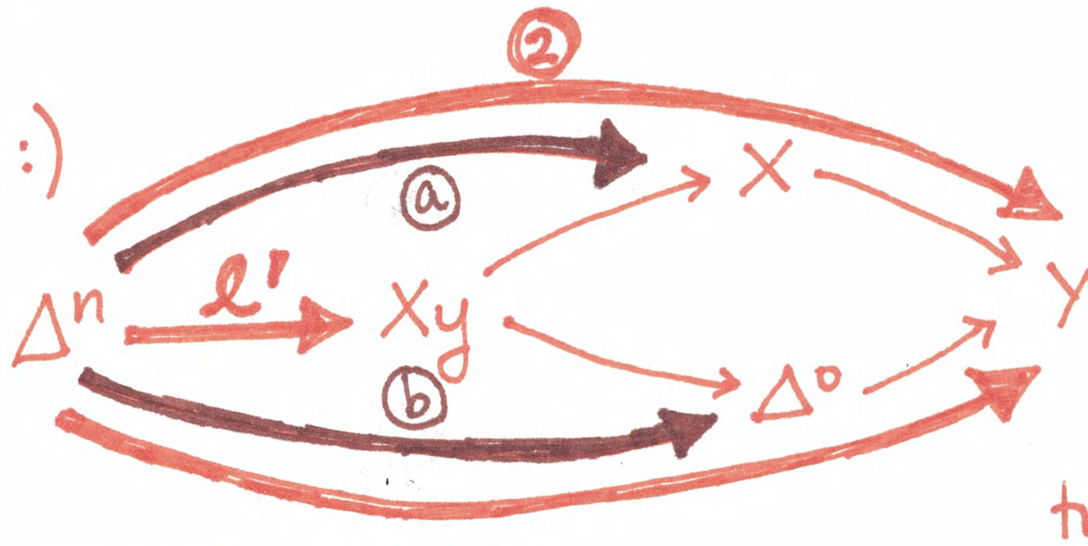


① is defined as the composition of the lower two maps, and

② is f∘l,

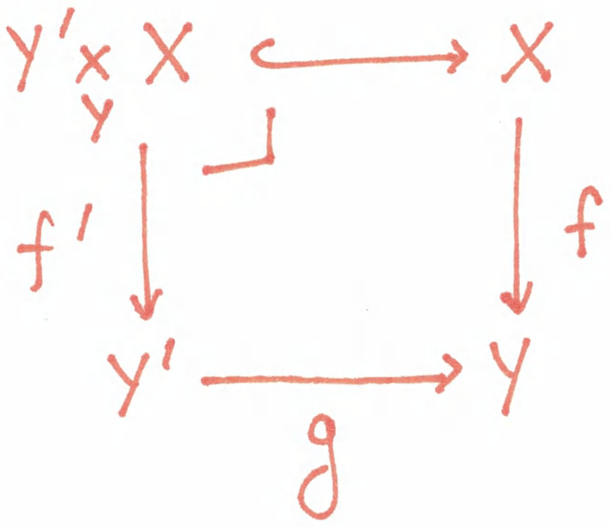
so l' exists, hence Xy (the fiber) is Kan! ▨

(that is:)

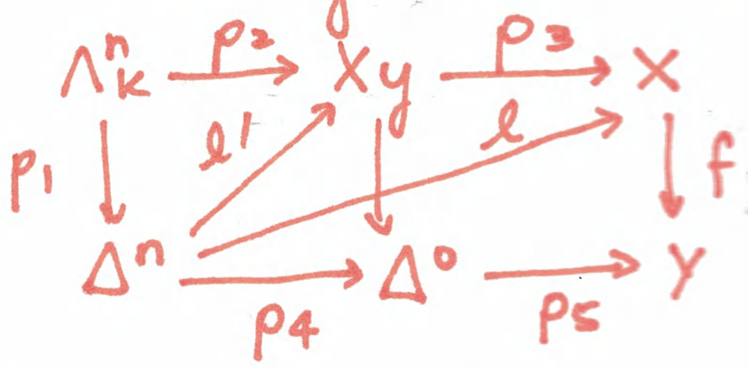


UNIVERSAL PROPERTY OF PULLBACK

NOTE: This extends to a "Generalized Fiber construction":



Now, the upper triangle commutes:



The composition:

$p_3 \circ l' \circ p_1 = l \circ p_1$

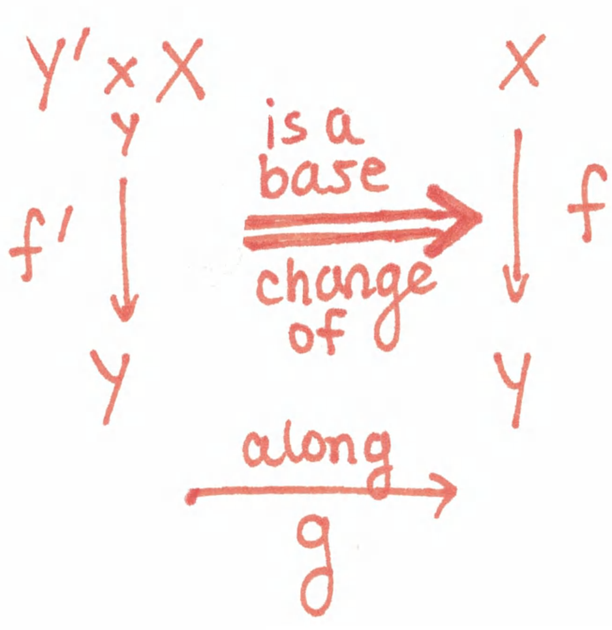
and this is equal to:

$p_3 p_2 = l \circ p_1$

so, $p_3 \circ l' \circ p_1 = p_3 p_2$ ⇒ $p_3 p_2 = l' \circ p_1$

The map f' is called the "base change" of f along g.

Kan fibrations are stable under base changes!



∀ f: X ↠ Y Kan fibration

↓

f': Y'×_Y X ⟶ Y' Kan fibration ▨

(b) Kan fibrations are closed under products (both finite and infinite products)

If X_i is a Kan fibration, then

$X_i \downarrow$

Y_i

by the UNIVERSAL PROPERTY OF PRODUCTS

$\prod_i X_i$

$\downarrow \prod_i f_i$ is a Kan fibration.

$\prod_i Y_i$

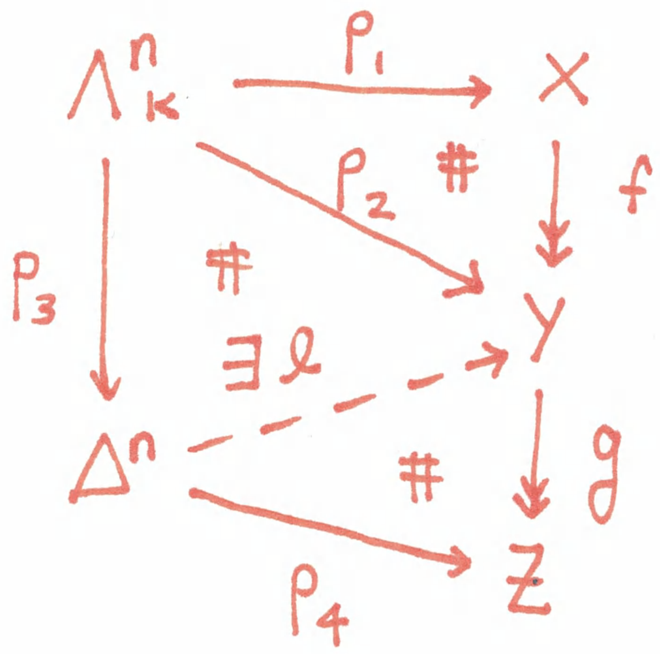
(c) Kan fibrations are closed under composition and contransfinite composition.

proof

$X \xrightarrow{f} Y$ Kan fibration

$Y \xrightarrow{g} Z$ Kan fibration

} consider $X \xrightarrow{g \circ f} Z$



By the Kan fibration property of the map g (lower square),

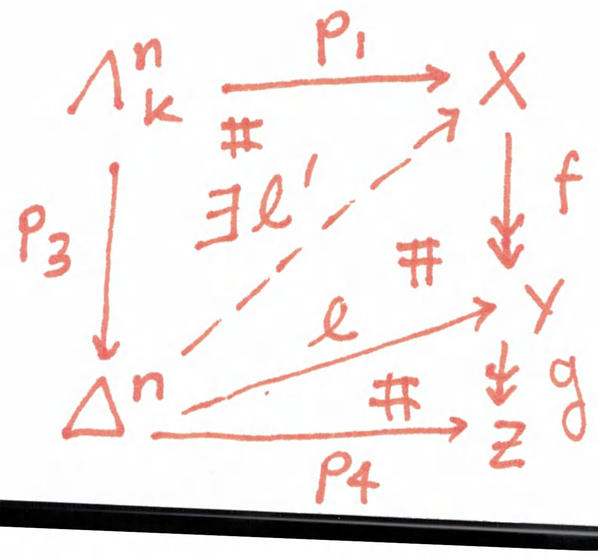
$\exists l : \Delta^n \longrightarrow Y$ such that the triangles formed by l commute.

$p_2 = l \circ p_3$ and $p_4 = g \circ l$

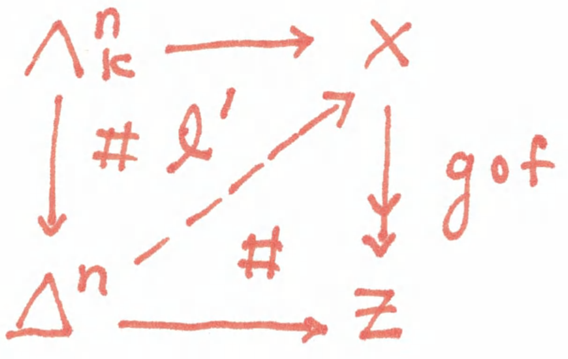
By the Kan fibration property of the map f (upper square)

$\exists l' : \Delta^n \longrightarrow X$ such that the triangles formed by l' commute

$p_1 = l' \circ p_3$ and $l = f \circ l'$



so,



The map $g \circ f$ is a Kan fibration. ▨

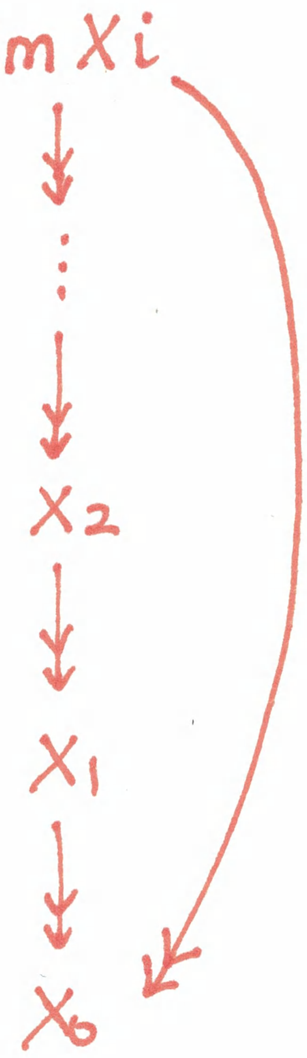
→

A "cotransfinite" composition is a "tower" of compositions:

suppose we have a tower of Kan fibrations

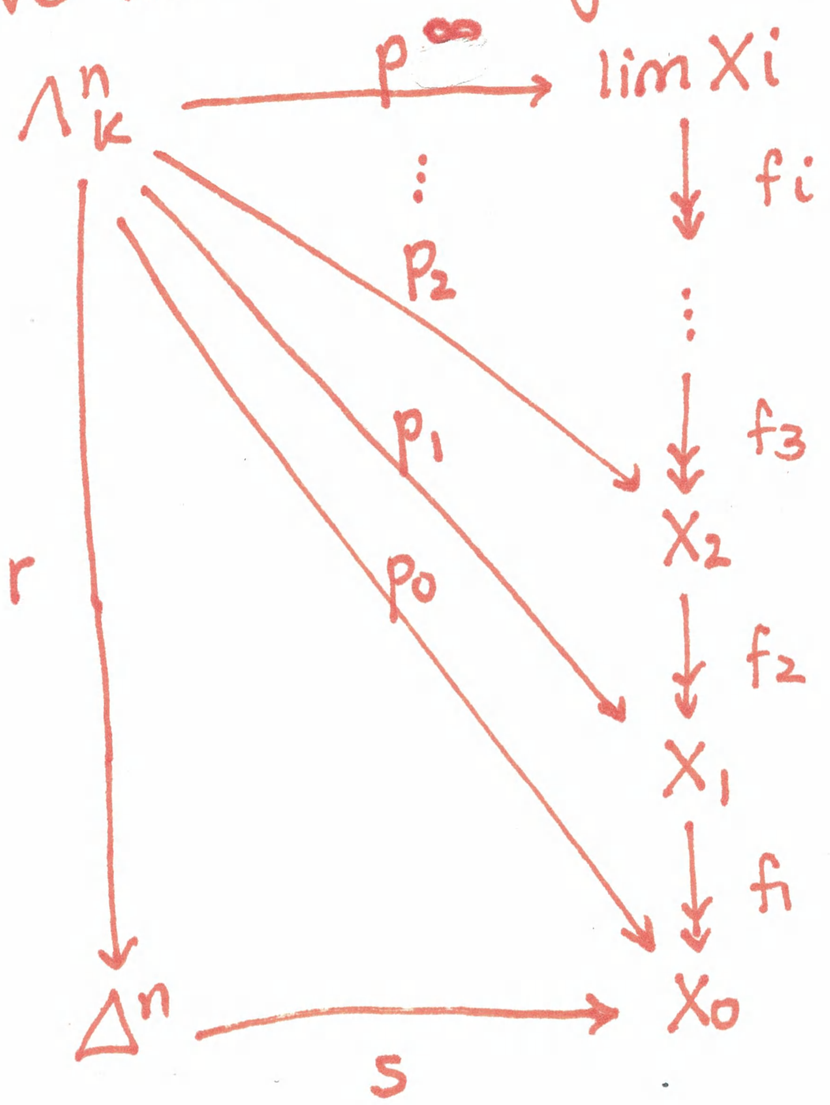
$X_{i+1} \to X_i$

we claim that this giant composition map is a Kan fibration!

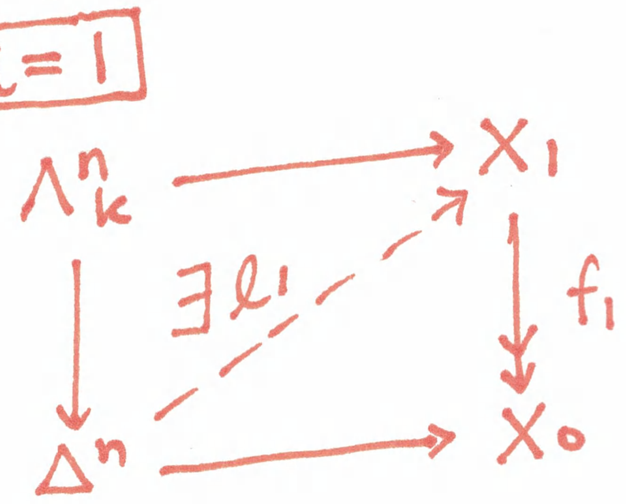


proof

We have a diagram, and we proceed by induction:

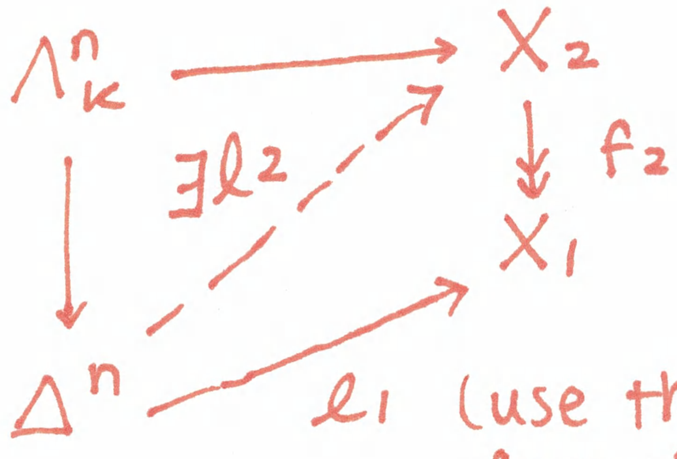


i = 1



f_1 is a Kan fibration, so $\exists l_1 : \Delta^n \to X_1$ such that the square commutes

i = 2



f_2 is a Kan fibration, so, $\exists l_2 : \Delta^n \to X_2$ such that the diagram commutes

(use the lift map from the previous step)

And an induction argument proves the case for the lift map $l^\infty : \Delta^n \to \lim X$ ∎

LEMMA If $B, C \in sSet$ are Kan, $C \xrightarrow{f} B$ then $C \times_B B^{\Delta^1}$ is a Kan fibration.

$$\begin{array}{c} C \times_B B^{\Delta^1} \\ \downarrow \\ B \end{array}$$

proof The product $C \times_B B^{\Delta^1}$ is a pullback, so we have the pullback diagram:

$$\begin{array}{ccc} C \times_B B^{\Delta^1} & \xrightarrow{\text{projection to path space}} & B^{\Delta^1} \xrightarrow{B^{l_0}} B \\ \downarrow & \lrcorner & \downarrow B^{l_1} \\ C & \xrightarrow{f} & B \end{array}$$

$l_0 : \Delta^0 \to \Delta^0$ picks starting vertex

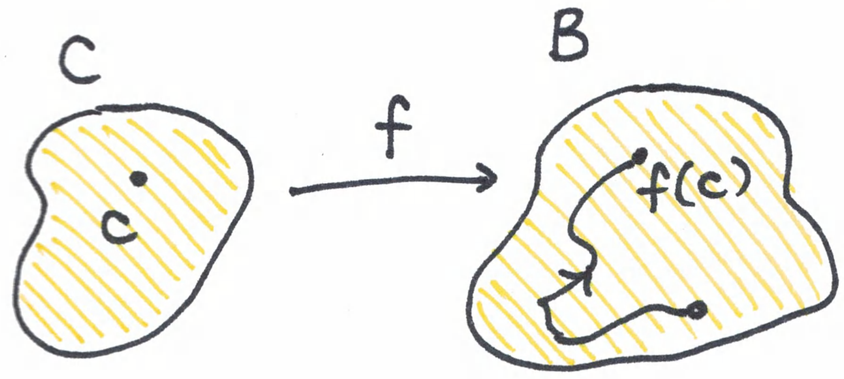
endpoint map

endpoint map

$l_1 : \Delta^1 \to \Delta^0$ picks ending vertex

pullback of C

Recall, a point in $C \times_B B^{\Delta^1}$ is a point $c \in C$ and a path in B whose endpoint is the image of c under f



The diagram above gives us a map given by the composition:

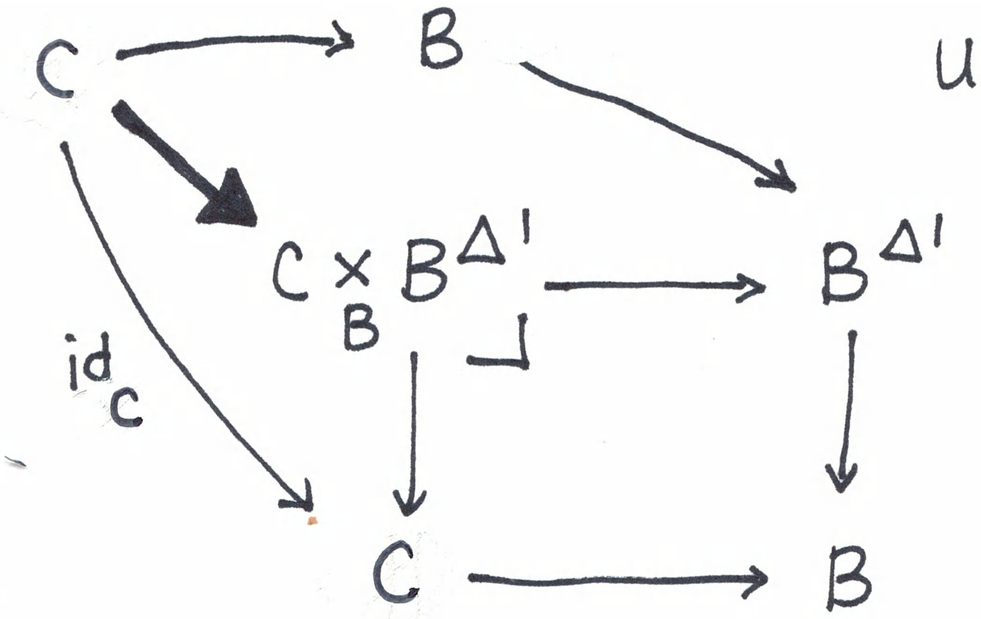
$$C \times_B B^{\Delta^1} \longrightarrow B$$

$$C \times_B B^{\Delta^1} \xrightarrow{\text{projection map}} B^{\Delta^1} \xrightarrow{\text{endpoint map}} B$$

NOTE: We also have a map:

$$C \longrightarrow C \times_B B^{\Delta^1}$$

induced by the universal property of pullbacks $\longrightarrow \cdots$



UNIVERSAL PROPERTY OF PULLBACKS

The map $C \longrightarrow C \times_B B^{\Delta^1}$

is induced by maps

map#1 $C \xrightarrow{id_C} C$

and

map#2 $C \longrightarrow B^{\Delta^1}$ (#) $B^{\Delta^1} \to B$

so, what we have is a "factorization"

$$C \longrightarrow C \times_B B^{\Delta^1} \longrightarrow B \qquad (\#) \; f$$

If we can show that the map $\downarrow$ is a Kan fibration, $C \times_B B^{\Delta^1}$ over B

we will have a result about simplicial maps (between Kan complexes)

$$C \xrightarrow{f} B$$

This map, as it turns out, will be a weak equivalence $\sim$

$C \to C \times_B B^{\Delta^1} \to B$ — Kan fibration

END OF NOTE

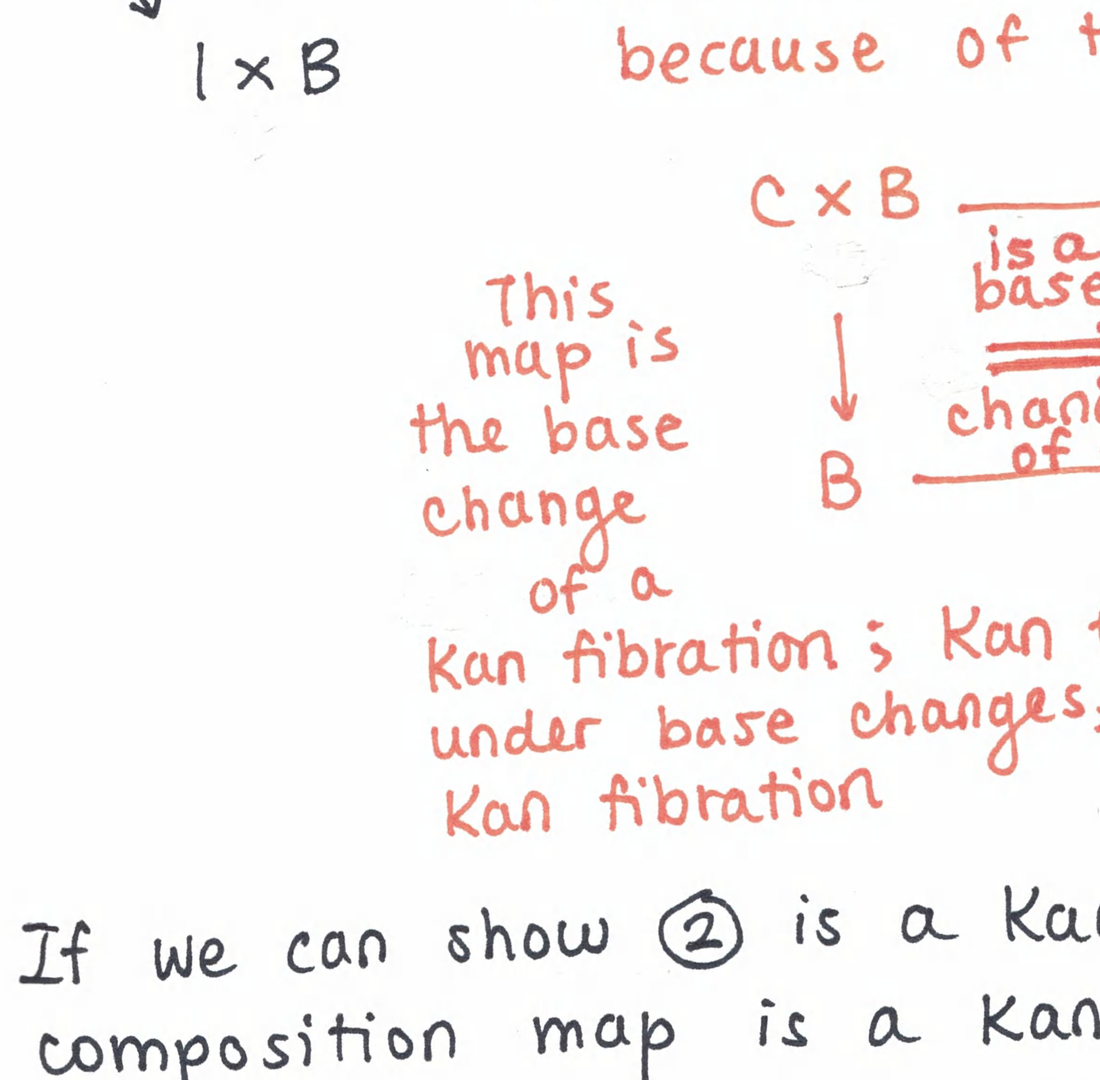
Now, back to showing that $C \times_B B^{\Delta^1} \downarrow B$ is a Kan fibration ... $\longrightarrow$

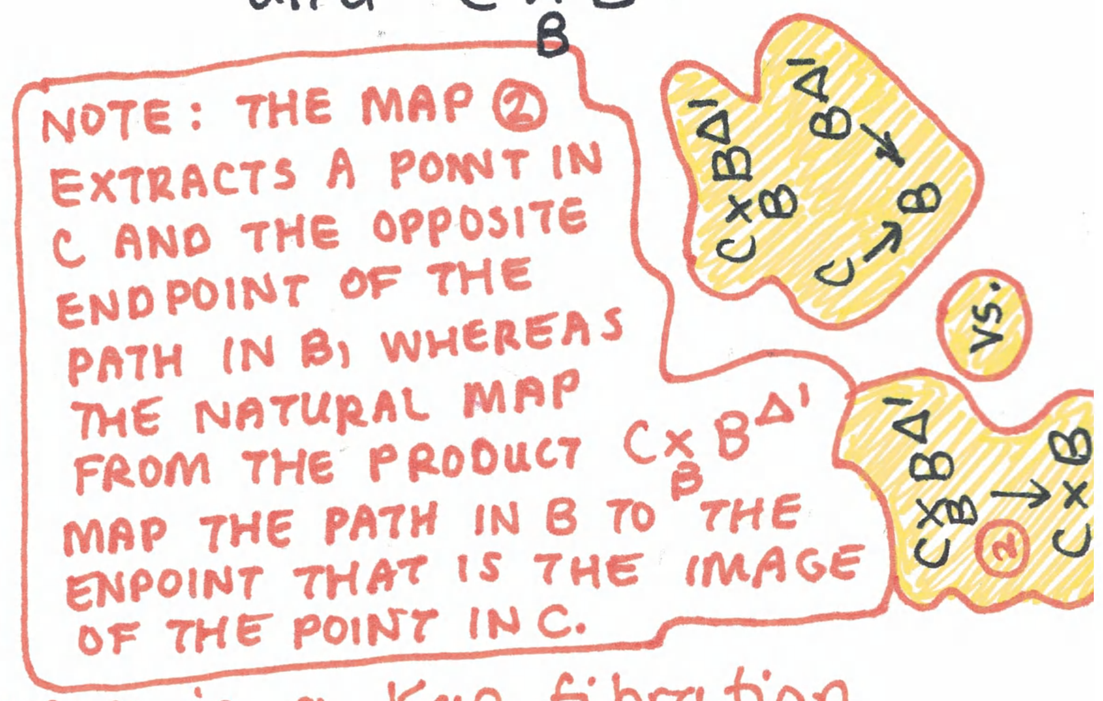
We constructed a composition map with the desired domain and codomain.

$$C \times_B B^{\Delta^1} \xrightarrow{\text{projection}} B^{\Delta^1} \xrightarrow{\text{endpoint}} B$$

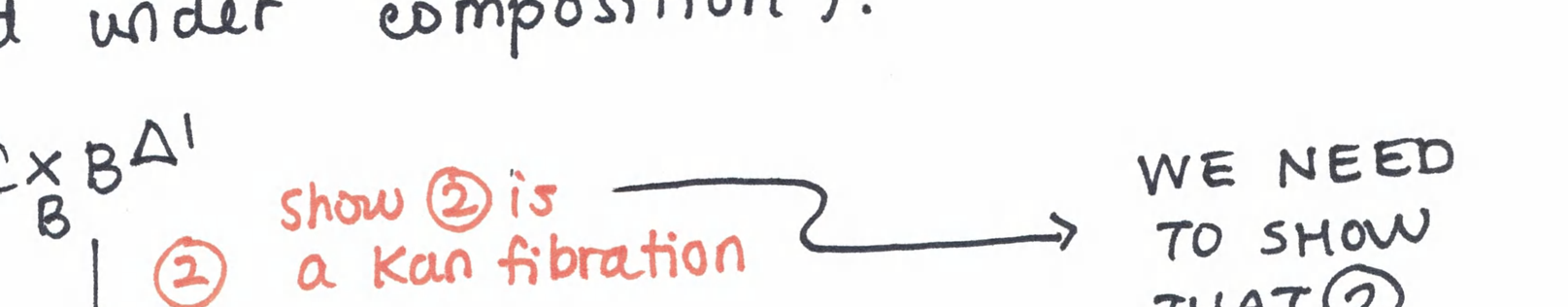
$$C \times_B B^{\Delta^1} \xrightarrow{\text{composition}} B$$

To show this map is a Kan fibration, consider $C \times_B B^{\Delta^1}$ as a pullback of B^{Δ^1} and $C \times_B B$.

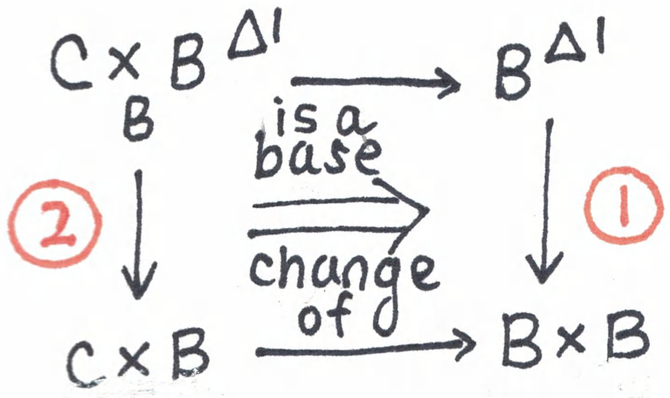




we know this map is a Kan fibration because of the pullback square:



If we can show ② is a Kan fibration, then the composition map is a Kan fibration (Kan fibrations are closed under composition).



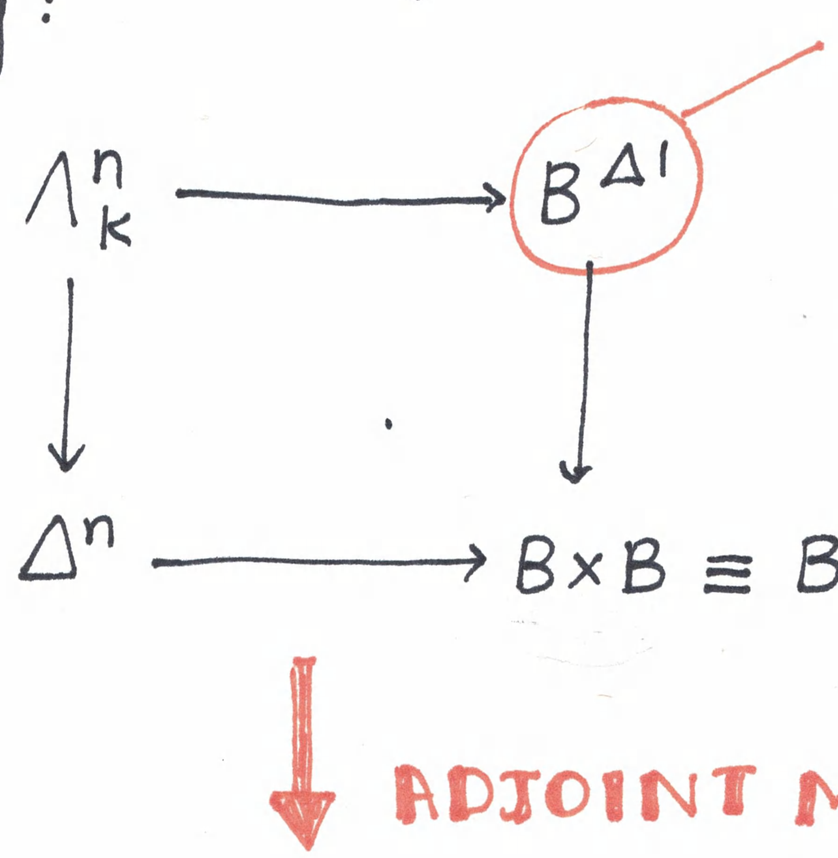
The map ② is a base change of the map ①

so...

IF WE CAN SHOW ① IS A KAN FIBRATION, THEN WE CAN SAY ② IS A KAN FIBRATION (since Kan fibrations are closed under base changes!)

Consider the map ① : $B^{\Delta^1} \longrightarrow B \times B$.

To show this map is a Kan fibration, we need to show that every horn inclusion has a lifting property:

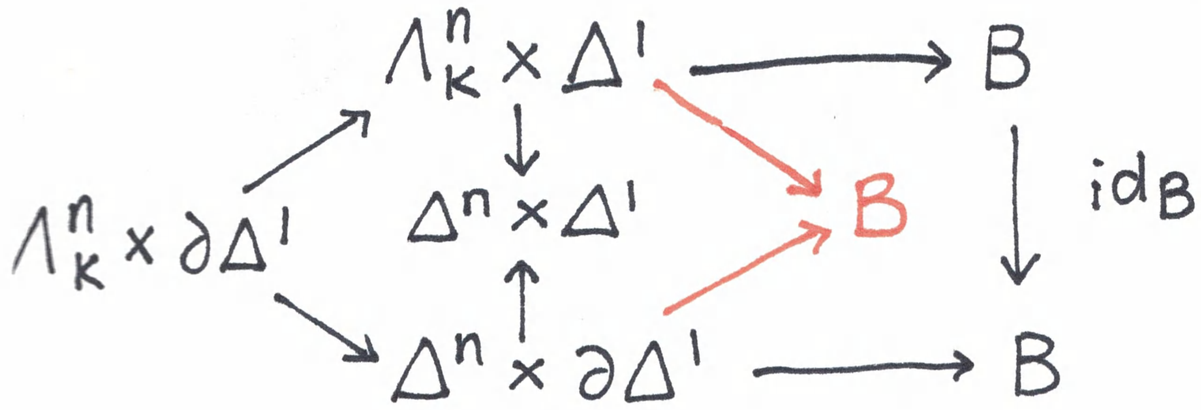


$B^{\Delta^1} = \mathrm{Hom}(\Delta^1, B)$
The internal hom has an adjoint functor for maps

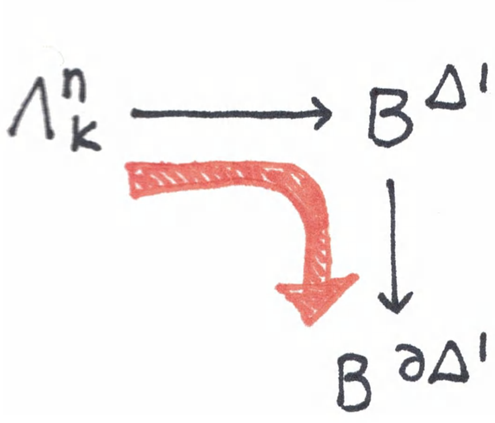
map $A \longrightarrow \mathrm{Hom}(\Delta^1, B)$
$A \times \Delta^1 \longrightarrow B$ adjoint map

USE ADJOINT PROPERTIES OF INTERNAL HOM!

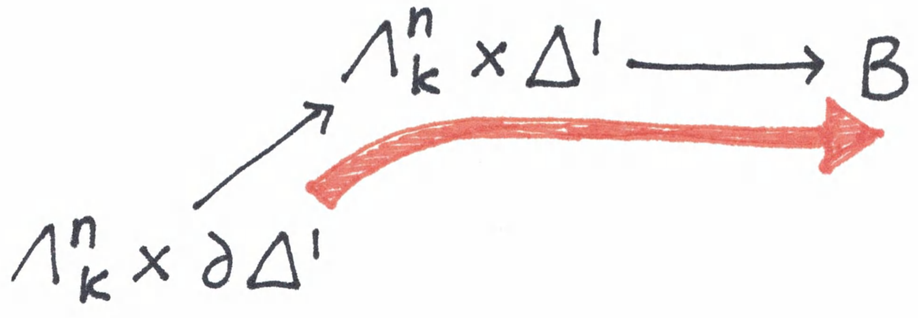
ADJOINT MAP



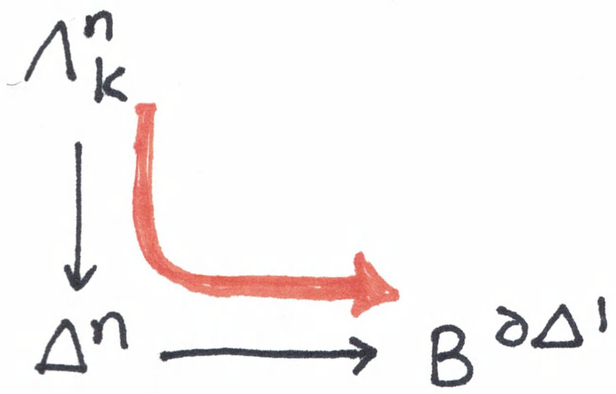
where the composition



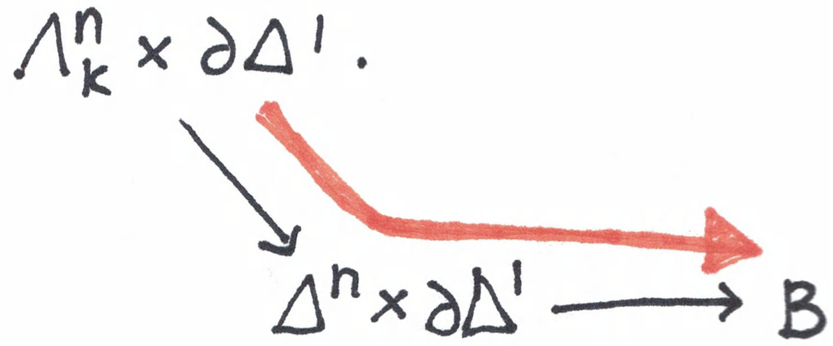
... is adjoint to...



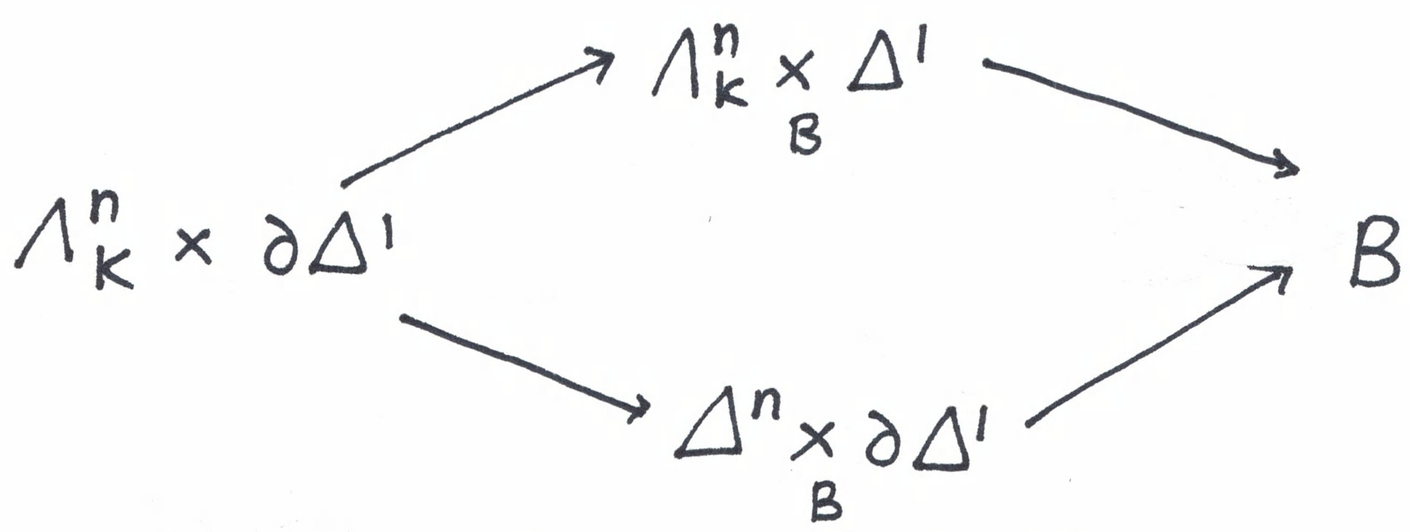
annnd... the composition



...is adjoint to...



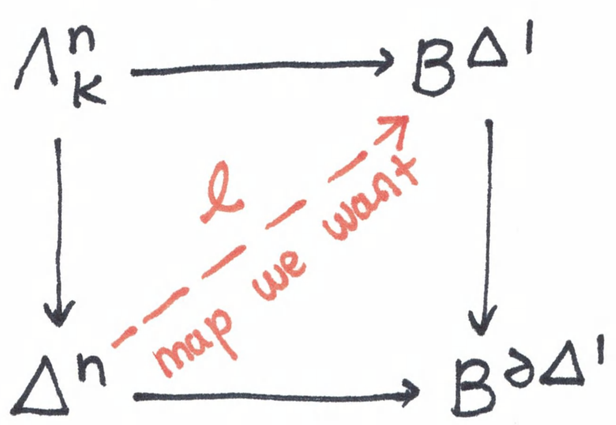
so we have a diagram:



So, we want to show there is a lift map

$$\exists\, l : \Delta^n \longrightarrow B^{\Delta^1}$$

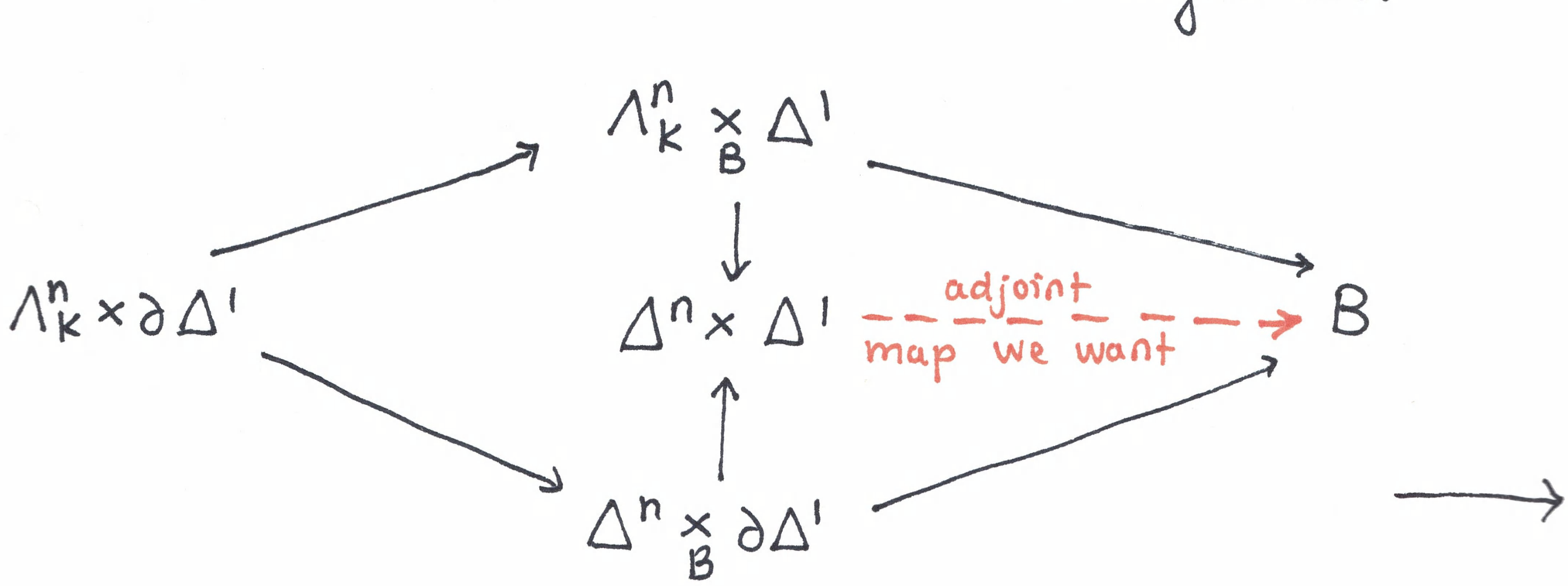
in the original diagram



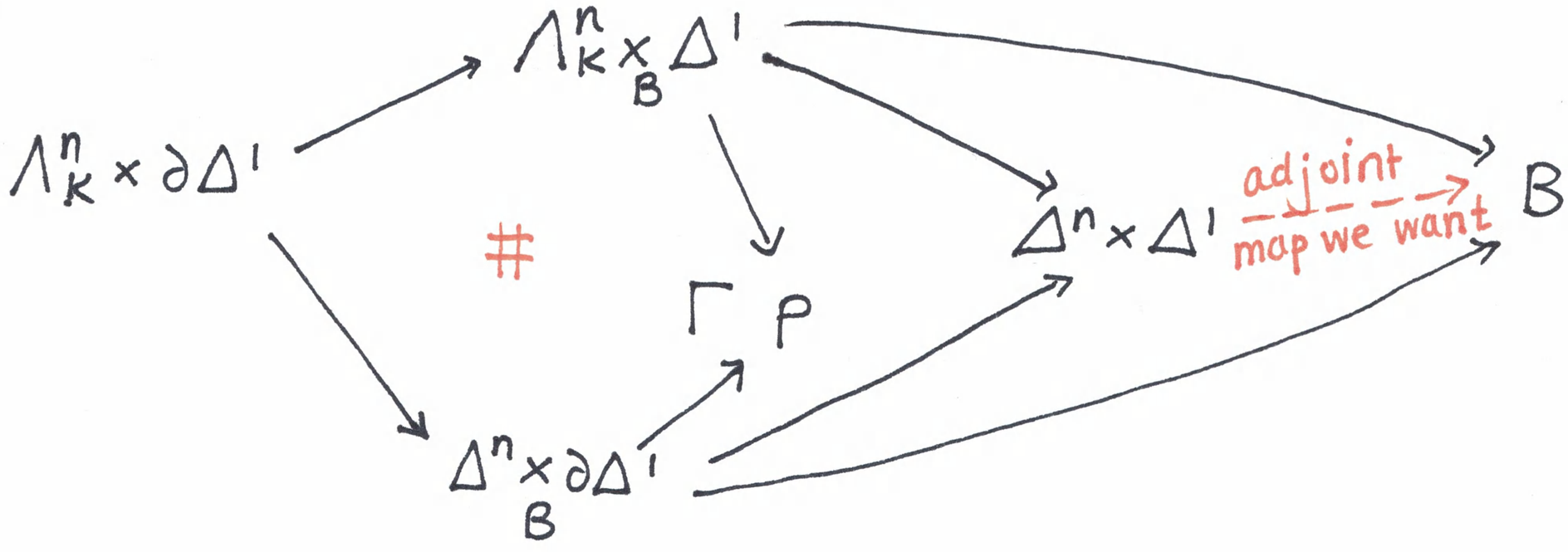
the adjoint of this lift map would be a map

$$\Delta^n \times \Delta^1 \xrightarrow[\text{map we want}]{\text{adjoint}} B$$

and we can see this in the diagram...

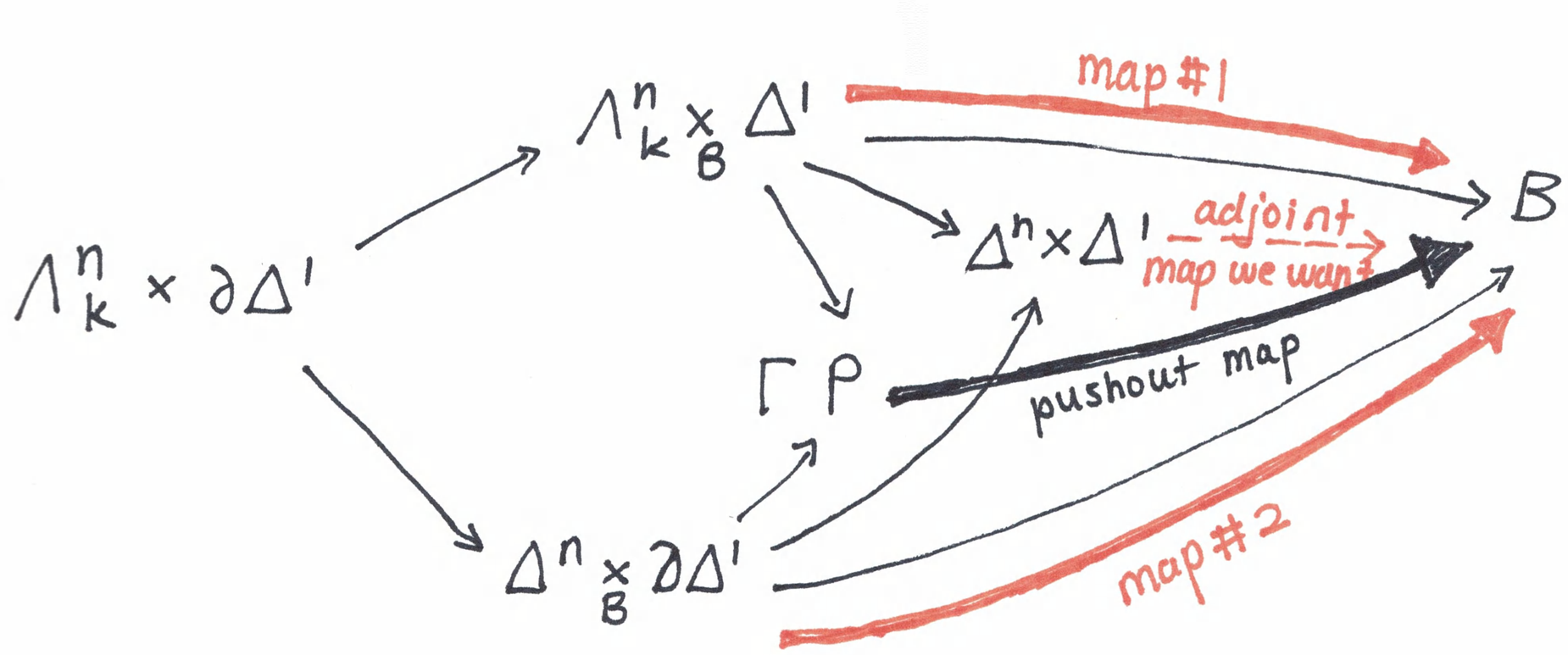


Now,
we can construct a pushout diagram:

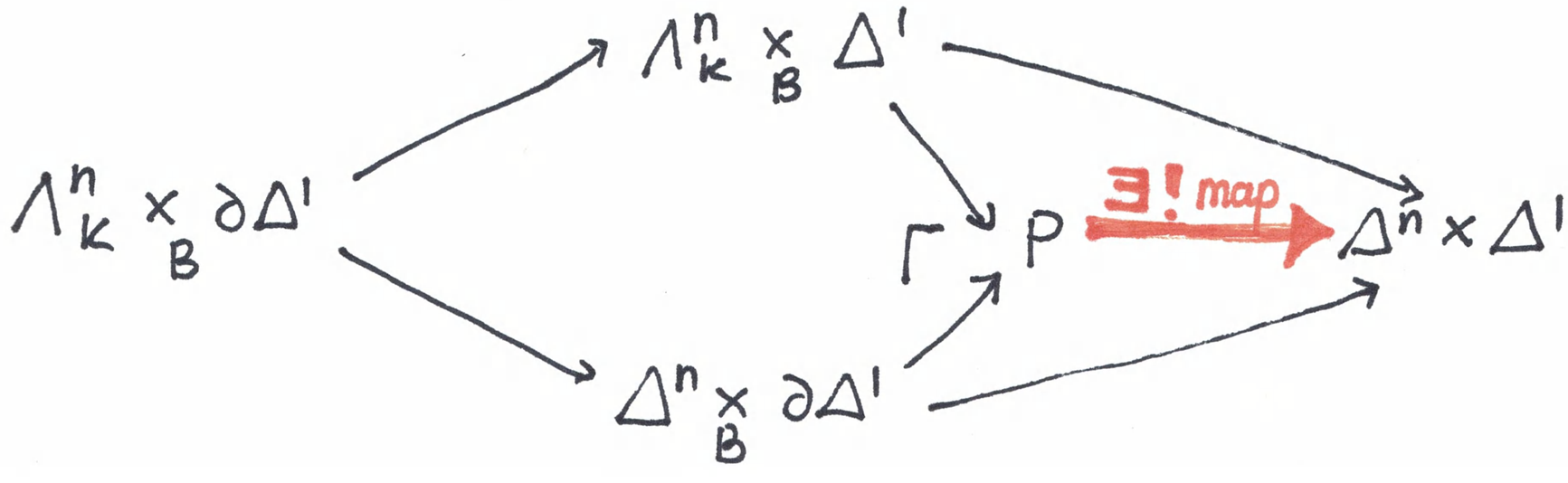


and by the UNIVERSAL PROPERTY OF PUSHOUTS,
the data from maps $\Lambda^n_k \times_B \Delta^1 \longrightarrow B$ map #1
and $\Delta^n \times_B \partial\Delta^1 \longrightarrow B$ map #2

defines a map $P \xrightarrow{\text{pushout map}} B$

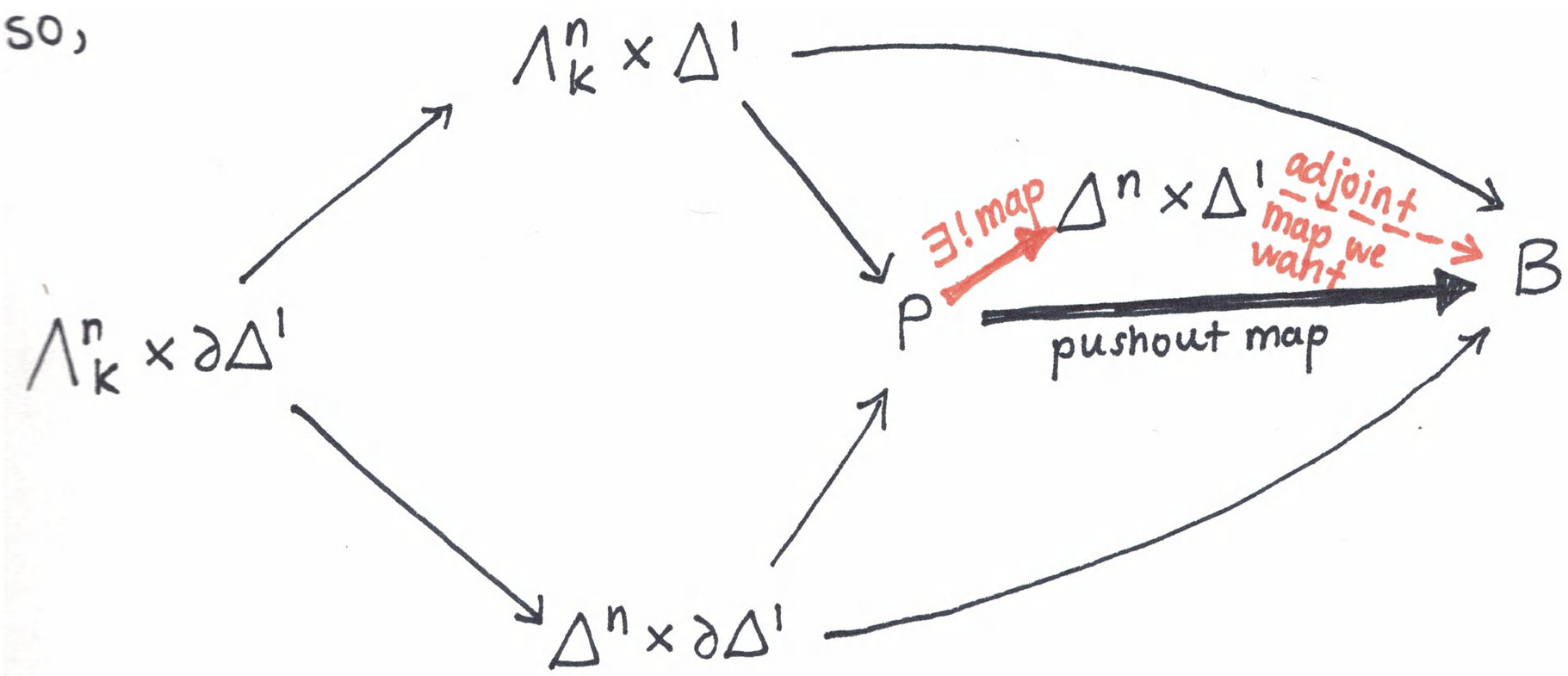


Also, by the UNIVERSAL PROPERTY OF PUSHOUTS,
between two commutative diagrams with the same "legs",
we can find a unique map $P \longrightarrow \Delta^n \times \Delta^1$ ...

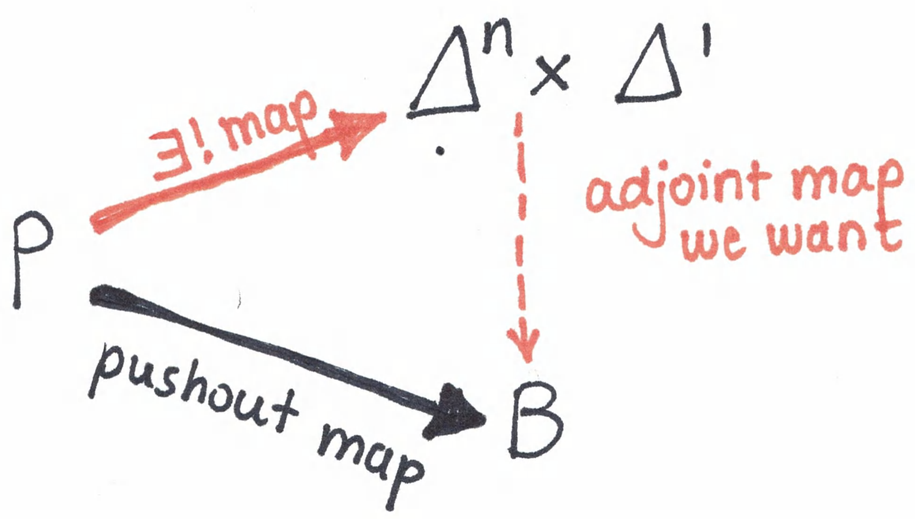


→

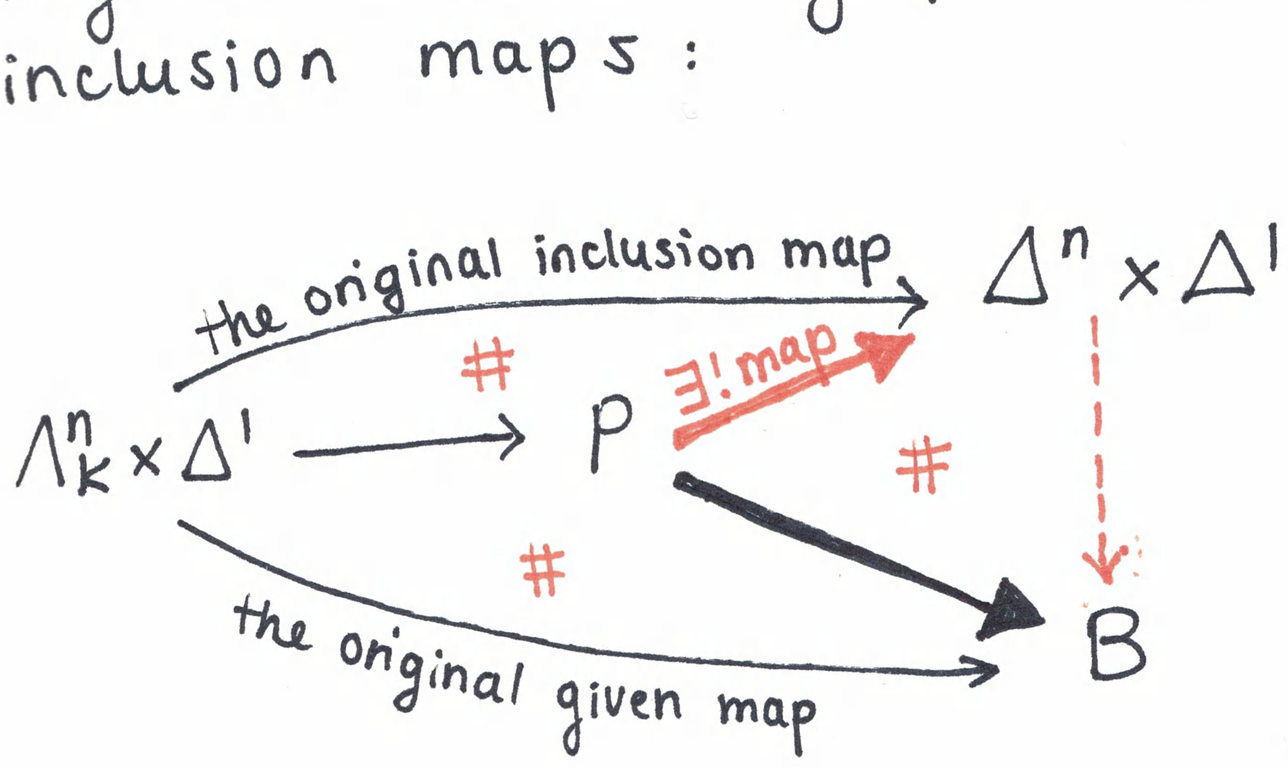
so,



we now need only to show that this small triangle commutes, for then we can define the adjoint map we want in terms of the pushout map!



the commutativity of this triangle is equivalent to the commutativity of the diagram obtained by precomposing with the inclusion maps:

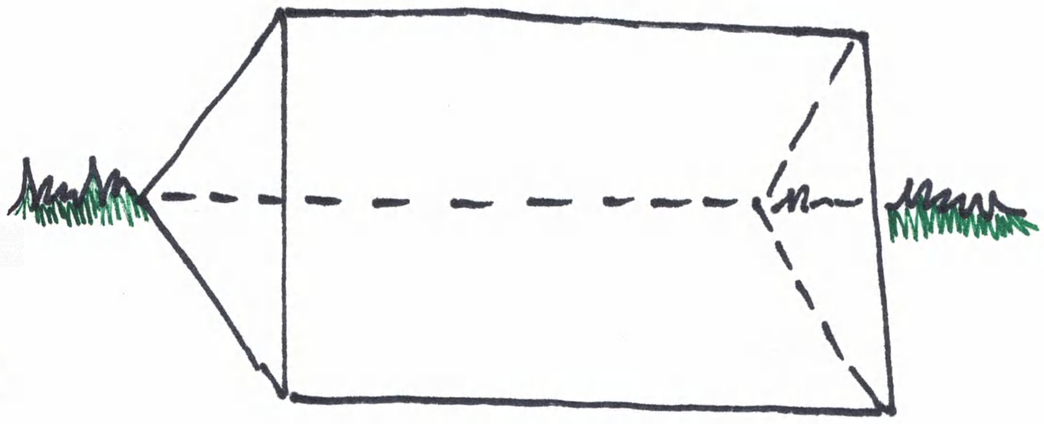


NOTE:
similarly, we have the other inclusion $\Delta^n \times_B \partial\Delta^1 \longrightarrow P$ and its corresponding diagram commutes by precomposing with the map $\Lambda^n_k \times_B \partial\Delta^1 \longrightarrow \Delta^n \times_B \partial\Delta^1$.

which commutes by the original diagram. $\longrightarrow$

Geometrically, this looks like ...

$\Delta^n \times \Delta^1$



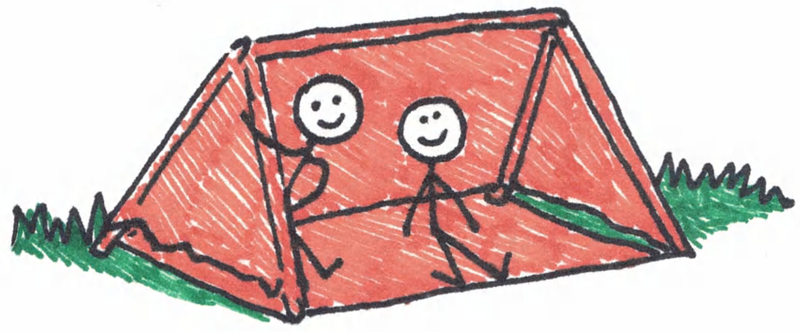
HOW TO CONSTRUCT THE UNIQUE MAP

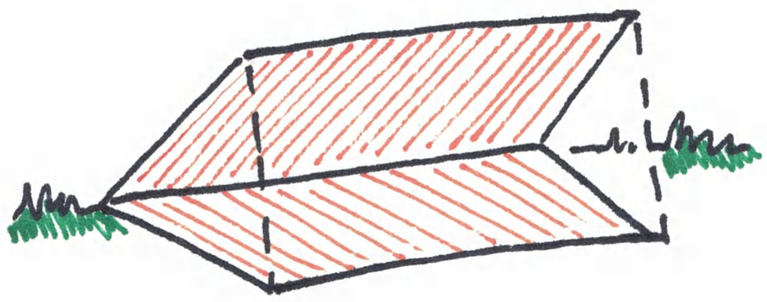
$P \xrightarrow{\exists! \text{ map}} \Delta^n \times \Delta^1$

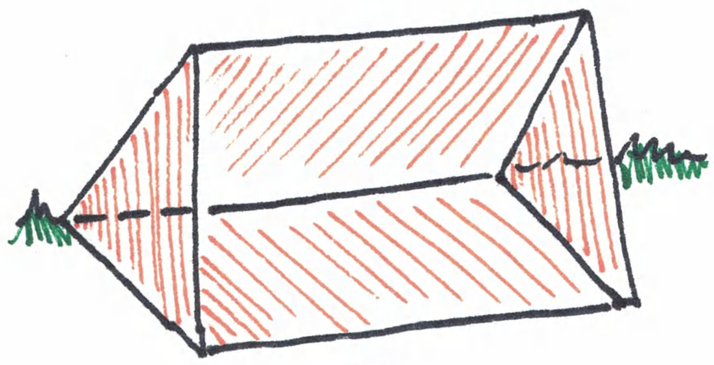
$\Lambda^n_k \times \Delta^1$

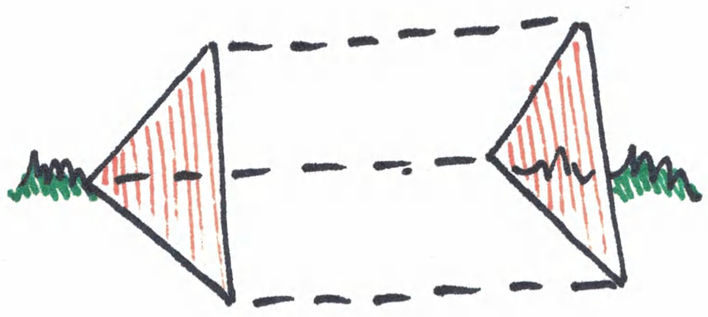
$\Delta^n \times \partial\Delta^1$

$\Gamma P \longrightarrow \Delta^n \times \Delta^1$

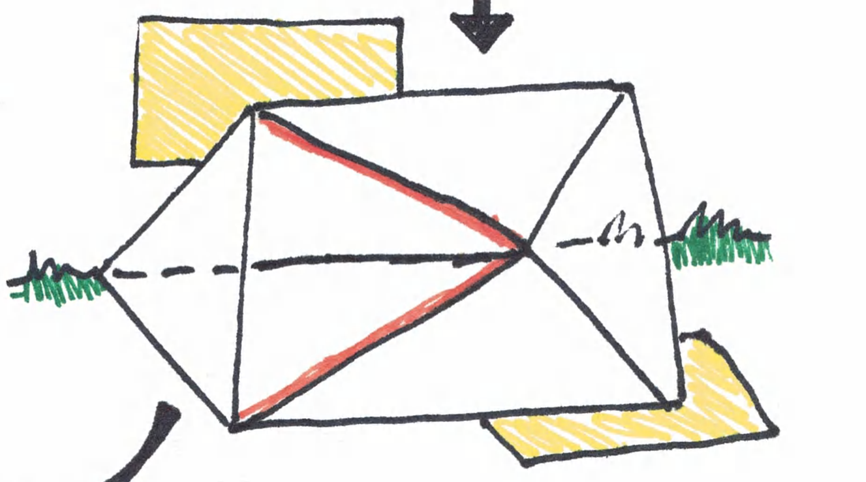


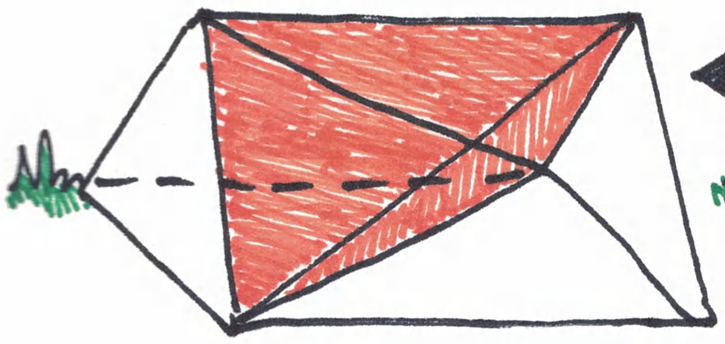


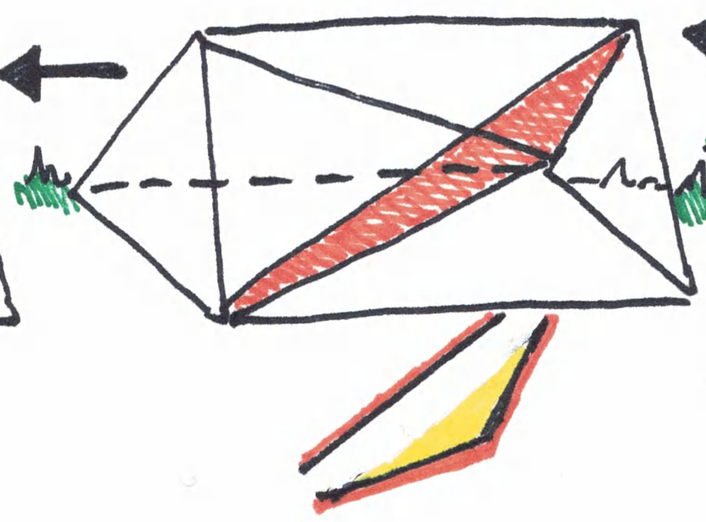


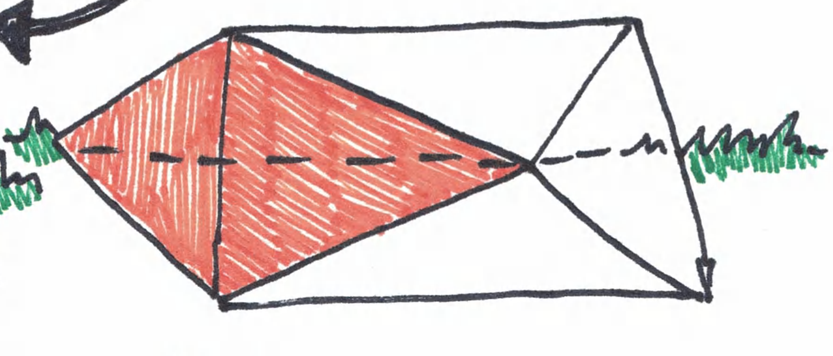


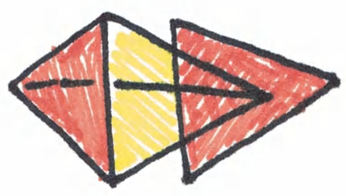


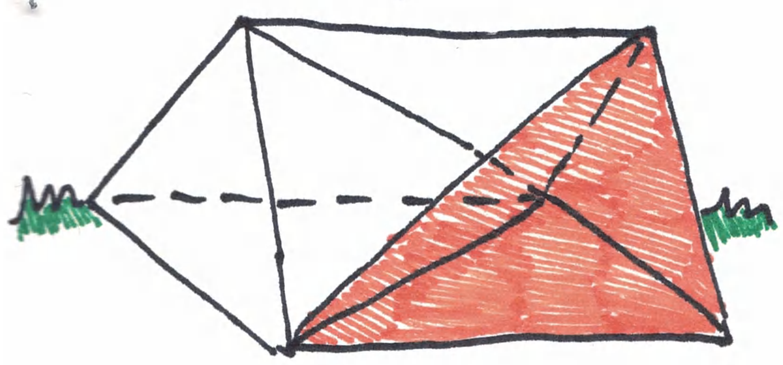


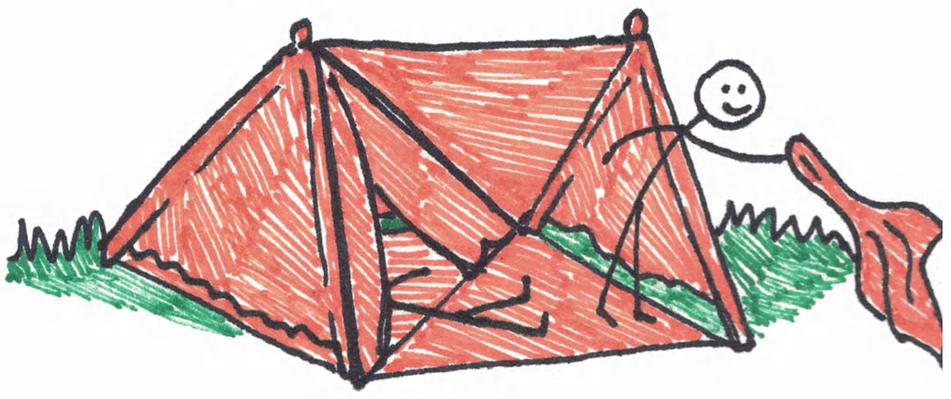


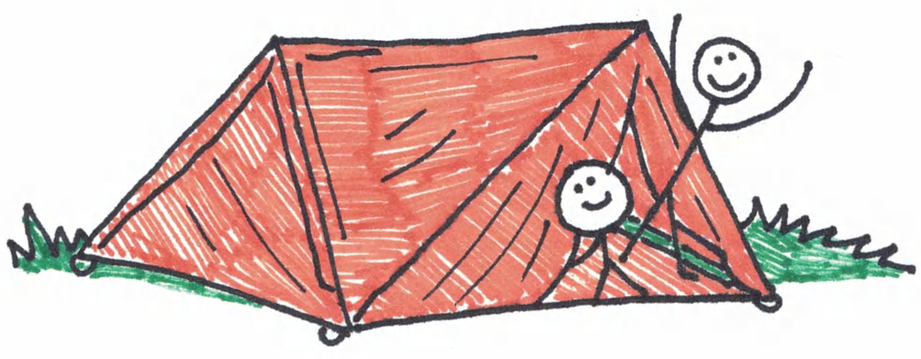






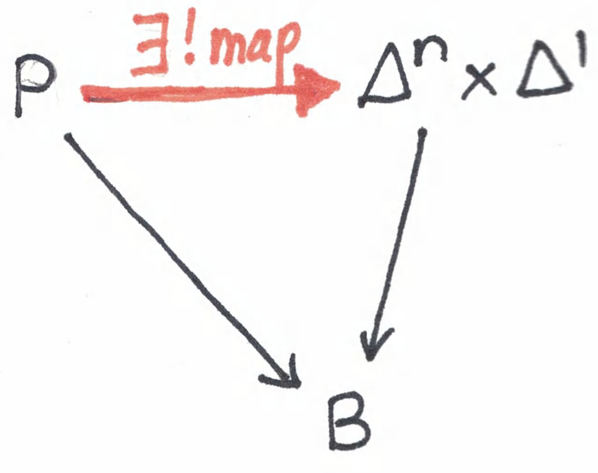






... That is, we have the inclusion map

$$P \xrightarrow{\exists!\ \text{map}} \Delta^n \times \Delta^1$$



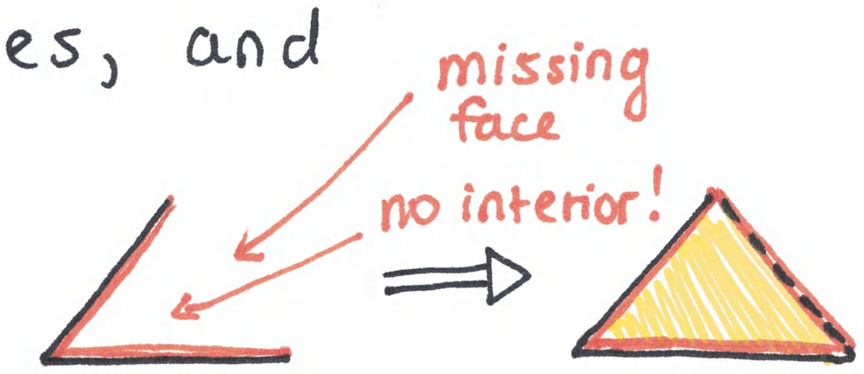
and we construct the map $P \longrightarrow B$ by "gluing together" (in B)

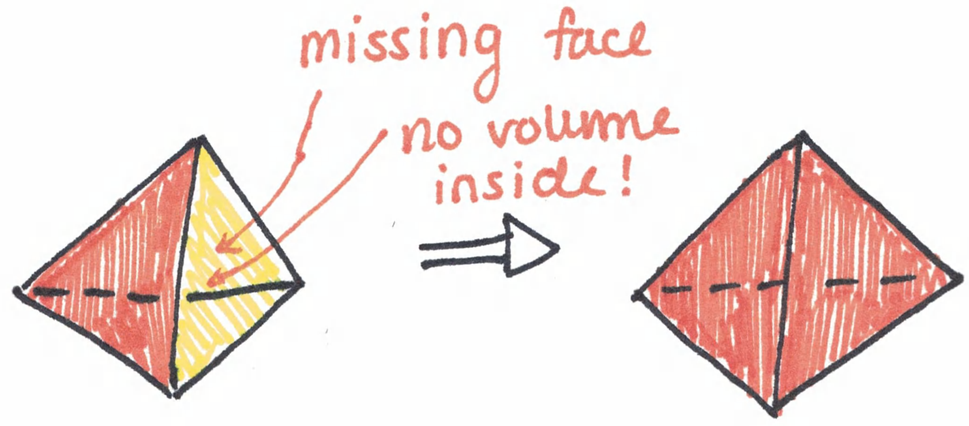
$\Lambda^n_k \times \Delta^1$ and $\Delta^n \times \partial\Delta^1$

along shared edges, and

"filling in" faces of [2]-simplices

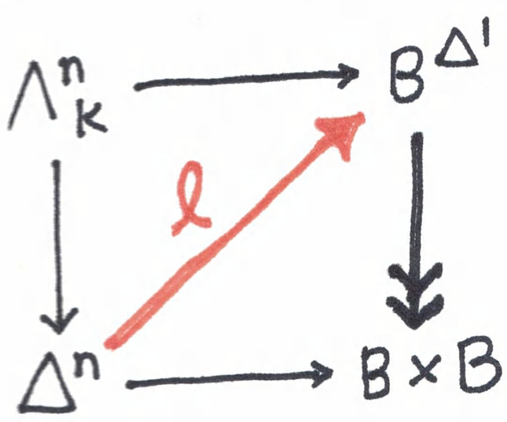
faces & volumes and of [3]-simplices





NOTE: We are allowed to perform these "horn fillings" because $B, \mathcal{C} \in$ sSet are Kan!

So... there is a lift map $\ell : \Delta^n \longrightarrow B^{\Delta^1}$, and



B^{Δ^1}

① ↓

$B \times B$

is a Kan fibration

(the base change of a Kan fibration is a Kan fibration)

Hence ② is a Kan fibration

Hence

$C \times_B B^{\Delta^1}$

↓ ②

$C \times_B B$

↓ ③

$1 \times_B B$

$C \times_B B^{\Delta^1}$

↓

B

is a Kan fibration. ▨

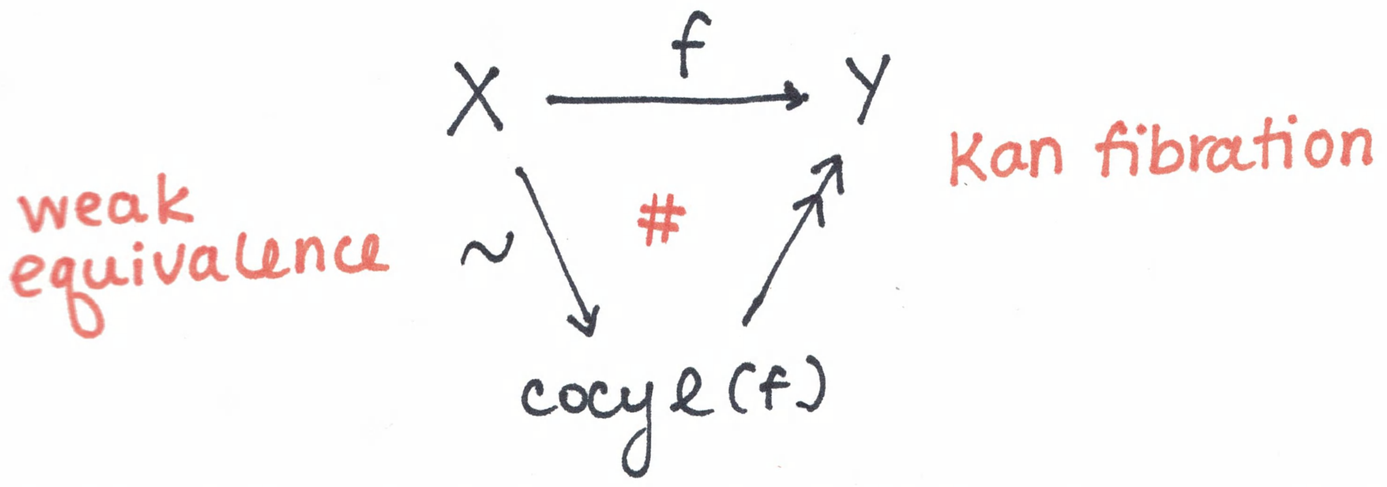
This lemma can be extended to a more general idea (which will be revisited later)...

DEFINITION The "cocylinder" of $f: X \longrightarrow Y$ is defined:

$$\mathrm{cocyl}(f) := X \underset{Y}{\times} Y^{\Delta^1}$$

LEMMA If $X, Y \in \mathrm{sSet}$ are Kan,
and $f: X \longrightarrow Y$ is a simplicial map,

then f can be factored as follows:



EXERCISE: Given $X \in \mathrm{sSet}$

and $X \downarrow \Delta^1$ a Kan fibration,

show that $X_0 \simeq X_1$

where $X_0 = X \underset{\Delta^1}{\times} 0$

and $X_1 = X \underset{\Delta^1}{\times} 1$

{MODEL STRUCTURE}

necessary BACKGROUND

① NOTATION:

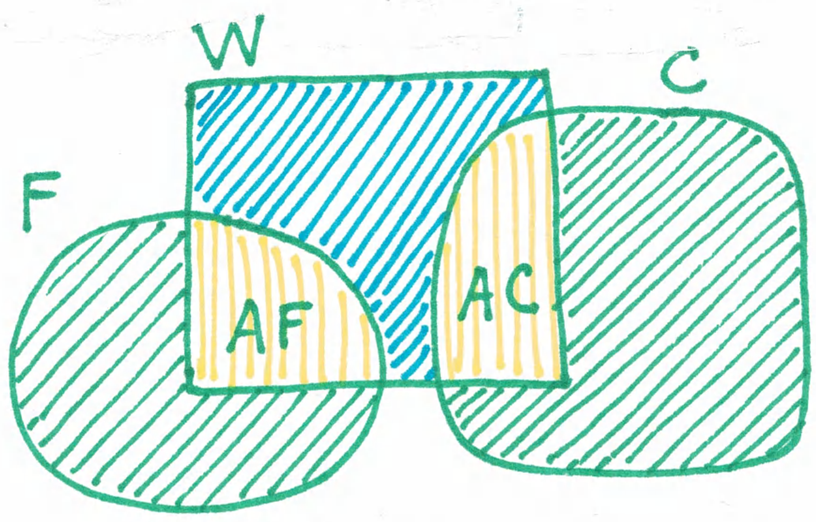
W: weak equivalences

F: fibrations

C: cofibrations

AF: acyclic fibrations = $F \cap W$

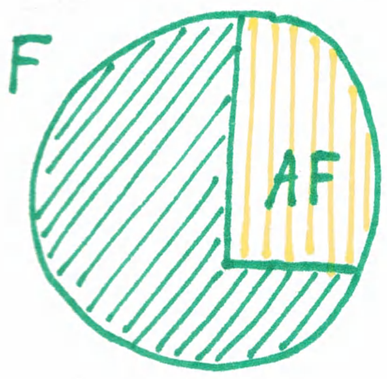
AC: acyclic cofibrations = $C \cap W$



NOTE: isomorphisms are both acyclic fibrations AND acyclic cofibrations



FIBRATIONS

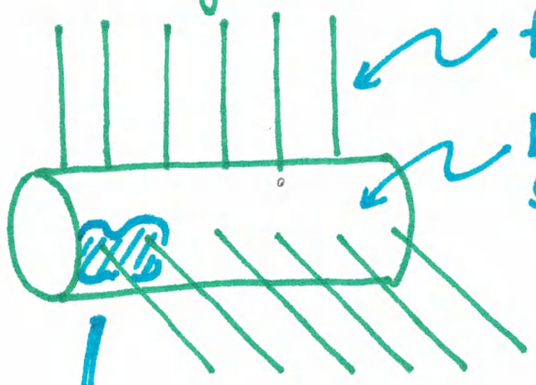


A fibration is a generalization of a fiber bundle.



fiberous hairbrush

FIBER BUNDLE





locally a product space

but globally, much different!

E: total space

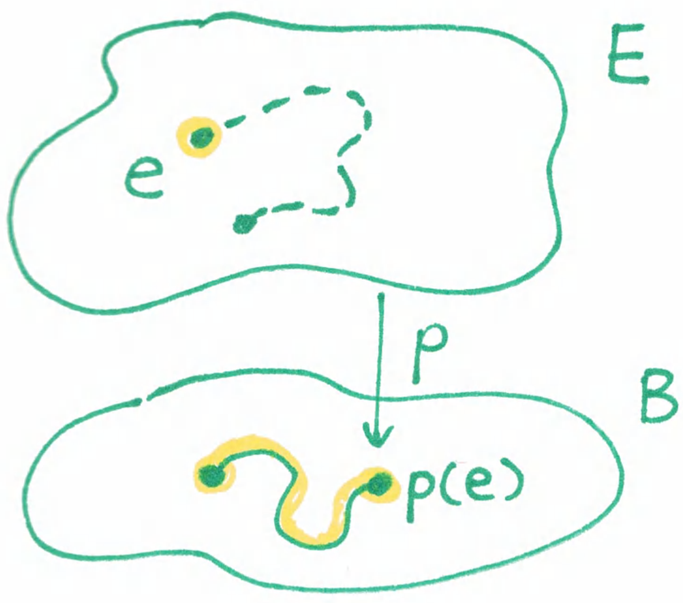
(in the trivial case of a fiber bundle, $E = B \times \{F\}$ and this is called the "trivial bundle"!

Each fiber can be "parameterized" by the base space.

There is a "projection map" (surjective)

$\pi : E \longrightarrow B$, which locally behaves like the projection $p : B \times F \longrightarrow B$.

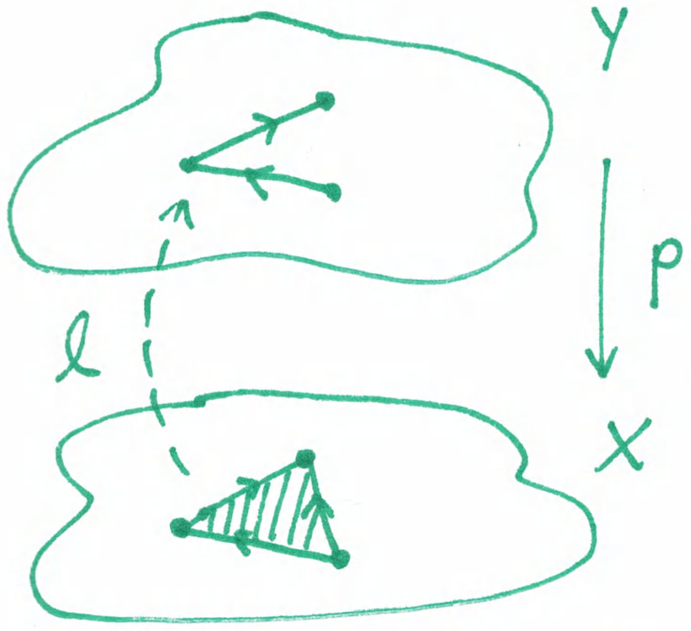
→ A fibration $p: E \longrightarrow B$ is a continuous function between topological spaces with a "lifting property" (similar to the "local" condition on $\pi: E \longrightarrow B$ in the definition of a fiber bundle):

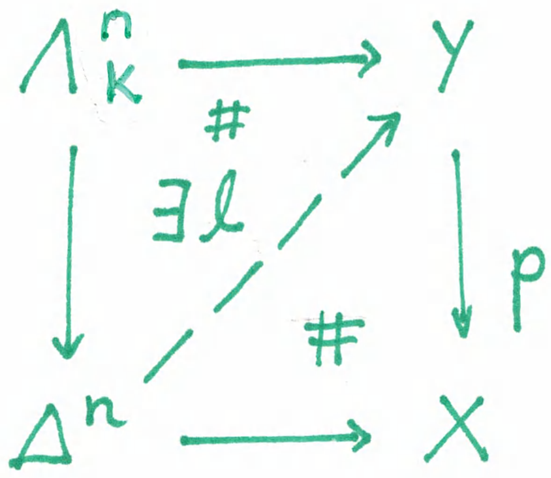


For Example:
Given $e \in E$
and a path $\Delta^1 \longrightarrow B$ in B
which begins at $p(e)$,
this path can be "lifted"
to a path $\Delta^1 \longrightarrow E$ in E
which begins at e,

~~and this path lifting is unique up to a homotopy.~~

In the case of a "Kan fibration", we have:





geometrically, the horn in Y can be completed to a simplex based on the information obtained from mapping it into X... this is the notion of "horn fillings!"

When the space Y is Kan, a n-horn can always be completed to an n-simplex.

Diagramatically, there is a lift map $\exists l: \Delta^n \longrightarrow Y$ such that the triangles formed by l commute with the map p and the horn inclusion.

SO... fibrations are those maps with a certain "lifting property."

"One generally assumes the lifting of additional structures, (including "higher homotopies") in B, which, in particular, imply that a path lifting is unique up to homotopy. Different choices of what can be lifted give rise to different notions of fibration, for example,

- In a Hurewicz fibration, all sorts of homotopies can be lifted
- In a serre fibration, topological n-cells can be lifted for all n
- In a Dold fibration, all homotopies can be lifted, but the lifting only has to agree with the given initial map up to vertical homotopy.

All three of these definitions give rise to a LONG EXACT SEQUENCE OF HOMOTOPY GROUPS."

-nLab

The difference here is that our focus is on fibrations as maps between simplicial sets with a certain lifting property.

In the case $X, Y \in sSet_{Kan}$,

A Lifting property with respect to horn inclusions yields a set of maps called "Kan fibrations"

A Lifting property with respect to boundary inclusions yields a set of maps called "acyclic Kan fibrations."

COFIBRATIONS

(dual to fibrations; can be thought of as injection maps)

② THE 2/3 CONDITION

For $(\mathcal{C}, W) \in$ Relcat such that

if $gf \in W$ & $f \in W$ then $g \in W$	or	if $gf \in W$ & $g \in W$ then $f \in W$

or

if $g \in W$ & $f \in W$ then $gf \in W$

The 2-out-of-3 condition states that if two out of three of the morphisms

$$f \;,\; g \;,\; gf$$

are weak equivalences, then the third is also a weak equivalence.

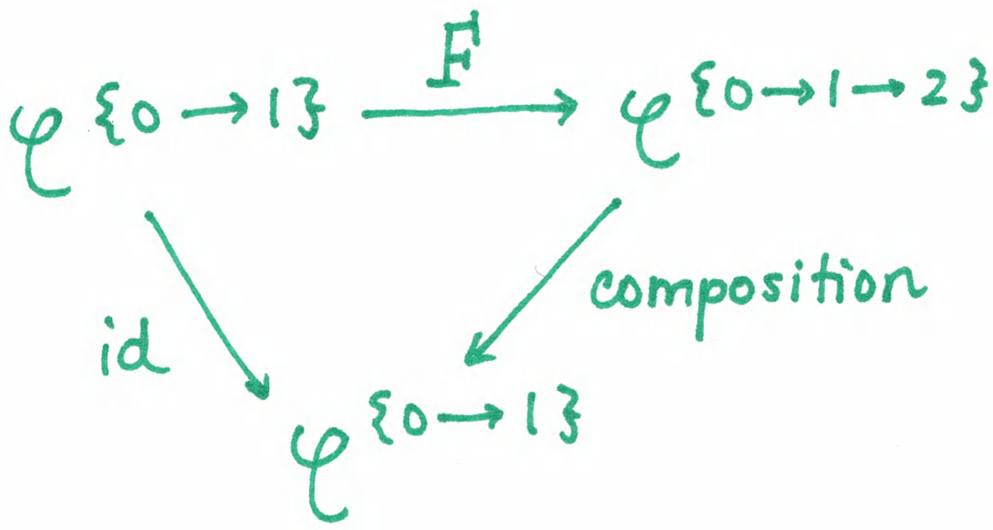
③ FUNCTORIAL FACTORIZATION SYSTEM

A functorial factorization on a category $\mathcal{C}$ is a functor

$$F : \mathcal{C}^{\{0\to 1\}} \longrightarrow \mathcal{C}^{\{0\to 1\to 2\}}$$

input is morphisms, $f \longmapsto$ output is composible pairs of morphisms $F_1(f), F_2(f)$

such that the diagram commutes:

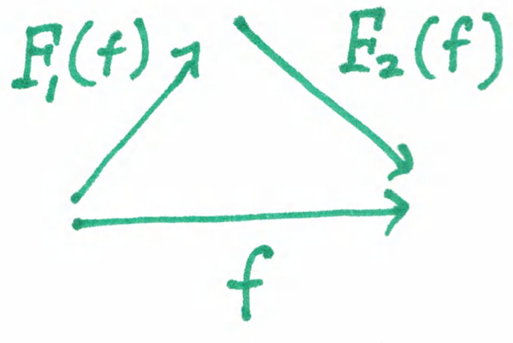


That is, maps can be "factored" into pairs of maps:

given $f \in \mathcal{C}^{\{0\to 1\}}$

$F(f) = \{F_1(f), F_2(f)\}$

such that $f = F_2(f) \circ F_1(f)$

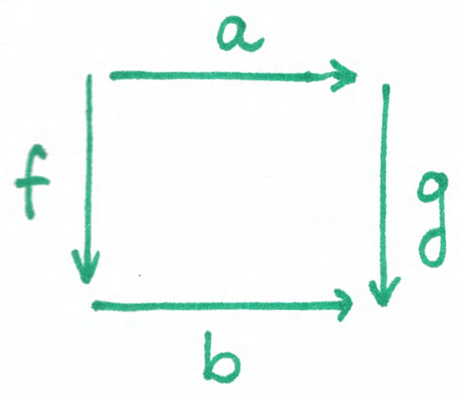


NOTE: A functorial factorization is a section of the composition functor

$\mathcal{C}^{\{0\to 1\}}$ is the category of morphisms in $\mathcal{C}$, whose objects are morphisms in $\mathcal{C}$, and whose morphisms are commutative diagrams $\mathcal{C}^{\{0\to 1\to 2\}}$ hu

④ LIFTING PROPERTY NOTATION & PROPERTIES

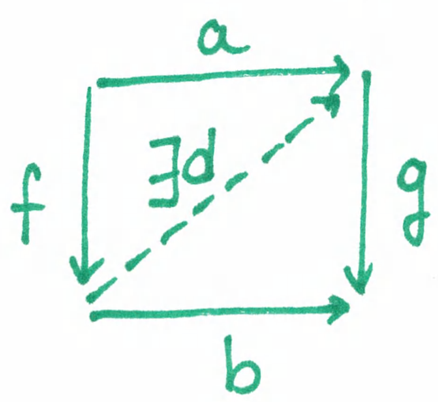
Consider the diagram



We say that f has a nonunique lifting property with respect to g

denoted $f \pitchfork g$

if $\forall$ maps a, b, $\exists$ d such that



$a = d \circ f$

$b = g \circ d$

hence, the square commutes:

$g a = b f$

$f \pitchfork g$	$f \perp g$
the pair f, g in any commutative square has a nonunique lifting property	the pair f, g in any commutative square has a unique lifting property

NOTATION

$$S^{\pitchfork} = \{ g \mid s \pitchfork g, \ \forall s \in S \}$$

$$^{\pitchfork}S = \{ g \mid g \pitchfork s, \ \forall s \in S \}$$

PROPOSITION: $S^{\pitchfork}$ and $^{\pitchfork}S$ contain all isomorphisms in the category $\mathcal{C}$ and are closed under compositions.

... and ...

$S^{\pitchfork}$ is closed under

- ☐ products
- ☐ base changes
- ☐ cotransfinite compositions

$^{\pitchfork}S$ is closed under

- ☐ coproducts
- ☐ cobase changes
- ☐ transfinite compositions

(5) WEAK FACTORIZATION

A weak factorization system on a category $\mathcal{C}$ is a pair (L, R),

$L \subset \mathcal{C}$
$R \subset \mathcal{C}$

such that there exists a functorial factorization

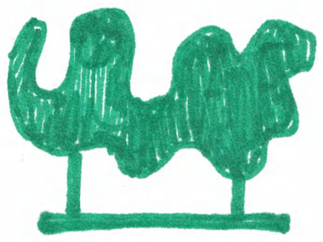
$$\hat{F}(f) = (\hat{F}_1(f), \hat{F}_2(f))$$

$\hat{F}_1(f) \in L$
$\hat{F}_2(f) \in R$

where $L = {}^{\pitchfork}R$ and $R = L^{\pitchfork}$

(6) MODEL STRUCTURE

A model structure on a relative category $(\mathcal{C}, W)$ is a pair of weak factorization systems



(C, AF) and (AC, F)

cofibrations — acyclic fibrations

acyclic cofibrations — fibrations

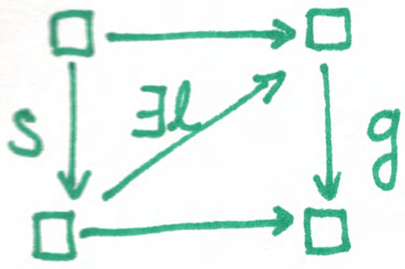
on $\mathcal{C}$ such that

$$AC = C \cap W$$
$$AF = F \cap W$$

NOTE: A model structure is similar to a coordinate system in that the choice of model structure does not affect the theoretical results. Hence, studying model structures is an efficient way to prove properties!

Example Let $S \subset \mathrm{Mor}(\mathcal{C})$, $S = \{\Lambda^n_k \longrightarrow \Delta^n\}$ and $\mathcal{C} = sSet$

Then $S^{\pitchfork} = \{g \in \mathrm{Mor}(\mathcal{C}) \mid \forall s \in S,\ s \pitchfork g\}$



s, g has a nonunique lifting property

Here, $S^{\pitchfork} = \{g \mid g \text{ is a Kan fibration}\}$

Example Let $S \subset \mathrm{Mor}(\mathcal{C})$, $S = \{\partial\Delta^n \longrightarrow \Delta^n\}$ and $\mathcal{C} = sSet$

Then $S^{\pitchfork} = \{g \mid g \text{ is an acyclic fibration}\}$

Remark: In sSet, acyclic fibrations are a subset of simplicial weak equivalences:

$$AF \subset \text{s.w.e.} \subset F$$

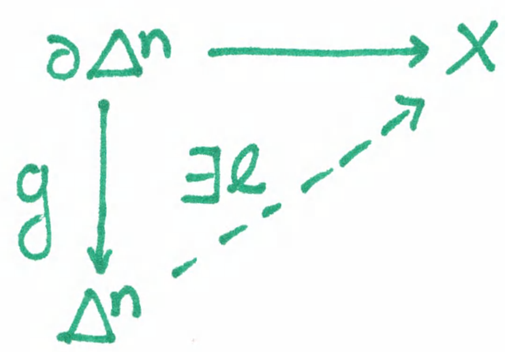
and all acyclic fibrations are Kan fibrations.

Lemma

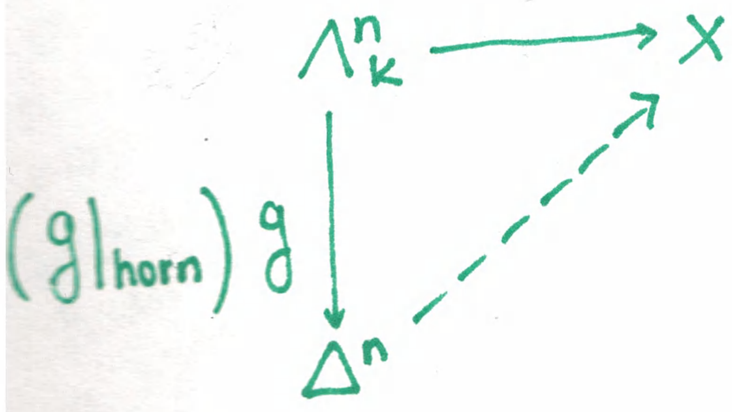
Acyclic fibrations are fibrations:

suppose we have an acyclic fibration, g

then we have a nonunique lifting property:

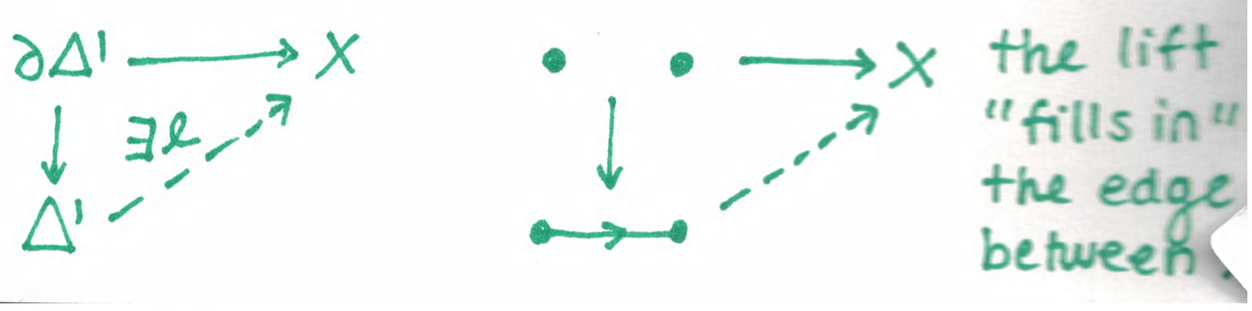


we need to show that g has a nonunique lifting property

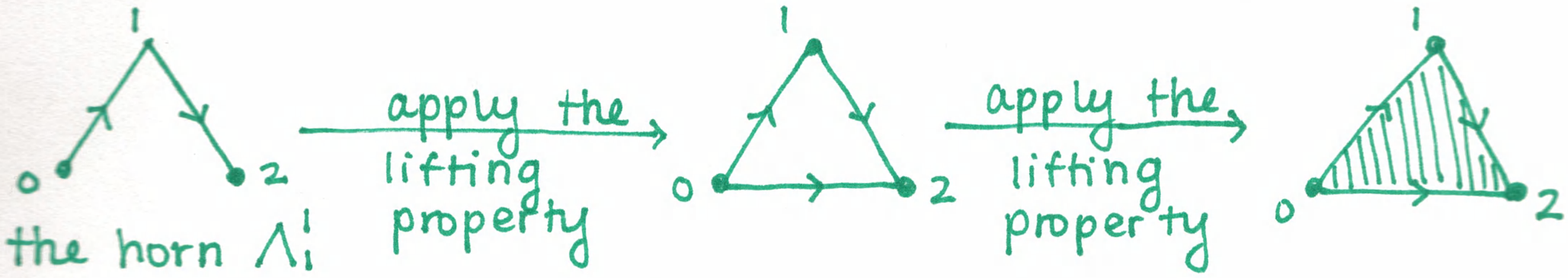


We can apply the lifting property of an acyclic fibration twice (in dimensions $n-1, n$)...

When $n=1$, we have a lifting property:



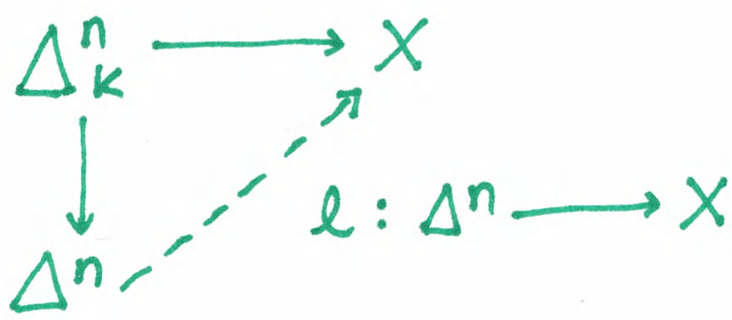
the lift "fills in" the edge between



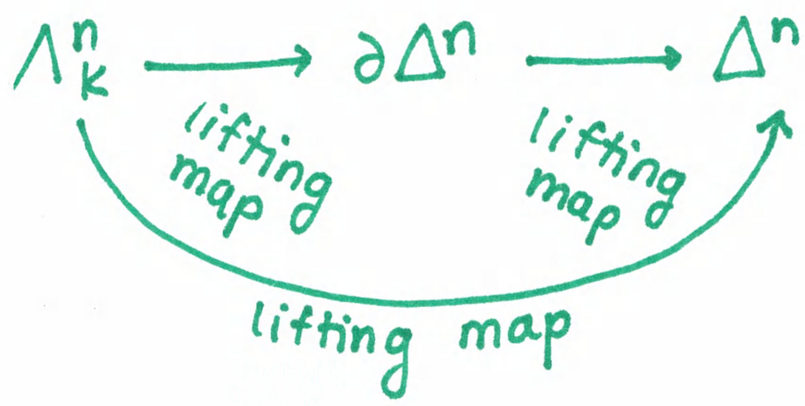
(i) Apply lifting property dimension $n-1$

(ii) Apply lifting property dimension n

In each dimension,
the composition of (i) and (ii) gives us a map l:

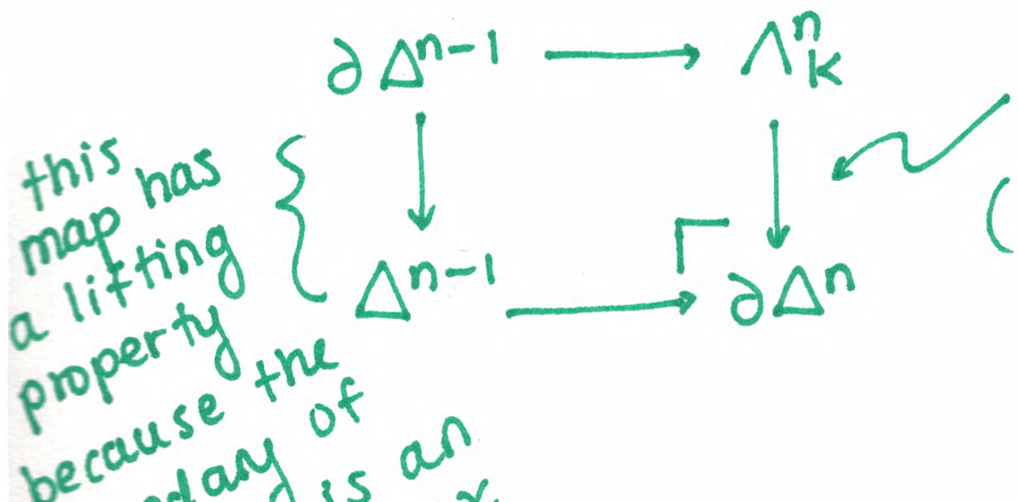


that is,



the composition of lifting maps gives us a lifting map.
(sth is closed under composition)

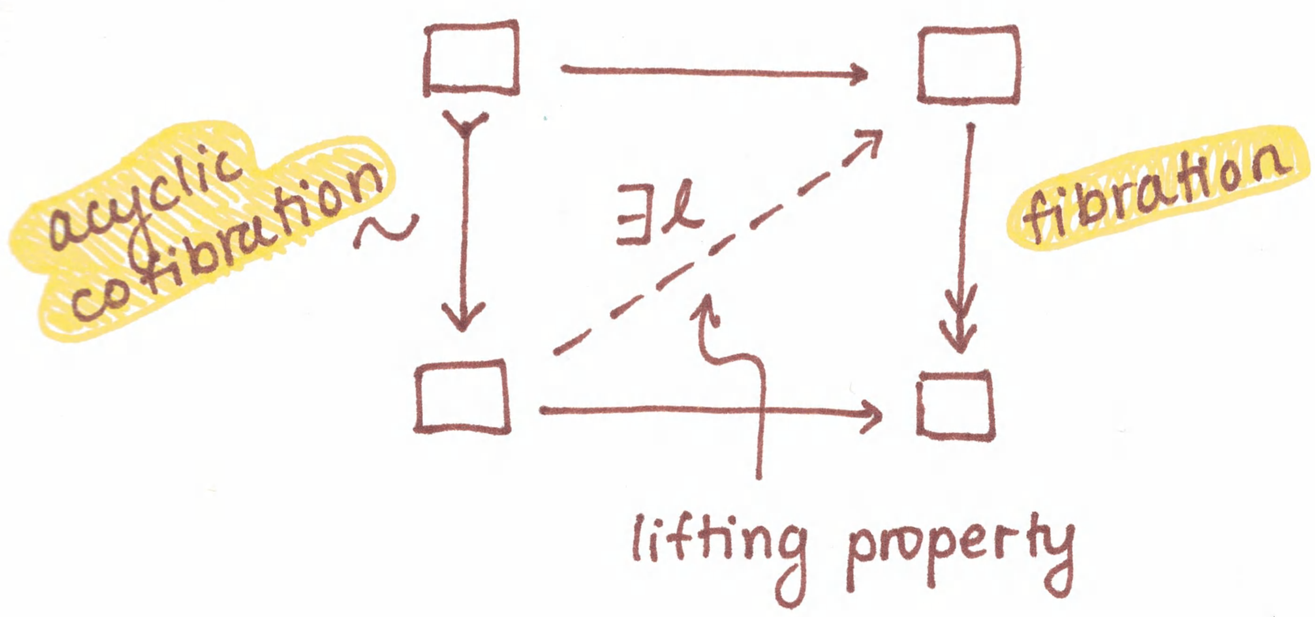
NOTE: the map $\Lambda^n_k \longrightarrow \partial\Delta^n$ has a lifting property from the pushout square:



This map is a cobase change of the left map.
(sth is closed under cobase changes, so this map also has a lifting property.) ▨

$$\boxed{AC = \pitchfork F} = C \cap W$$

acyclic cofibrations are those morphisms which have a left-lifting property with respect to fibrations.



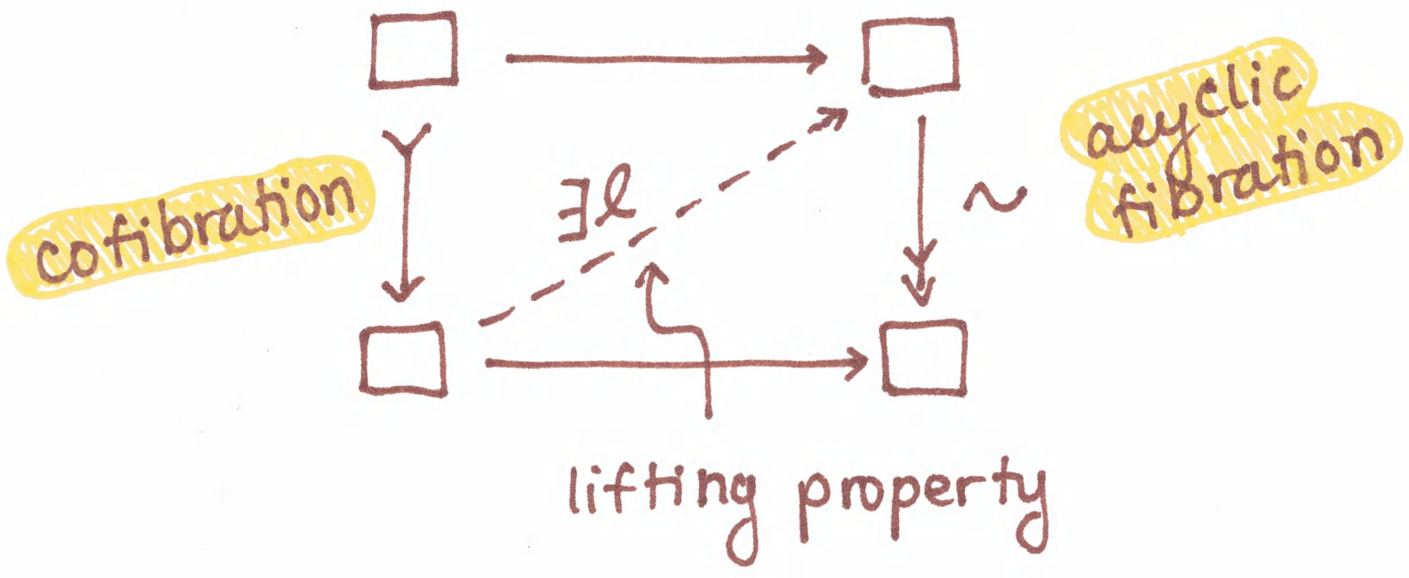
$$AC = \{\, g \mid \forall f \in F,\ g \pitchfork f \,\}$$

morphisms ... ... left-lifting property

fibrations

$$\boxed{AF = C^{\pitchfork}} = F \cap W$$

acyclic fibrations are those morphisms which have a right-lifting property with respect to cofibrations.



$$AF = \{\, h \mid \forall c \in C,\ c \pitchfork h \,\}$$

morphisms ... ... right-lifting property

cofibrations

properties

These are the two lifting properties of a model category

DEFINITION Let $\mathcal{C}$ be a co-complete category and $S \subset \mathrm{Mor}(\mathcal{C})$

S is called "weakly saturated" if it is closed under the following properties:

(1) COPRODUCTS

(2) COBASE CHANGES

(3) TRANSFINITE COMPOSITIONS

(4) RETRACTS

(5) PASSAGE TO ISOMORPHIC MAPS

Notation: $\mathrm{sat}(S) :=$ the weak saturation of S

To determine $\mathrm{sat}(S)$, start with $S \subset \mathrm{Mor}(\mathcal{C})$, and repeatedly apply properties (2), (3), and (4).

EXERCISE Show that properties (2), (3), and (4) imply (1) and (5).

[that is, show that coproducts and passage to isomorphic maps can be ~~info~~ derived by repeated application of cobase changes, transfinite compositions, and/or retracts]

Remark

The weak saturation of S, $\mathrm{sat}(S)$, is the SMALLEST weakly saturated class that contains S.

For $S \subset S_i$, S_i weakly saturated, consider $\{S_i\}$

$\bigcap_i S_i$ is nonempty (each S_i contains S)

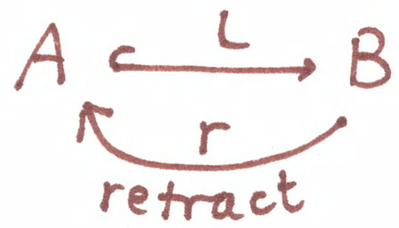
and $\bigcap_i S_i$ is weakly saturated (since each of (2), (3), and (4) are closed under intersection).

RESOURCES:

(i) HOVEY

(ii) JOYAL

Recall, a "retract" of a map $A \xrightarrow{L} B$ (inclusion) is a map r,



defined by

$$\text{retract} := \operatorname*{colim}_{I} \quad \text{where} \quad I = \{ * \circlearrowleft^{id}_{e} \}$$

such that $e \cdot e = e$.

Here, $* = B$ and $e = L \circ r$

since L is the inclusion map, we know $r \circ l = id_A$

so, we have $e \cdot e = (L \circ r) \circ (L \circ r) = L \circ (r \circ l) \circ r$

$= L \circ id_A \circ r = L \circ r = e$

EXERCISE Show that $A = \operatorname*{colim}_{I}$

where $I = \{ B \circlearrowleft^{id_B}_{L \circ r} \}$

Example $\mathcal{C} = sset$, $S \subset Mor(\mathcal{C})$

$S = \{ \partial\Delta^n \longrightarrow \Delta^n \}$

$sat(S) =$ cofibrations

To show this equality, we need to show :

(i) cofibrations are weakly saturated

(ii) any element of S can be obtained from cofibrations

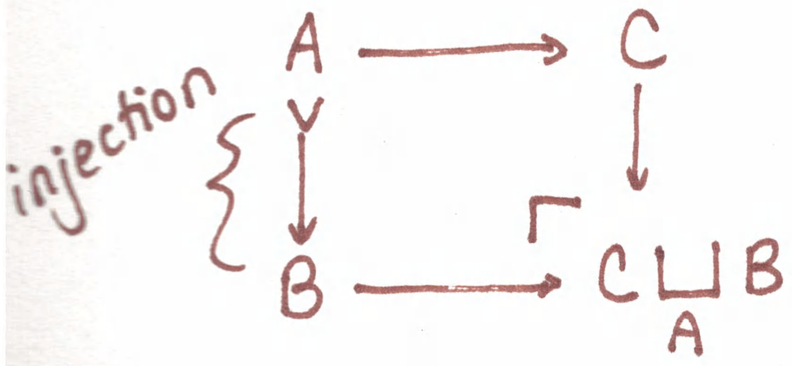
To show (i), note the following ... $\longrightarrow$

→ Colimits in sSet are computed degreewise
(and cofibrations are degreewise injections)
so, it suffices to show injections of simplicial sets are closed under ~~satisfy~~ properties (2), (3), and (4) in ~~def~~ the definition of "weakly saturated".

(Notes for property (1), closure under coproducts,
if $A_i \hookrightarrow B_i$ is an injection,
then $\bigsqcup_i A_i \hookrightarrow \bigsqcup_i B_i$ is also an injection,
since if two elements of $\bigsqcup_i A_i$ map to the same element in $\bigsqcup_i B_i$, they must be the same in $\bigsqcup_i A_i$ (they must come from the same component, A_i, and $A_i \hookrightarrow B_i$ is an injection.)

For property (2): We want to show closure under cobase changes...

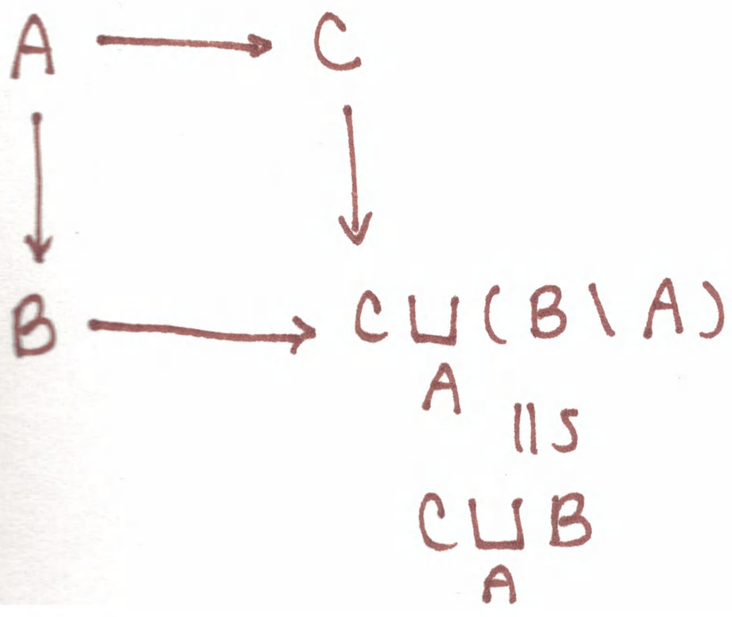
If we have a diagram



where the map $A \longrightarrow B$ is an injection, we can "remove" these elements that map from B into the coproduct $C \sqcup_A B$ by writing the coproduct

$$C \sqcup_A B = C \sqcup_A (B \setminus A)$$

so...



For property (3): We want to show closure under transfinite compositions

Given injections,

$$\begin{array}{ccccccccc} A_0 & \rightarrowtail & A_1 & \rightarrowtail & A_2 & \rightarrowtail & \cdots & \rightarrowtail & \operatorname{colim} A_i \\ \| & & \| & & \| & & & & \| \\ A_0' & \subset & A_1' & \subset & A_2 & \subset & \cdots & \subset & \bigcup_i A_i' \end{array}$$

there is an isomorphism to sets
that sends injections to inclusions,
and the transfinite composition of such inclusions
is an inclusion

$$A_0' \subset \bigcup_i A_i'$$

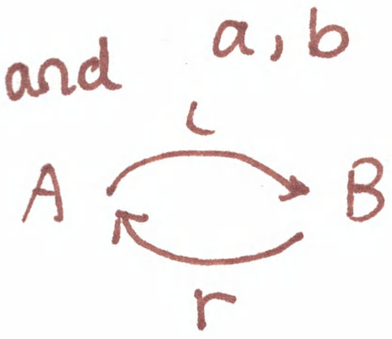
so the transfinite composition of injections
is an injection.

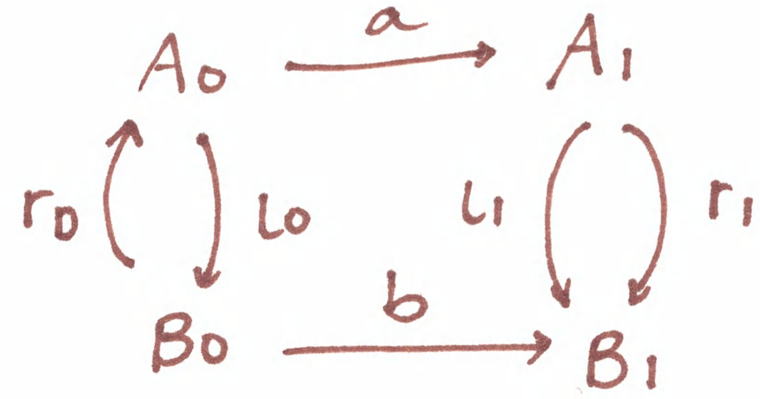
For property (4): We want to show closure under retracts

Given a retract

between two elements

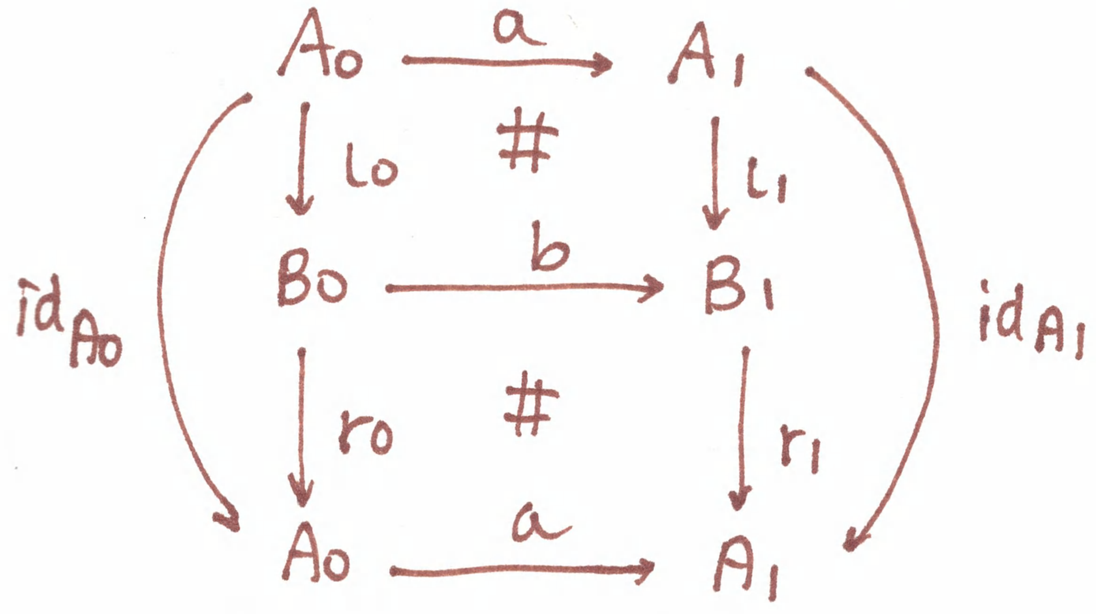
and





we know $r_0 l_0 = id_{A_0}$ and $r_1 l_1 = id_{A_1}$

and we have a diagram



we want to show

$a: A_0 \longrightarrow A_1$

is an injection.

suppose $p, q \in A_0$

$a(p) = a(q)$

then commuting w/ L_1
still maintains equality

$L_1 \circ a(p) = L_1 \circ a(q)$ →

→ and by commutativity of the diagram:

$$\begin{array}{ccc} L_1 a(p) & = & L_1 a(q) \\ \| & & \| \\ b\, l_0(p) & = & b\, l_0(q) \end{array}$$

since b is an injection, $b\, l_0(p) = b\, l_0(q)$

$$\Rightarrow b(l_0(p)) = b(l_0(q))$$

$$\Rightarrow l_0(p) = l_0(q)$$

Then, composing with r_0 yields identity on A_0,

$$r_0\, l_0(p) = r_0\, l_0(q) \Rightarrow p = q$$

hence, $a : A_0 \longrightarrow A_1$ is an injection.

Hence, cofibrations are weakly saturated (injections) [closed under (2), (3), & (4)]

$$\text{cofibrations} \subset \text{Sat}(S)$$

For (ii), given $X \overset{f}{\rightarrowtail} Y$,

let $Y_{f,n} \subset Y$ be the subset generated by f and all n-simplices of Y.

Then we have a skeletal filtration:

$$\begin{array}{ccccccccc} Y_{f,-1} & \hookrightarrow & Y_{f,0} & \hookrightarrow & Y_{f,1} & \hookrightarrow & \cdots & \hookrightarrow & Y \\ \| & & \| & & \| & & & & \\ f(X) & & f(X) \sqcup \Delta^0_Y & & f(X) \sqcup \Delta^1_Y & & & & \end{array}$$

and the map $Y_{f,-1} \longrightarrow Y$ is a transfinite composition (as $Y = \operatorname{colim}_i Y_{f,i}$).

It is enough to show that $Y_{f,n-1} \hookrightarrow Y_{f,n}$ is in $\text{Sat}(S)$, $\forall n$, for then the composition will be a cofibration.

→

$$\begin{array}{ccc} \sqcup \partial\Delta^n & \longrightarrow & Y_{f,n-1} \\ \downarrow & & \downarrow \\ \underset{Y'}{\sqcup}\Delta^n & \longrightarrow & Y_{f,n} \end{array}$$

from the map $Y_{f,n-1} \hookrightarrow Y_{f,n}$, we have the pushout diagram

where $Y' = Y_n^{\text{nondegenerate}} \setminus f(X_n)$

To show the map $Y_{fn-1} \hookrightarrow Y_{f,n} \in sat(S)$, it is enough to show the map

$$\sqcup \partial\Delta^n \hookrightarrow \underset{Y'}{\sqcup}\Delta^n \quad \text{is in } sat(S)$$

(this was shown in a previous exercise)

so, $sat(S) \subset$ cofibrations, as desired. ▨

MORAL OF THE EXAMPLE:

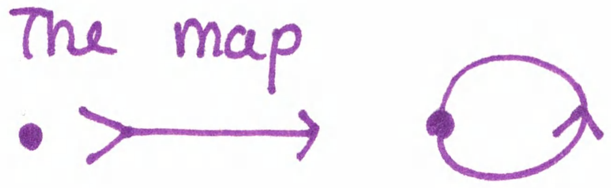
When $S = \{\partial\Delta^n \longrightarrow \Delta^n\}$
(boundary inclusions)

$Sat(S) =$ cofibrations

ALSO

When $S = \{\Lambda^n_k \longrightarrow \Delta^n\}$ $n \geq 1$
(horn inclusions)

$Sat(S) =$ acyclic cofibrations

Example The map 

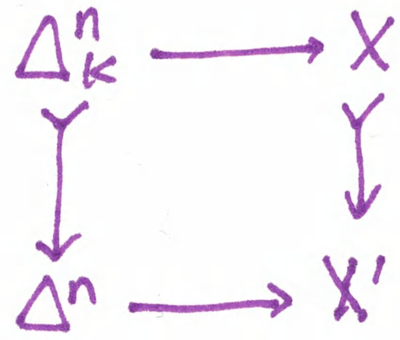
is a cofibration, but NOT an acyclic cofibration!

Consider the operations for weakly saturated:

- coproduct closure
- cobase change closure
- transfinite composition closure

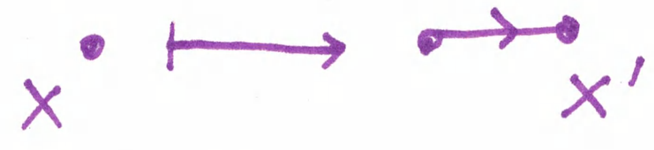
$AC = C \cap W$

so, for this map to be an acyclic cofibration, it must also be a weak equivalence.

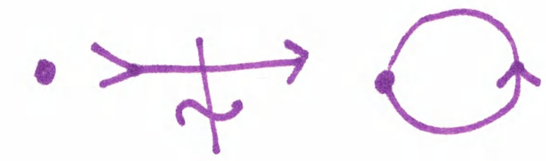
For such cobase changes: 

$X \setminus X' = \Delta^n \setminus \Lambda^n_k$

is the top dimensional face, and at each cobase change, we add a nondegenerate $n-1$ simplex and a nondegenerate n-simplex

For $n=1$, 

there is no injective map 

so 

this cofibration is not an acyclic cofibration.

NOTE: acyclic cofibrations are the solution to a very general lifting problem:

Q: Along which class of maps to a Kan complex can we find an extension map?

$$A \xrightarrow{\sim} B \in \text{Kan}$$

A: Along acyclic cofibrations!

Given a fibration

$$\begin{array}{c} X \\ \downarrow \\ Y \end{array}$$

and a 1-simplex in Y, which by Yoneda yoga is a map $\sigma : \Delta^1 \longrightarrow Y$

$$\begin{array}{ccc} X & & \\ \downarrow & & \\ Y & \xleftarrow[\sigma]{} & \Delta^1 \end{array}$$

consider the fiber over σ in X:

$$\begin{array}{ccc} X & \longleftarrow & X_\sigma \\ \downarrow & & \downarrow \\ Y & \xleftarrow[\sigma]{} & \Delta^1 \end{array}$$

the square forms a pullback square

$$\begin{array}{ccc} X & \longleftarrow & X_\sigma \\ \downarrow & & \downarrow \\ Y & \longleftarrow & \Delta^1 \end{array}$$

We have attaching maps and a pair of "inverse maps" (inverse up to homotopy).

$$\begin{array}{ccccc} & & X_0 & & \\ & & \sim\downarrow & \nearrow\searrow & \\ X & \longleftarrow & X_\sigma & \longleftarrow & X_1 \\ \downarrow & & \lrcorner\downarrow & & \sim\downarrow \\ Y & \longleftarrow & \Delta^1 & & \end{array}$$

If $X \xrightarrow{f} Y$,
then all fibers ~~over~~ of f over ~~a~~ the same connected component of Y are simplicially homotopy equivalent Kan complexes.

That is,

Kan fibrations are "homotopy bundles" of Kan complexes.

Exercise

$$A \underset{\exists r}{\overset{\sim f}{\rightleftarrows}} B$$

Given a Kan complex A and an acyclic cofibration $f : A \to B$, show $\exists r : B \to A$ (inverse map) such that

$$r \circ f = id_A$$

Proposition $X \xrightarrow{f} Y$

a fibration f is an ~~acyclic~~ acyclic fibration

iff

any fiber of the fibration is contractible

($\forall y \in Y_0$, X_y is contractible)

NOTE: A fiber is a kan complex, so "contractible" means simplicial homotopy equivalent to a point.

Acyclic cofibrations are "homotopy bundles" of contractible Kan complexes.

Geometrically, we can translate this as follows:

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow \\ B & \longrightarrow & Y \end{array}$$

we have a cofibration

$$A \rightarrowtail B$$

and a fibration

$$X \twoheadrightarrow Y,$$

one of which will be a weak equivalence (hence, acyclic)

Either

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow \sim & & \downarrow \\ B & \longrightarrow & Y \end{array}$$

(acyclic cofibration; fibration)

CASE 1

or

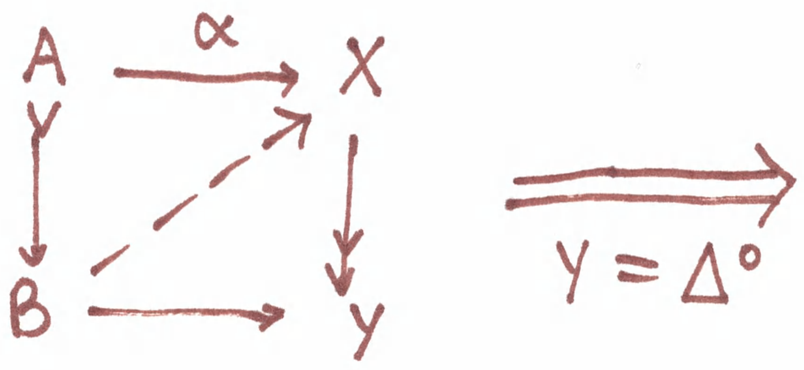
$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow \sim \\ B & \longrightarrow & Y \end{array}$$

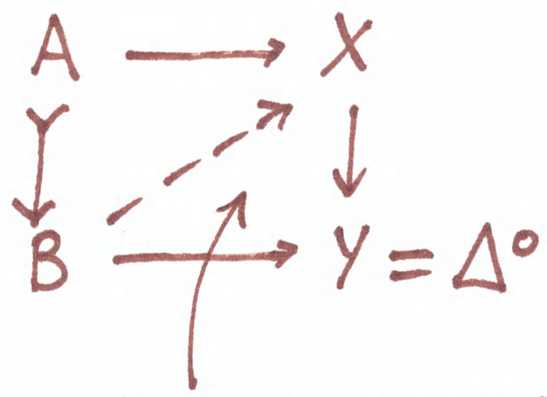
(cofibration; acyclic fibration)

CASE 2

$\longrightarrow$

Consider when $Y = \Delta^0$ (a point)
and the general case (for $Y = \Delta^n$) amounts to performing this lifting in Y-indexed families

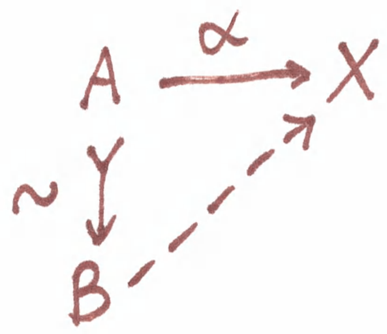




this lower triangle will always commute!

(X is a Kan complex)

Case 1





Then in X,
we can "grow" contractible stuff on the image of A
(the Kan property always allows horn fillings)



Then, acyclic cofibrations lift along fibrations!

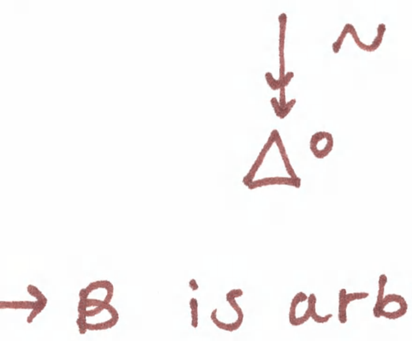
Acyclic cofibrations can be thought of as "growing" contractible stuff on A (you cannot produce new topological features, only "thicken" the existing features)

Case 2:

arbitrary inclusion



and



The inclusion $A \rightarrowtail B$ is arbitrary, so we can change the topology (fill in holes, etc.), however X is contractible to a point, so $\alpha(A)$ can be "filled in"



cannot have a hole inside α(A) (X contractible)

$\Rightarrow$

so,



α(A) can be filled in!

So... for a model category:

MODEL CATEGORY STRUCTURE

① WEAK EQUIVALENCES

② COFIBRATIONS — weak factorization system #1 — ④ FIBRATIONS

weak factorization system #2

③ ACYCLIC COFIBRATIONS — ⑤ ACYCLIC FIBRATIONS

Any map can be written as a composition of
a cofibration followed by an
acyclic fibration

OR

an acyclic fibration followed by
a fibration

sSet MODEL CATEGORY STRUCTURE

(sSet, swe) has a model structure:

sSet: simplicial sets
swe: simplicial weak equivalences

① WEAK EQUIVALENCES := simplicial weak equivalences

② COFIBRATIONS := inclusions (monomorphisms)
[weak saturation of boundary inclusions, $\mathrm{sat}(\partial\Delta^n \longrightarrow \Delta^n)$]

③ ACYCLIC COFIBRATIONS := weak saturation of horn inclusions
[$\mathrm{sat}(\Lambda^n_k \longrightarrow \Delta^n)$]

④ FIBRATIONS := the set ~~of maps~~ $(\Lambda^n_k \longrightarrow \Delta^n)\pitchfork$
(right-lifting wrt horn inclusions)

⑤ ACYCLIC FIBRATIONS := the set $(\partial\Delta^n \longrightarrow \Delta^n)\pitchfork$ (wrt boundary inclusions)

NOTE: We can only define a model category structure
on
(i) RELATIVE CATEGORIES
that satisfy the
(ii) 2/3 PROPERTY!

(i) $(sSet, swe)$ is a RELATIVE CATEGORY!

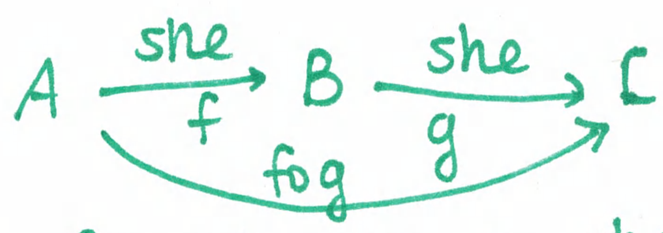
NOTATION

swe := simplicial weak equivalence

she := simplicial homotopy equivalence

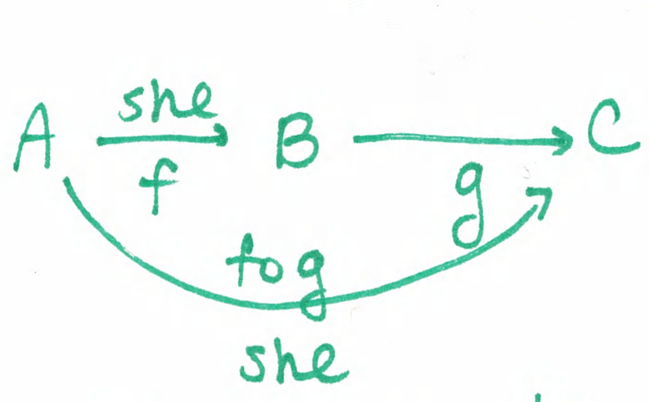
(ii) Recall, f swe $\Leftrightarrow$ $Ex^{\infty} f$ is a she,
so it suffices to show
the 2/3 property for a she.

Case #1

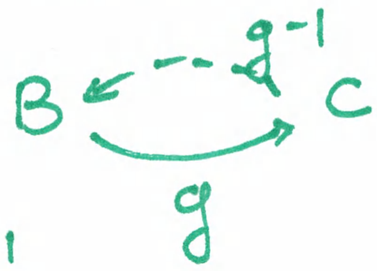


the map $A \to C$ is a she since she's are closed under composition!

Case #2



the map $B \to C$ is a she since we can construct an inverse map



by $\bar{g}^{1} = f \circ (g \circ f)^{-1}$

Then, $g \circ \bar{g}^{1} = g \circ (f \circ (g \circ f)^{-1}) = (g \circ f) \circ (g \circ f)^{-1} \cong id_C$

and $\bar{g}^{1} \circ g = f \circ (g \circ f)^{-1} \circ g \cong f \circ (g \circ f)^{-1} \circ g \circ (f \circ f^{-1})$

$= f \circ (id_A) \circ f^{-1} \cong id_B$

Now, we need to show
② ⑤ and ③ ④
are weak factorization systems

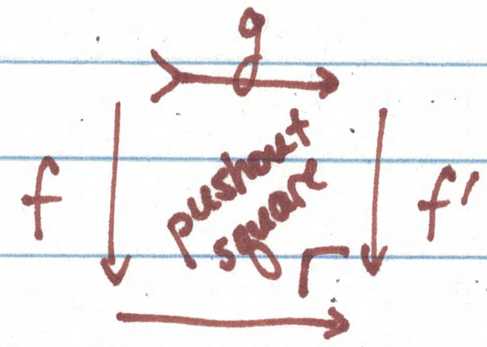
} (skipped for now; will finish time permitting)

Homological Algebra [10/17]

Prop (from last time)

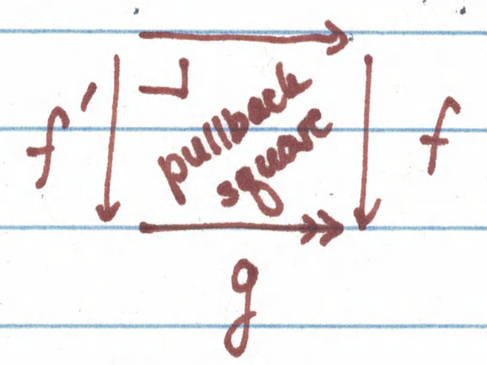
s.h.e. are closed under base changes along cofibrations

(a)



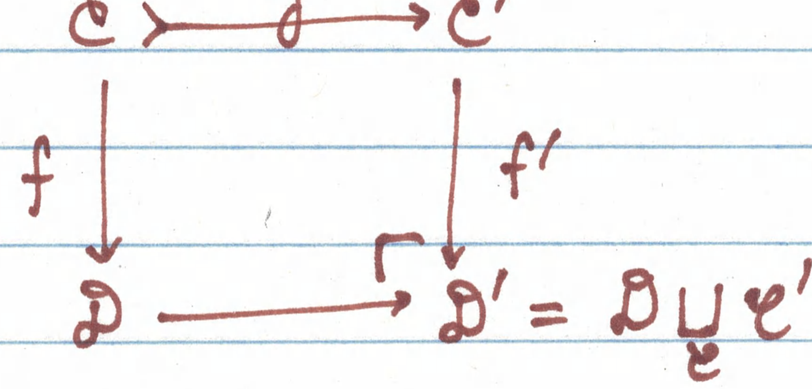
f s.h.e.
g cofibration
⇓
f' s.h.e.

(b)



f s.h.e.
g fibration
⇓
f' s.h.e.

Proof (a)



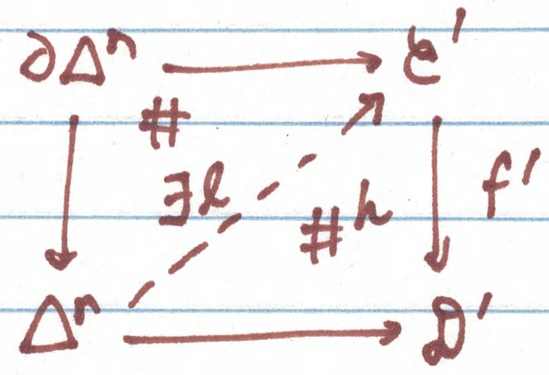
f s.h.e
g cofibration

Recall simplicial Whitehead Thm

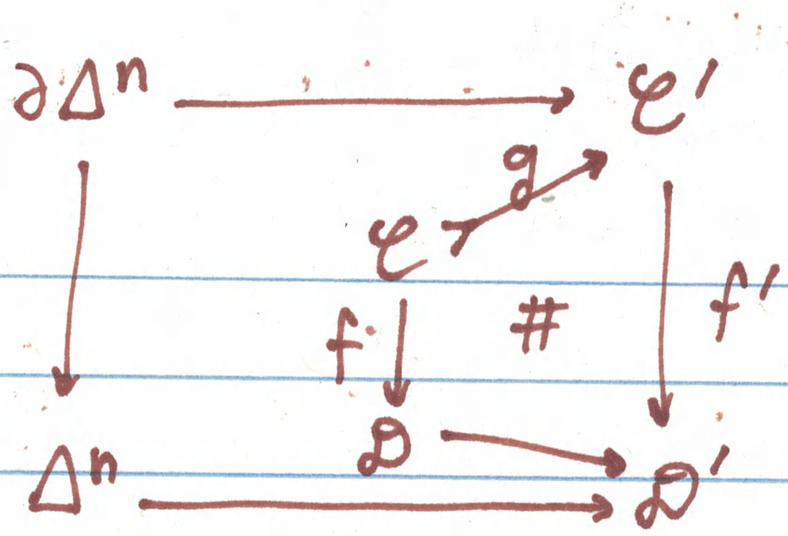
f' s.h.e.

iff.

$\forall$ comm. square, $\exists l : \Delta^n \to \mathcal{C}'$ such that $\# / \#h$ commutes



→



cobase change from $\mathscr{C}$ to $\mathscr{C}'$
will take $\mathscr{D}$ and "add" to it
the part of $\mathscr{C}'$ that is not in $\mathscr{C}$

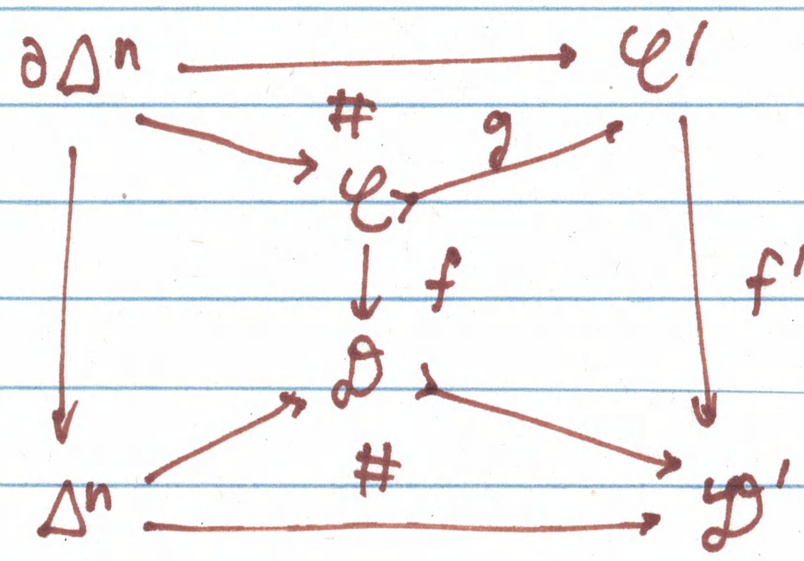
$$\mathscr{D} \underset{\mathscr{C}}{\sqcup} \mathscr{C}'$$

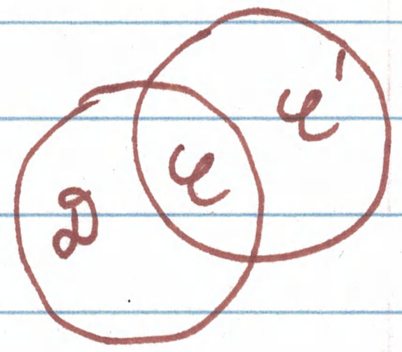
The map $\mathscr{D} \longrightarrow \mathscr{D}'$ is a cofibration
because it is cobase change of map $\mathscr{C} \xrightarrow{g} \mathscr{C}'$

CASE 1 [redacted]

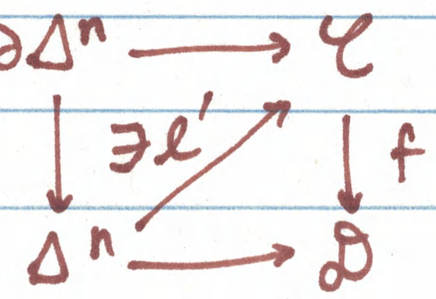
If there is a map $\Delta^n \longrightarrow \mathscr{D}$,
then the map $\Delta^n \longrightarrow \mathscr{D}'$ factors through $\mathscr{D}$, (uniquely factors!)
hence, does not touch the part of $\mathscr{C}'$ not in $\mathscr{C}$

similarly, the map $\partial\Delta^n \longrightarrow \mathscr{C}'$
factors through $\mathscr{C}$



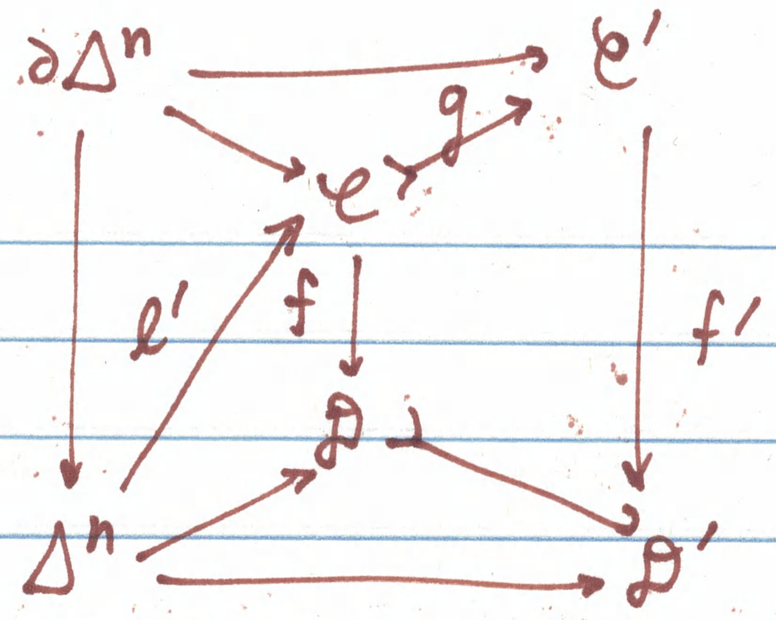


Now,



simplicial WHT
f s.h.e. $\Rightarrow \exists\, \ell' : \Delta^n \to \mathscr{C}$

$\longrightarrow$



Then define $l : \Delta^n \longrightarrow \mathcal{E}$

by $g \circ l' = l$

and this is the lift map we want.

To show upper triangle # and lower triangle $\#^h$,

we use the map

$$\Delta^n \longrightarrow \mathcal{D} \longrightarrow \mathcal{D}'$$

for the homotopy h.

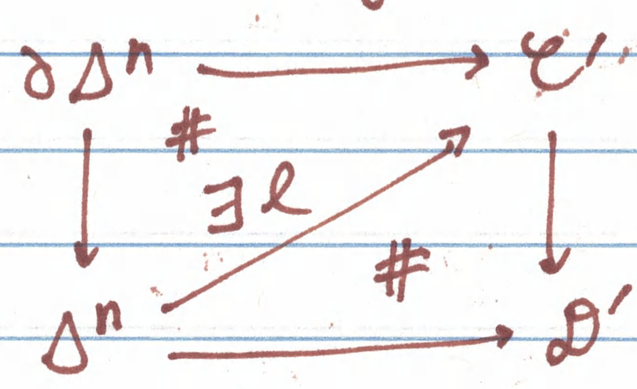
Q: Why is it necessary for g to be a cofibration?

CASE 2 If the map $\Delta^n \longrightarrow \mathcal{D}'$ does NOT factor through $\mathcal{D}$,

then it must land in the part of $\mathcal{E}'$ not in $\mathcal{E}$, hence, the map

$$\Delta^n \longrightarrow \mathcal{D}'$$

factors through $\mathcal{E}'$, and we have a map



which is exactly the map we wanted.

(and in this case, the lower triangle commutes strictly! Not even up to homotopy) ▨

NOTE: $(\mathcal{D} \underset{\mathcal{C}}{\sqcup} \mathcal{C}')_n = \mathcal{D}_n \underset{\mathcal{C}_n}{\sqcup} \mathcal{C}'_n$ only when the maps

$$\cong \mathcal{D}_n \sqcup (\mathcal{C}'_n \setminus \mathcal{C}_n)$$

Recall, we compute $\mathcal{D} \sqcup \mathcal{C}'$ in $\mathcal{C}$ every degree as...

$\mathcal{C} \to \mathcal{C}'$ and $\mathcal{D} \to \mathcal{D}'$ were injective (cofibrations are injective!)

Return to Model Categories!

Applications We want to have an "easy way" to derive functors!

Remarks Definitions of a Model Category

original Quillen (1969)

$\mathcal{C}$ is finitely complete and finitely co-complete and factorizations do not have to be functorial

USED MOST OFTEN! ★ Hovey-Hirshhorn (2001-2002)

$\mathcal{C}$ is complete & cocomplete and factorizations are functorial.

We will use this definition!

Quillen Functors

Def) $\mathcal{C}, \mathcal{D}$ model categories

~~Assume~~

$F: \mathcal{C} \longrightarrow \mathcal{D}$ is a functor

F is a "left quillen functor" if $\longrightarrow$

→ categorical requirement: F is a left adjoint functor (in particular, F preserves colimits)

model structure requirement:

F preserves cofibrations and acyclic cofibrations

F is a "right quillen functor"

if • preserves limits

• preserves fibrations and acyclic fibrations

i.e. F(fibration) = fibration

F(acyclic fibration) = acyclic fibration

THEOREM A left quillen functor can be (easily!) left derived as follows:

dual! "right"

$F: \mathcal{C} \longrightarrow \mathcal{D}$

A left quillen functor is left derivable

(i) identify "nice" subcategory of $\mathcal{C}$

$\mathcal{C}' \subset \mathcal{C}$

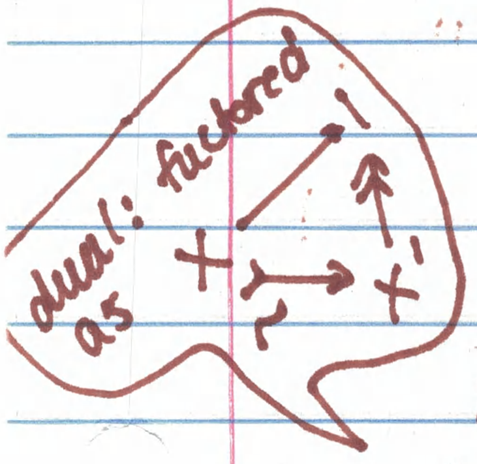
full subcategory

dual! "fibrant objects"

⇒ let $\mathcal{C}'$ be the full subcategory of cofibrant objects

$0 \rightarrowtail X$ (objects X such that map $0 \to X$ is cofibration)

dual: factored as



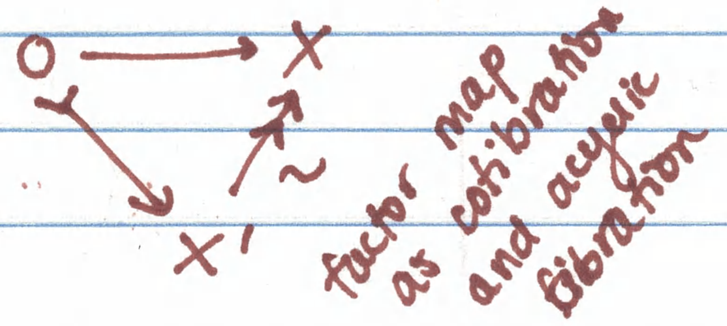
(ii) define resolution functor

take object in $\mathcal{C}$ and replace w/ weakly equivalent object in $\mathcal{C}'$

$X \in \mathcal{C}$

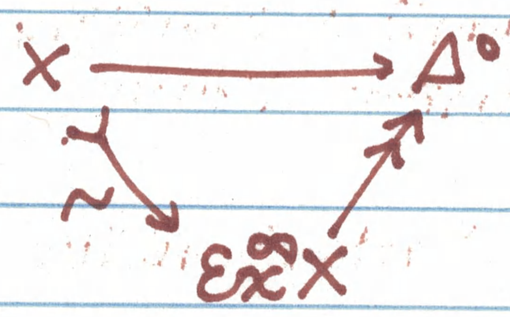
$0 \longrightarrow X$

(not necessarily cofibration)



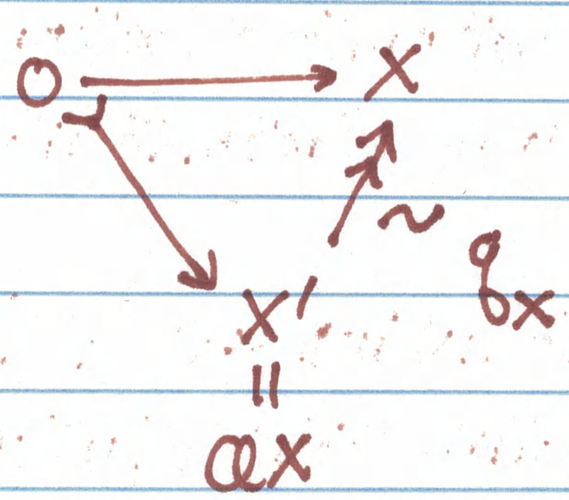
factor map as cofibration and acyclic fibration

⇒ NOTE: Ex^∞ is an example of a fibrant replacement functor for (sSet, swe)



⇒ Resolution functor: Cofibrant Replacement

$X \in \mathcal{C}$



where $q: Q \xrightarrow{\sim} id_{\mathcal{C}}$

Then cofibrant

Left derived functor

$LF = F \circ Q$

~~Right~~ Left

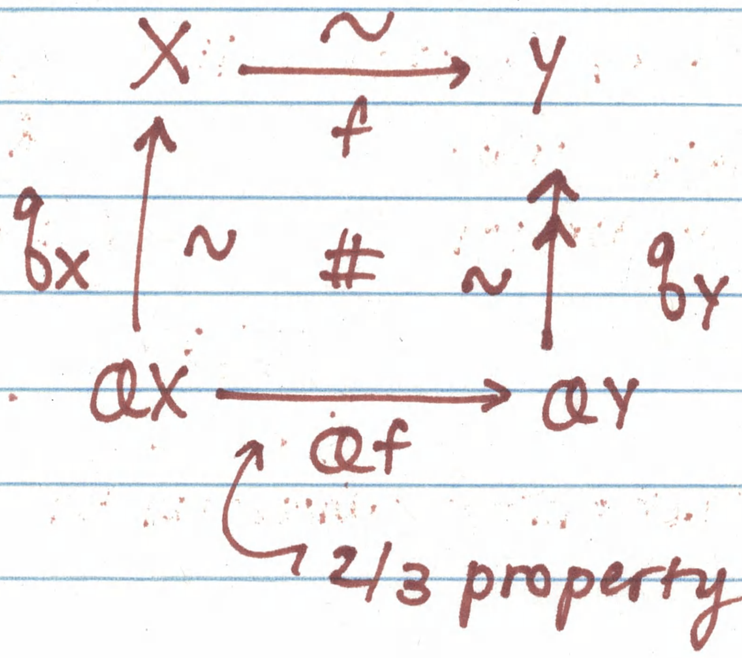
~~RFF~~

~~(ii) Restriction of functor LF to $\mathcal{C}'$~~
~~coincides with~~

~~(ii) Prop Statement of theorem~~

(iii') Show resolution functor
(a) preserves w.e.'s and
(b) restriction to nice subcategory $\mathcal{C}'$ ~~gives same as~~ preserves w.e.'s.

(a) Cofibrant replacement preserves w.e.'s:



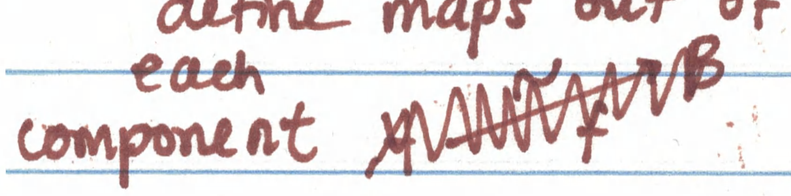
F/e' preserves w.e.'s
(Ken Brown's Lemma)

A, B cofibrant	Need to show
$A \xrightarrow[f]{\sim} B$	$FA \xrightarrow{\sim} FB$

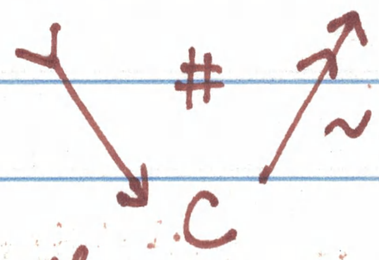
① consider the map

$$A \amalg B \longrightarrow B$$

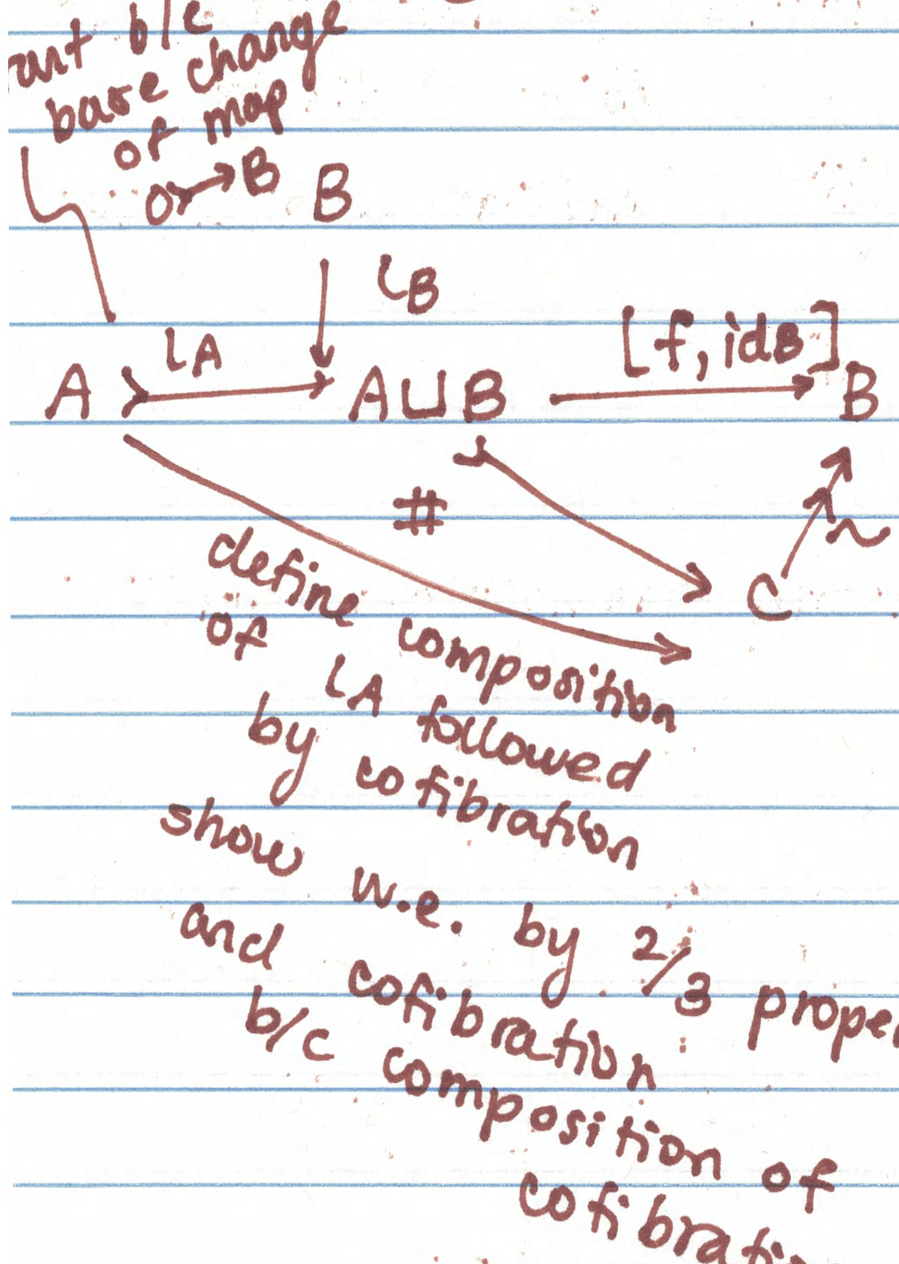
out of a coproduct
define maps out of each component

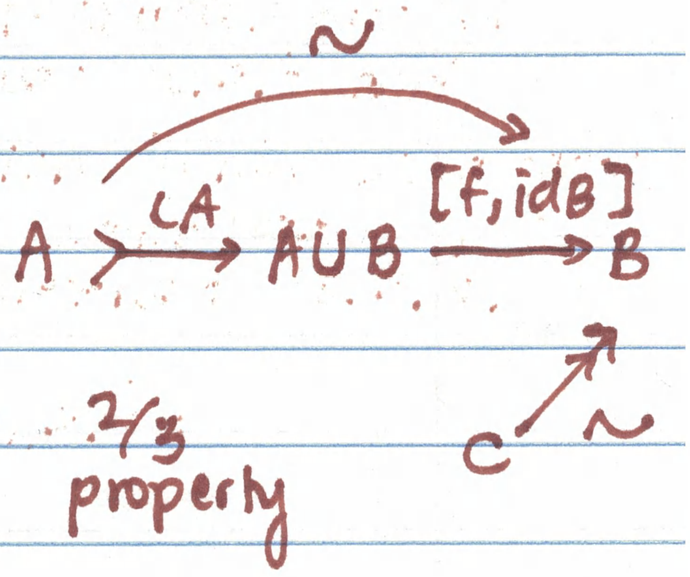


$$A \amalg B \xrightarrow{[f, id_B]} B$$



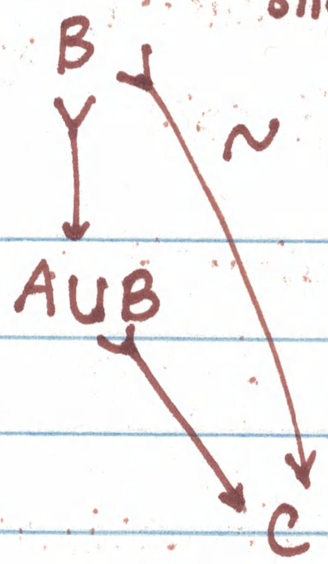
② factor as cofibration followed by fibration augelir





→

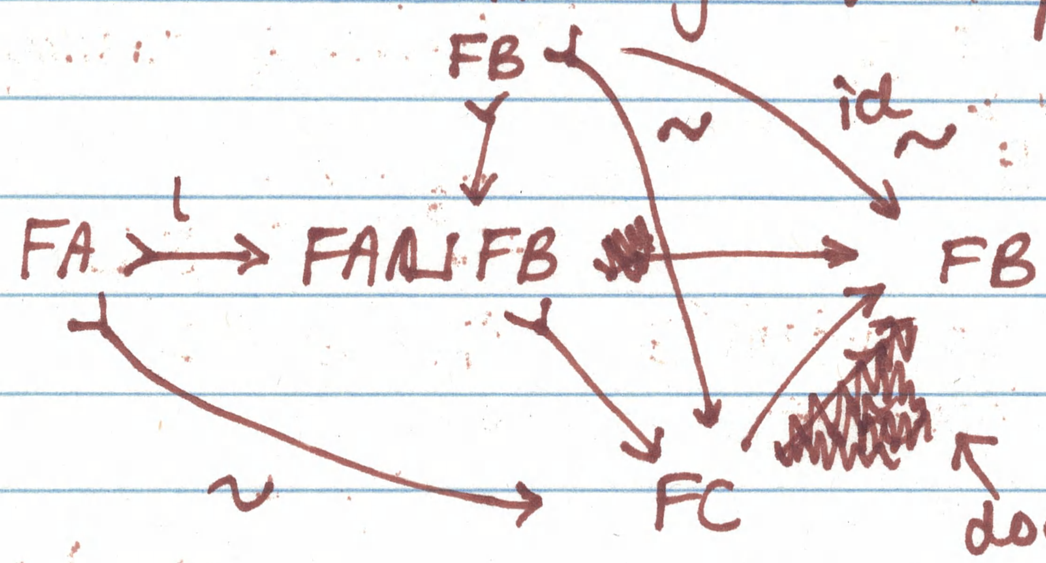
similarly, show $B \xrightarrow{\sim} C$



Then $A \xrightarrow{\sim} C$ and $B \xrightarrow{\sim} C$

apply the left Quillen Functor to the diagram

Recall, F is left adjoint, so preserves coproducts and cofibrations/ acyclic cofibrations



F does NOT preserve acyclic fibrations!

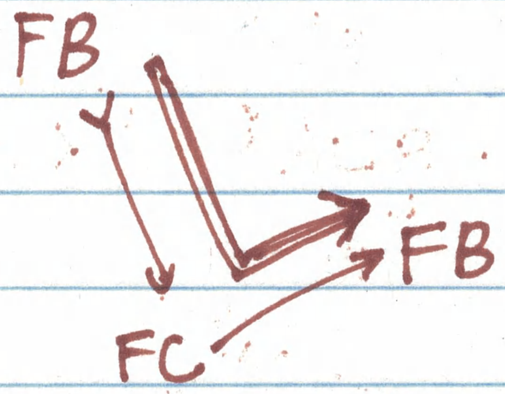
But, it is a w.e. by 2/3 property

$FB \xrightarrow[\sim]{id} FB$

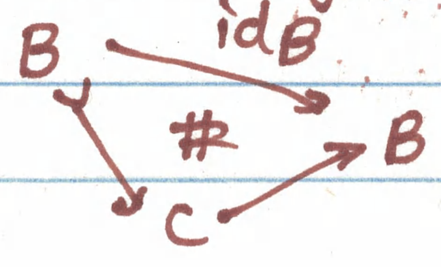
$FB \xrightarrow[\sim]{} FC$

so

$FC \xrightarrow[\sim]{} FB$



composition is identity map because it is the image of the identity map



Hence, 2/3 property again

$FA \rightarrowtail FA \sqcup FB \to FB$, with $FA \rightarrowtail FA \sqcup FB \xrightarrow{\sim} FB$

□

Homological Algebra [10/22]

Remark: $\mathcal{C} \xrightarrow{L} \mathcal{D}$

$L \dashv R$ L left Quillen functor $\Rightarrow$ R right Quillen functor

The right adjoint functor to a left Quillen functor is a right quillen functor

Proof: $f: X \twoheadrightarrow Y$ in $\mathcal{D}$

NTS: $Rf: RX \longrightarrow RY$ is a fibration in $\mathcal{C}$

that is, $\forall g: A \xrightarrow{\sim} B$ in $\mathcal{C}$ (acyclic cofibration)

we have a lifting property.

$$\begin{array}{ccc} A & \longrightarrow & RX \\ {\scriptstyle g}\downarrow{\scriptstyle \sim} & \overset{\text{we need this map}}{\nearrow} & \downarrow{\scriptstyle Rf} \\ B & \longrightarrow & RY \end{array}$$

To find this lift map, we can...

Take left adjoint diagram

$$\begin{array}{ccc} LA & \longrightarrow & X \\ {\scriptstyle Lg}\downarrow{\scriptstyle \sim} & \overset{\ell}{\nearrow} & \downarrow{\scriptstyle f} \\ LB & \longrightarrow & Y \end{array}$$

in $\mathcal{D}$.

$Lg \in AC$ because L is Left Quillen functor

f is a fibration by definition

so we have a lift

$\exists \ell: LB \longrightarrow X$

Passing back to the previous diagram, we have the lift map

$\exists \ell': B \longrightarrow RX$

(the adjoint of ℓ) ▨

"Quillen adjunction"

"Quillen pair" the existance of the left Quillen functor guarentees the existence of the right Quillen functor

These terms are used to refer to ~~describe~~ the previous remark

Examples (left Quillen functors and their adjoints)

$- \times X : sSet \longrightarrow sSet$

where $X \in sSet$

product functor that takes product with a fixed simplicial set

① Claim: $- \times X$ functor preserves cofibrations

$Y \rightarrowtail Z$

$X \times Y \rightarrowtail X \times Z$

product doesn't change injectivity; product preserves C (cofibrations)

② Claim: $Y \xrightarrow{\sim} Z$

$X \times Y \xrightarrow{\sim} X \times Z$

product preserves AC (acyclic cofibrations)

③ Claim: The product functor $- \times X$ is left adjoint to the hom functor $Hom(X, -)$ by the hom-product adjunction

$Hom(X, -)$ is a right Quillen functor

We must verify ①, ②, ③ to prove that $- \times X$ is a Quillen functor (left Quillen)

→

So, we have a "Quillen pair"

$- \times X$ and $\mathrm{Hom}(X, -)$

left Quillen right quillen

Now, compute the derived Quillen functors
(by procedure in last lecture)

Derived adjunction

$W \overset{L}{\times} X := W \times X$

$\mathrm{RHom}(X,Y) := \mathrm{Hom}(X, \mathrm{Ex}^\infty Y)$

Previously, we defined $\mathrm{RHom}(X,Y) := \mathrm{Hom}(\mathrm{Ex}^\infty X, \mathrm{Ex}^\infty Y)$

There is a map $\mathrm{Lx}: X \hookrightarrow \mathrm{Ex}^\infty X$ (the canonical inclusion)

So, we have a map (a restriction)

$\mathrm{Hom}(\mathrm{Ex}^\infty X, \mathrm{Ex}^\infty Y) \longrightarrow \mathrm{Hom}(X, \mathrm{Ex}^\infty Y)$

$\mathrm{Hom}(\mathrm{Lx}, \mathrm{Ex}^\infty Y)$
restriction map

More Generally,

$\mathrm{sSet} \longleftarrow \mathrm{sset}^{op}$

$\mathrm{Hom}(-, \mathrm{Ex}^\infty Y)$

Recall, $\mathrm{Hom}(\mathrm{colim} X_i, Y) = \lim \mathrm{Hom}(X_i, Y)$

when we consider hom (a set)
$\mathrm{hom}(\mathrm{colim} X_i, Y) = \lim \mathrm{hom}(X_i, Y)$
is the Universal property of colimits!

What is the left adjoint functor to $\mathrm{Hom}(-, \mathrm{Ex}^\infty Y)$

For $X \in \mathrm{sset}$

$LX \longrightarrow Z$ should correspond to $X \longrightarrow \mathrm{Hom}(Z, \mathrm{Ex}^\infty Y)$

$\uparrow$ sset^{op} $\uparrow$ sset^{op}

$\longrightarrow$

The map $\begin{array}{ccc} LX & \longrightarrow & Z \\ \Uparrow & & \Uparrow \\ sSet^{op} & & sSet^{op} \end{array}$

is the same as

$\begin{array}{ccc} LX & \longleftarrow & Z \\ \Uparrow & & \Uparrow \\ sSet & & sSet \end{array}$

so, we need to define L such that

$\begin{array}{ccc} LX & \longleftarrow & Z \\ \Uparrow & & \Uparrow \\ sSet & & sSet \end{array}$ correspond to

$X \longrightarrow Hom(Z, \mathcal{E}\overset{\infty}{x} Y)$

$\Downarrow$ by the Hom product adjunction,
this is equivalent to

$X \times Z \longrightarrow \mathcal{E}\overset{\infty}{x} Y$

$\Downarrow$ by the Hom product adjunction AGAIN!

$Z \longrightarrow Hom(X, \mathcal{E}\overset{\infty}{x} Y)$

so, we want to show that maps

$LX \longleftarrow Z$ correspond to $\underline{Z} \longrightarrow Hom(X, \mathcal{E}\overset{\infty}{x} Y)$

$\Vert$

$\underline{Z} \longrightarrow LX$

so, we can see what the definition of LX needs to be:

$$LX = Hom(X, \mathcal{E}\overset{\infty}{x} Y)$$

needed because target is $sSet^{op}$

so define the left adjoint functor $L(-) = Hom(-, \mathcal{E}\overset{\infty}{x} Y)^{op} \longrightarrow$

so, we have a pair of adjoint categories

$$sSet \underset{Hom(-, Ex^\infty Y)}{\overset{L(-) = Hom(-, Ex^\infty Y)^{op}}{\rightleftarrows}} sSet^{op}$$

model category ↑ ↑ model category

[Remark: if $(\mathcal{D}, W, C, F)$ is a model category, then so is $(\mathcal{D}^{op}, W^{op}, F^{op}, C^{op})$

weak equivalence, cofibrations, fibrations

[all arrows are reversed, and cofibrations and fibrations are reversed]]

~~A pair of Adjoint model categories~~

Lemma A pair of adjoint cartesian closed model categories is a Quillen adjunction

Lemma

$$sSet \underset{Hom(-, Ex^\infty Y)}{\overset{Hom(-, Ex^\infty Y)^{op}}{\rightleftarrows}} sSet^{op}$$

[Left Quillen] [Right Quillen]

is a Quillen adjunction

$Hom(-, Ex^\infty Y)$ preserves all weak equivalences!

Right Quillen functors preserve w.e. of fibrant objects.

Everything in $sSet^{op}$ is fibrant, because everything in sset is cofibrant (ssets are always cofibrant)

→

The map $l_X : X \xrightarrow{\sim} \mathrm{Ex}^\infty X$

is a w.e. between fibrant objects,

so, $\mathrm{Hom}(l_X, \mathrm{Ex}^\infty Y) : \mathrm{Hom}(\mathrm{Ex}^\infty X, \mathrm{Ex}^\infty Y)$

$$\downarrow \sim$$

$$\mathrm{Hom}(X, \mathrm{Ex}^\infty Y)$$

is a weak equivalence.

Definition

An adjunction in two variables is a functor

$$\otimes : \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{E}$$

It admits adjoint functors

$\mathrm{Hom}_l : \mathcal{C}^{op} \times \mathcal{E} \longrightarrow \mathcal{D}$

$\mathrm{Hom}_r : \mathcal{D}^{op} \times \mathcal{E} \longrightarrow \mathcal{C}$

such that $\exists$ natural bijections

$c \in \mathcal{C}$, $d \in \mathcal{D}$, $e \in \mathcal{E}$

$$c \otimes d \longrightarrow e$$

$$d \longrightarrow \mathrm{Hom}_l(c, e)$$

$$c \longrightarrow \mathrm{Hom}_r(d, e)$$

Example 1 If $\mathcal{C}, \mathcal{D}, \mathcal{E}$ are cartesian closed,
and $\otimes := \times$
(define adjunction functor to be ordinary product)
then

$\mathrm{Hom}_l = \mathrm{Hom}$

$\mathrm{Hom}_r = \mathrm{Hom}$

Example 2 If $\mathcal{C}, \mathcal{D}, \mathcal{E}$ are closed monoidal categories
and $\otimes :=$ monoidal product
then
Hom_l, Hom_r are adjoint to $\otimes$. (for example...)

→ $Vect_K$ Mod_R

$\otimes := \otimes_R$ Then

adjunction functor is defined as tensor product over Ring R

$Hom_\ell = Hom_R(-,-)$

$Hom_r = Hom_R(-,-)$

Example 3

regular set

$\mathcal{C} \in set$

$\mathcal{D} \in sSet$

$\mathcal{E} \in sSet$

categories don't have to be of the same type

simplicial sets

Then $\mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{E}$

$set \times sSet \longrightarrow sSet$

$Hom_\ell : Set^{op} \times sSet \longrightarrow sSet$

$S \times X \longmapsto \prod_S X$

$Hom_r : sSet^{op} \times sSet \longrightarrow set$

$\|$

hom

↳ the ordinary hom-set!

(takes a pair $X \in sSet^{op}$, $Y \in sSet$) $\longrightarrow hom(X,Y)$

$\Uparrow$

set

Definition A left Quillen bifunctor is

(i) $\mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{E}$

A 2-variable adjunction

(ii) that preserves properties:

Take P,

P is the pushout of the two maps...

$c \xrightarrow{f} c'$

$d \xrightarrow{g} d'$

construct diagram ⇒

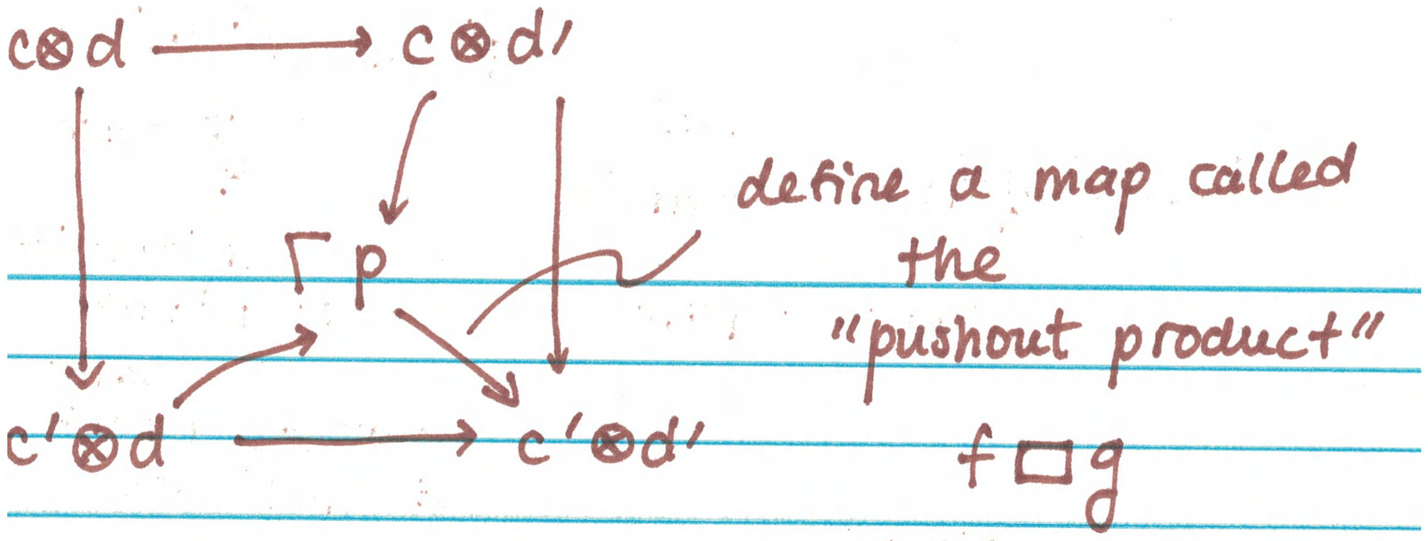
$c \otimes d \longrightarrow c \otimes d'$

$\downarrow \qquad \searrow \quad \downarrow$

$\qquad P \quad \downarrow$

$c' \otimes d \longrightarrow c' \otimes d'$

→



define a map called the "pushout product"

$f \square g$

If $f \in C_{\mathcal{C}}$ (cofibration in $\mathcal{C}$)

$g \in C_{\mathcal{D}}$ (cofibration in $\mathcal{D}$)

Then $f \square g \in C_{\mathcal{E}}$ (cofibration in $\mathcal{E}$)

And

If $f \in AC_{\mathcal{C}}$

$g \in C_{\mathcal{D}}$

Then $f \square g \in AC_{\mathcal{E}}$

If $f \in C_{\mathcal{C}}$

$g \in AC_{\mathcal{D}}$

Then $f \square g \in AC_{\mathcal{E}}$ ▨

Homological Algebra 10/24

Last time!...

Adjunction in two variables

we want to derive product ($\times$), internal Hom (Hom)
tensor product over ring ($\otimes_R$), etc...

→ all are examples of adjunction in two variables!

$c \otimes d \longrightarrow e$ $\qquad$ $\otimes : \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{E}$

$d \longrightarrow \mathrm{Hom}_l(c, e)$ $\qquad$ $\mathrm{Hom}_l : \mathcal{C}^{op} \times \mathcal{E} \longrightarrow \mathcal{D}$

$c \longrightarrow \mathrm{Hom}_r(d, e)$ $\qquad$ $\mathrm{Hom}_r : \mathcal{D}^{op} \times \mathcal{E} \longrightarrow \mathcal{C}$

Example Chain complex of right R-modules $\quad {}_R\mathrm{Ch} \times \mathrm{Ch}_R$ $\quad$ Chain complex of left R-modules

$\mathrm{Ch}_R \times {}_R\mathrm{Ch} \xrightarrow{\otimes} \mathrm{Ch}_{\mathbb{Z}}$ ← abelian group

R-noncommutative rings

left hom — $\mathrm{Hom}_l : \mathrm{Ch}_R^{op} \times \mathrm{Ch} \longrightarrow {}_R\mathrm{Ch}$

right hom — $\mathrm{Hom}_r : {}_R\mathrm{Ch}^{op} \times \mathrm{Ch} \longrightarrow \mathrm{Ch}_R$

that is, take M_R (right R-module)
A (abelian group)

Claim: $\mathrm{Hom}(M_R, A) \in {}_R\mathrm{Mod}$

$M \in \mathrm{Mod}_R$ $\quad$ $A \in \mathrm{Ab}$

Why is $\mathrm{Hom}(M_R, A) \in {}_R\mathrm{Mod}$?

$f \in \mathrm{Hom}(M_R, A)$, $\quad f : M \longrightarrow A$

We need to define

$r \cdot f$, $r \in R$ $\qquad$ $rf : M \longrightarrow A$ $\qquad \longrightarrow$

define $r \cdot f : M \longrightarrow A$

$m \longmapsto f(m \cdot r)$

Then $r_2(r_1 f) = r_2 \cdot (m \longmapsto f(m \cdot r_1))$

$= (m \longmapsto f(m r_2 \cdot r_1))$

$(r_2 r_1) f = (m \longmapsto f(m(r_2 \cdot r_1))$

Example (bimodule)

$Q, R, S \in$ Rings (not necessarily commutative)

$${}_Q Ch_R \times {}_R Ch_S \xrightarrow{\otimes_R} {}_Q Ch_S$$

abelian group with additional left, right action structure

left hom: $\text{Hom}_\ell : {}_Q Ch_R^{op} \times {}_Q Ch_S \longrightarrow {}_R Ch_S$

right hom: $\text{Hom}_r : {}_R Ch_S^{op} \times {}_Q Ch_S \longrightarrow {}_Q Ch_R$

left hom - same action on left
different actions on right

$\text{Hom}_\ell({}_Q L_R, {}_Q N_S)$

$= {}_Q\text{Hom}(L, N)$

Q-linear maps from L to N

And from previous example,

$${}_R\left({}_Q\text{Hom}(L, N)\right)_S$$

action on the second component is covariant (S-action doesn't change)

action in first component contravariant (change from right R to left R)

Similarly,

$$\mathrm{Hom}_r({}_R M_S, {}_Q N_S) = {}_Q(\mathrm{Hom}_S(M,N))_R$$

$\otimes : \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{E}$

colimits of $\mathcal{C}, \mathcal{D}$ are preserved

$\mathrm{Hom}_\ell : \mathcal{C}^{op} \times \mathcal{E} \longrightarrow \mathcal{D}$

$\mathrm{Hom}_r : \mathcal{D}^{op} \times \mathcal{E} \longrightarrow \mathcal{C}$

limits in $\mathcal{C}^{op}, \mathcal{E}, \mathcal{D}^{op}$ are preserved

and recall, limits in $\mathcal{C}^{op}$ are colimits in $\mathcal{C}$.

Derived Adjunctions in two variables.

(applies to $\times$, $\otimes_R$, etc...)

Recall "pushout product"

$f : A \to B$

$g : X \to Y$

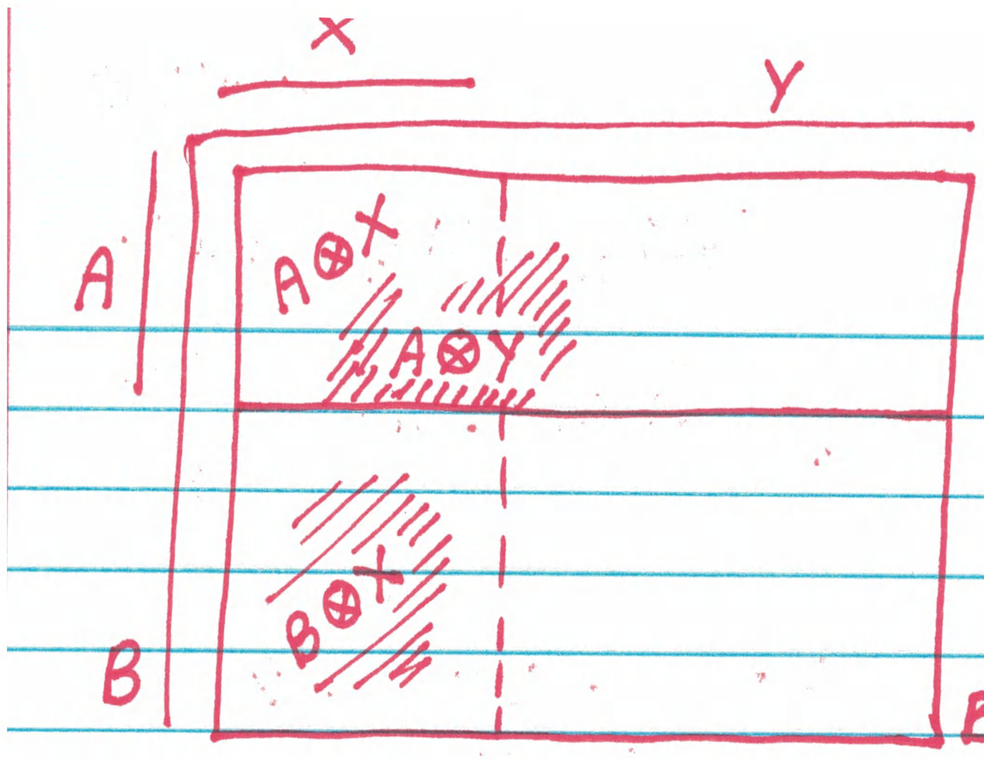
$$\begin{array}{ccc} A \otimes X & \longrightarrow & A \otimes Y \\ \downarrow & & \downarrow \\ B \otimes X & \longrightarrow & B \otimes Y \end{array}$$

compute pushout

$$P = B \otimes X \sqcup_{A \otimes X} A \otimes Y$$

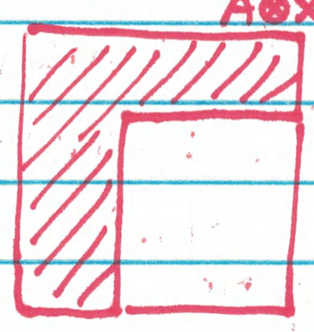
$$\begin{array}{ccc} A \otimes X & \longrightarrow & A \otimes Y \\ \downarrow & \ulcorner & \downarrow \\ B \otimes X & \longrightarrow & P \end{array} \qquad \begin{array}{ccc} & & \\ & \searrow & \\ & & B \otimes Y \end{array}$$

the map $P \to B \otimes Y$ is called the pushout product $f \square g$



$g : X \longrightarrow Y$

$P = B \otimes X \sqcup_{A \otimes X} A \otimes Y$



$\longrightarrow$ pushout product includes P into $B \otimes Y$

$f \square g : P \longrightarrow B \otimes Y$

Example (closed) Enriched categories

$\mathcal{V}$: monoidal category , $\mathcal{C}$: bicomplete category

A category $\mathcal{C}$ is called "$\mathcal{V}$-enriched" if

$\otimes$

$\mathcal{V} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}$ "tensoring" "copowering"

$\mathrm{Hom}_{\mathcal{C}} : \mathcal{V}^{op} \times \mathcal{C} \longrightarrow \mathcal{C}$ "powers" "cotensors"

$\mathrm{Hom}_{\mathcal{V}} : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathcal{V}$ "enriched hom"

For example : $\mathcal{V} = (set, \times)$

"copowering" $S \in set$

$S \otimes X := \coprod_{S} X$ (coproduct)

If $\mathcal{C} = set$

$S \otimes X = \coprod_{S} X = S \times X$

"powers" $X^S := \prod_S X$

If $\mathcal{C} = Set$, $X^S = \prod_S X = \{S \to X\}$

"enriched hom" $Hom(X,Y) := hom(X,Y) = \{X \to Y\}$

For another example

Chain complexes are enriched over abelian groups.

Ch enriched over Ab

Plug $\mathcal{C} = Ch$ and $\mathcal{V} = Ab$ into the previous definition, and justify the new operations "copower", "power", "enriched hom"

Recall, ... Left Quillen bifunctor

$\mathcal{C}, \mathcal{D}, \mathcal{E}$ model categories

A "left Quillen bifunctor" is an adjunction in two variables

$$\mathcal{C} \times \mathcal{D} \xrightarrow{\otimes} \mathcal{E}$$

(from which we can determine $Hom_\mathcal{E}$, $Hom_\mathcal{C}$)

Recall, Left Quillen preserves C, AC

Here,

$C_\mathcal{C} \square C_\mathcal{D} \subset C_\mathcal{E}$

pushout product of cofibrations in $\mathcal{C}$ and $\mathcal{D}$ ↑ cofibrations in $\mathcal{E}$

→

$AC_{\mathcal{C}} \square C_{\mathcal{D}} \subset AC_{\mathcal{E}}$

$C_{\mathcal{C}} \square AC_{\mathcal{D}} \subset AC_{\mathcal{E}}$

If one component of pushout product it an acyclic cofibration, then the result is an acyclic cofibration

Lemma: $\mathcal{C}$ = monoidal category

a) The category of arrows in $\mathcal{C}$
$\mathcal{C}^{\to}$: Functors ($\{0 \to 1\}, \mathcal{C}$)
(commutative squares)

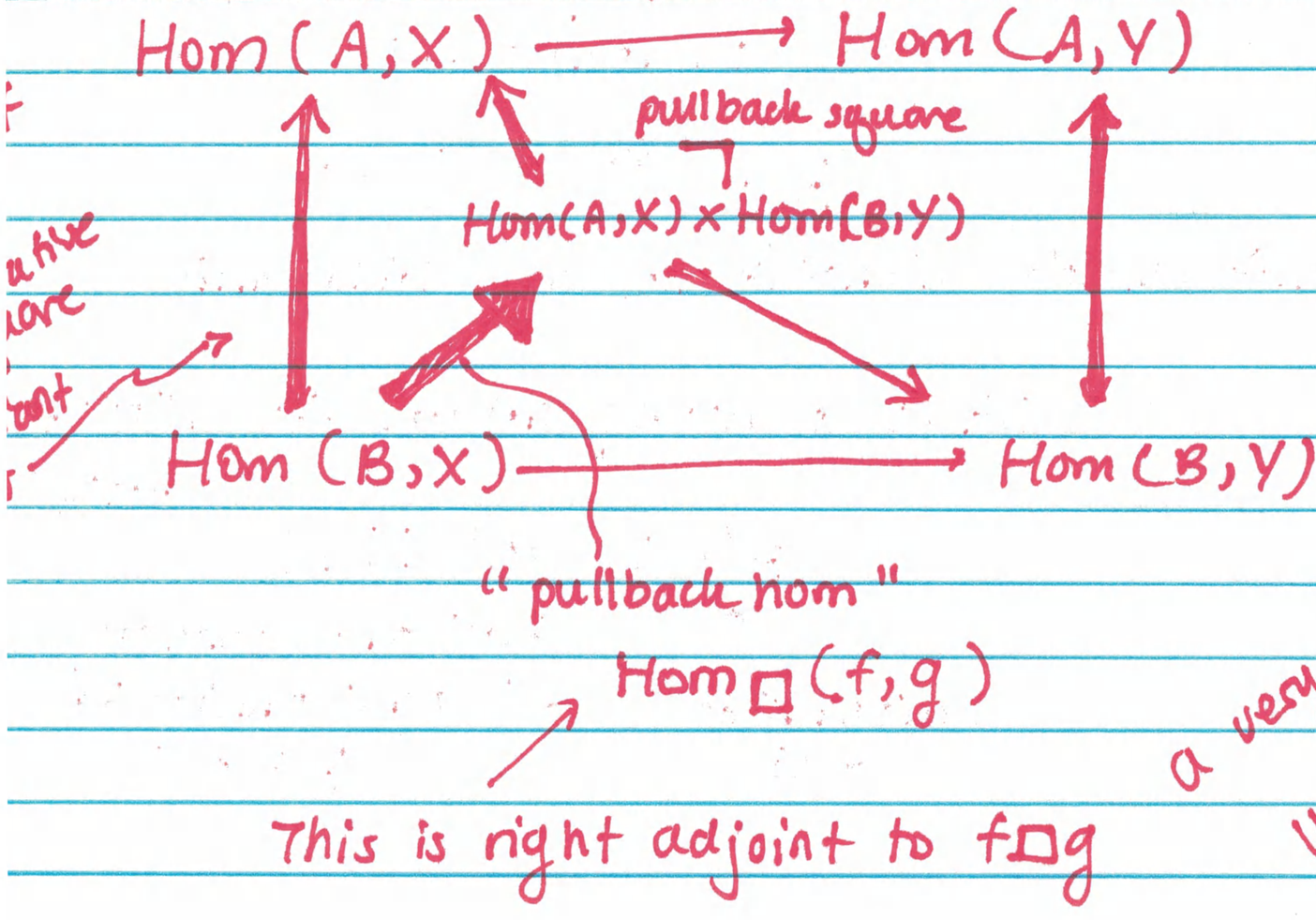
equipped with $\square$ is a monoidal category

b) If $\mathcal{C}$ is closed
then $\mathcal{C}^{\to}$ is closed

In particular, the pushout product is cocontinuous in each variable
($f \square g$ preserves colimits in each variable)

internal hom in the category of arrows:

"pullback hom" (or pullback power)
(the closed structure)



a very tedious proof with Universal property, etc.

Homological Algebra (10/29)

Corollary $\square$ is separately cocontinuous in each variable

(Corollary)^2 $C_{\mathcal{C}} \square C_{\mathcal{D}} \subset C_{\mathcal{E}}$

$AC_{\mathcal{C}} \square C_{\mathcal{D}} \subset AC_{\mathcal{E}}$

$C_{\mathcal{C}} \square AC_{\mathcal{D}} \subset AC_{\mathcal{E}}$

$C_{\mathcal{C}}$ = weak saturation of $S_{\mathcal{C}} \subset \mathrm{Mor}\,\mathcal{C}$

the set of cofibrations is the weak saturation of a class of morphisms of a category $\mathcal{C}$.

$C_{\mathcal{C}} = {}^{\pitchfork}((S_{\mathcal{C}})^{\pitchfork})$

$AC_{\mathcal{C}}$ = weak saturation of $AS_{\mathcal{C}}$

$S_{\mathcal{C}} \square S_{\mathcal{D}} \subset C_{\mathcal{E}}$

$AS_{\mathcal{C}} \square S_{\mathcal{D}} \subset AC_{\mathcal{E}}$

$S_{\mathcal{C}} \square AS_{\mathcal{D}} \subset AC_{\mathcal{E}}$

Def] $\mathcal{C}$ model category & monoidal category

we can impose a relation between

model structure and monoidal structure

$\mathcal{C}$ is a "monoidal-model category"

if ⓐ $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ is a left, Quillen bifunctor

ⓑ $\mathbb{1}_{\mathcal{C}}$ (the monoidal unit)

is cofibrant

Def] A cartesian model category

is a model category $\mathcal{C}$ such that $\mathcal{C}$, equipped with

Cartesian structure $\rightarrow (\mathcal{C}, \times, 1)$

is a monoidal

model category.

monoidal product $\otimes := \times$

monoidal unit $\mathbb{1}_{\mathcal{C}}$ is the terminal object of $\mathcal{C}$

Corollary: In a monoidal, model category, we have

$$A \overset{L}{\otimes} B = QA \otimes QB$$

cofibrantly replace both components

~~$R\mathrm{Hom}(A,B) = \mathrm{Hom}(QA,QB)$~~

$$R\mathrm{Hom}(A,B) = \mathrm{Hom}(QA, RB)$$

Q: cofibrant replace domain

R: fibrant replace codomain

$$Q : \mathcal{C} \longrightarrow \mathcal{C}_{\text{cofibrant}} \qquad R : \mathcal{C} \longrightarrow \mathcal{C}_{\text{fibrant}}$$

Ex sSet, swe, $\times$

$$A \overset{L}{\times} B = \overset{QA \times QB}{QA \otimes QB} = A \times B$$

cofibrant replacement does nothing in sSet

$$R\mathrm{Hom}(A,B) = \mathrm{Hom}(QA, RB) = \mathrm{Hom}(A, \mathrm{Ex}^{\infty} B)$$

cofibrant replacement does nothing

fibrant replacement functor (use Ex^{∞} functor)

Prop sSet, swe is a cartesian model category

proof By (corollary)^2,

define $S = \{ \partial\Delta^n \hookrightarrow \Delta^n \mid n \geq 0 \}$

$$AS = \{ \Lambda^n_k \hookrightarrow \Delta^n \mid \begin{matrix} n \geq 1 \\ 0 \leq k \leq n \end{matrix} \}$$

Look at pushout products of S and AS $\longrightarrow$

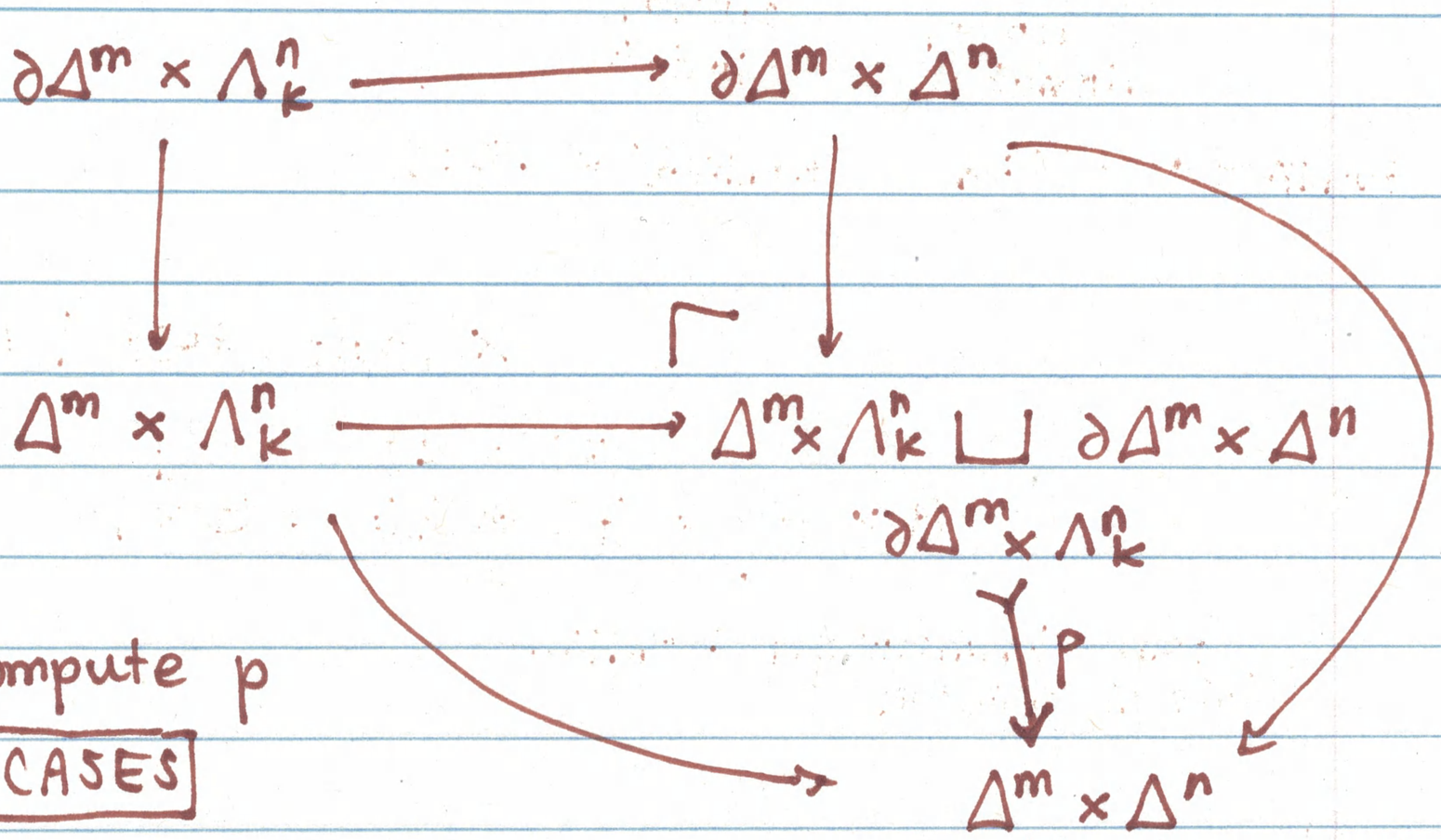
We previously showed

$$C_{sset} \square C_{sset} \subset C_{sset}$$

(showed that the pushout product of cofibrations is a cofibration, when $\mathcal{C} = sset$)

Now, compute the pushout product

$$(\partial\Delta^n \longrightarrow \Delta^n) \square (\Lambda^n_k \longrightarrow \Delta^n)$$



compute p

CASES

$m=0$, $n \geq 1$

$p : \Lambda^n_k \longrightarrow \Delta^n$

$m=1$, $n=1$

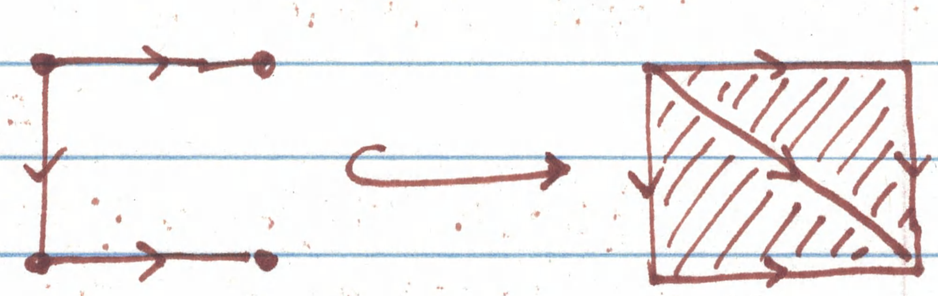
$\Delta^m \times \Lambda^n_k \sqcup_{\partial\Delta^m \times \Lambda^n_k} \partial\Delta^m \times \Delta^n$

$\partial\Delta^1 =$ • •

$\Delta^1 =$ •⟶•

$\Lambda^1_k \simeq$ •

$\Delta^1 =$ •⟶•



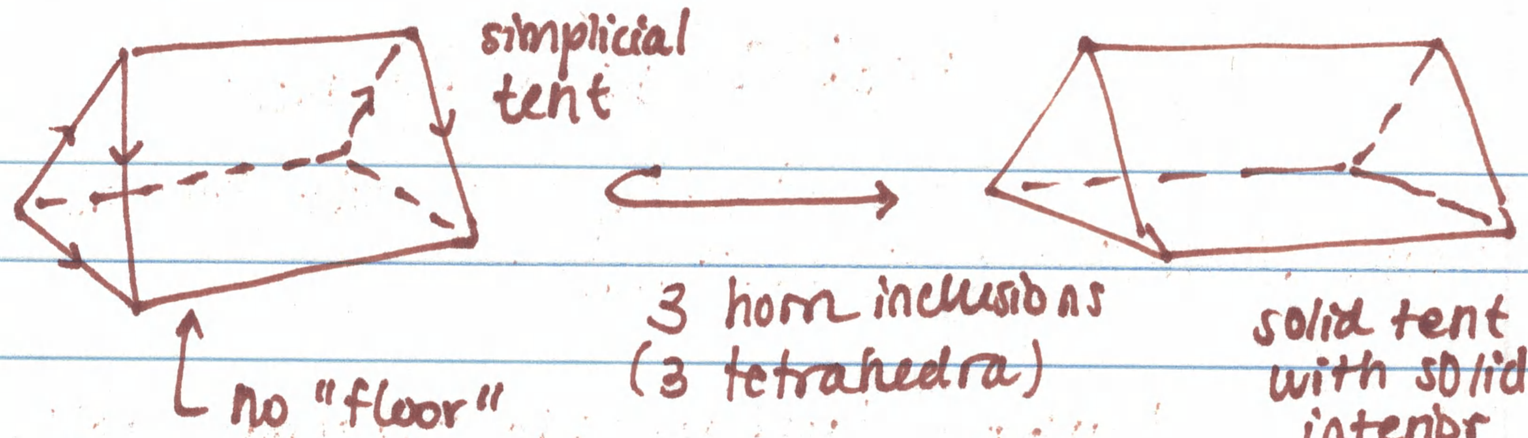
Can be obtained via a sequence of horn inclusions (cofibrations)

add horn

add horn

so this is a cofibration (composition of cofibrations is cofibration)

$m=1$, $n=2$



$\Rightarrow$ $S \underset{sset}{\square} A \underset{sset}{S} C \underset{sset}{A} G$

and similarly,

$\Rightarrow$ The Dold-Kan correspondence $\Leftarrow$

How to construct normalized chains out of a simplicial set

simplicial abelian groups $sAb \rightleftarrows Ch_{\geq 0}$ ~~pos~~ non-negatively graded chain complexes

adjunction (actually, an equivalence) of categories

Recall,

$sAb := Fun(\Delta^{op}, Ab)$

free abelian group $\mathbb{Z}[-]$ / U underlying set } weak eq.) Fibrations, and acyclic fibrations are created by U

sset

$sAb \underset{\Gamma}{\overset{N}{\rightleftarrows}} Ch_{\geq 0}$ (N: normalized chains)

We have a model structure on sset;

We use the ~~equivalence~~ adjunction between $sSet \rightleftarrows sAb$ and then use equivalence $sAb \rightleftarrows Ch_{\geq 0}$ to define a model structure on $Ch_{\geq 0}$

$sAb \underset{\Gamma}{\overset{N}{\rightleftarrows}} Ch_{\geq 0}$

apply N to all 5 classes of maps to get 5 classes in $Ch_{\geq 0}$

→ all 5 classes :

- are created by Γ
- are images under N of corresponding classes in sAb

for example:

Weak equivalences in sAb maps (under N) to quasi-isomorphisms

recall, quasi-isomorphisms induce isomorphisms on all homology groups

sAb monoidal / degreewise tensor abelian groups

BUT...

$Ch_{\geq 0}$ monoidal / tensor product of chain complexes

monoidal structures do not agree

we will use Eilenberg-Zilber and Alexander-Witney maps to resolve this issue

Def

$$\begin{array}{ccc} \Delta & \longrightarrow & Ch_{\geq 0} \\ \Psi & & \Psi \\ m & \longmapsto & C_*(\Delta^m) \end{array}$$

simplicial chains on Δ^m

construction is the chain complex version of the construction of singular simplicial set functor!

look up this construction!

$$m \longmapsto n \longmapsto C_*(\Delta^m) \longrightarrow C_*(\Delta^n)$$

a) define Γ:

$$sAb \ni \Psi$$

$$C \in Ch_{\geq 0},\ \Gamma C := (m \longmapsto \hom(C_*(\Delta^m), C))$$

looks like the singular simplicial set functor

∎

Transfer of model structures

Def $\mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}$

$\mathcal{C}, \mathcal{D}$ model categories

In this case $L \dashv R$ is a Quillen adjunction

The model structure on $\mathcal{D}$ is transferred from $\mathcal{C}$ if $WE_{\mathcal{D}} = R^* WE_{\mathcal{C}}$

(R^* is the preimage under R of weak equivalences on $\mathcal{D}$)

- fibrations $F_{\mathcal{D}} = R^* F_{\mathcal{C}}$
- acyclic fibrations $AF_{\mathcal{D}} = R^* AF_{\mathcal{C}}$

Def A model category $\mathcal{C}$ is cofibrantly generated if $\exists$ sets $G_{C_{\mathcal{C}}} \subset C_{\mathcal{C}}$, $G_{AC_{\mathcal{C}}} \subset AC_{\mathcal{C}}$ such that

$$F_{\mathcal{C}} = G_{AC_{\mathcal{C}}}^{\pitchfork}$$
$$AF_{\mathcal{C}} = G_{C_{\mathcal{C}}}^{\pitchfork}$$

Example: sSet (w/ standard model structure, "Kan-Quillen") is cofibrantly generated;

$$G_C = \{ \partial\Delta^n \longrightarrow \Delta^n \mid n \geq 0 \}$$
$$G_{AC} = \{ \Lambda^n_k \longrightarrow \Delta^n \mid n \geq 0, 0 \leq k \leq n \}$$

DEF A combinatorial model category is a cofibrantly generated model category whose underlying category is locally presentable

Theorem: $\mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}$

If $\mathcal{C}$ is a combinatorial model category, then the transferred model structure on $\mathcal{D}$ ($\mathcal{D}$ locally presentable) exists iff the following properties are satisfied:

point #1 • transfinite compositions of cobase changes of elements of $L(G_{AC_{\mathcal{C}}})$ are elements of $WE_{\mathcal{D}}$; in particular... (point #2)

(from now on write $W_{\mathcal{D}}$)

point #2 • $L(G_{C_{\mathcal{C}}})^{\pitchfork} \subset W_{\mathcal{D}}$

maps with right lifting property w.r.t. images of generating cofibrations in $\mathcal{C}$ are weak equivalences in $\mathcal{D}$

In this case, $\mathcal{D}$ is also combinatorial and $G_{C_{\mathcal{D}}} = L(G_{C_{\mathcal{C}}})$

$G_{AC_{\mathcal{D}}} = L(G_{AC_{\mathcal{C}}})$

Ex) $sSet \underset{U}{\overset{\mathbb{Z}[-]}{\rightleftarrows}} sAb$

since... ↑

combinatorial
- ☑ cofibrantly generated
- ☑ locally presentable
- ☑ small

$\Rightarrow$ Then...

sAb is a combinatorial model category!

$$GC_{sAb} = \{ \mathbb{Z}[\partial\Delta^n \longrightarrow \Delta^n] \mid n \geq 0 \}$$

$$GAC_{sAb} = \{ \mathbb{Z}[\Lambda^n_k \longrightarrow \Delta^n] \mid \begin{matrix} n \geq 1 \\ 0 \leq k \leq n \end{matrix} \}$$

Corollary $Ch_{\geq 0}$ is a combinatorial model category

(since $sAb \underset{\Gamma}{\overset{N}{\rightleftarrows}} Ch_{\geq 0}$) where $C_* = N \circ \mathbb{Z}[-]$

$sSet \rightleftarrows sAb \rightleftarrows Ch_{\geq 0}$

$$GC_{Ch_{\geq 0}} = \{ C_*(\partial\Delta^n \longrightarrow \Delta^n) \mid n \geq 0 \}$$

$$GAC_{Ch_{\geq 0}} = \{ C_*(\Lambda^n_k \longrightarrow \Delta^n) \mid n \geq 1, 0 \leq k \leq n \}$$

We will now investigate all 5 categories of morphisms on $Ch_{\geq 0}$ as obtained by transferring along the map

$$sSet \xrightarrow{\mathbb{Z}[-]} sAb \xrightarrow{N} Ch_{\geq 0}$$

$$\searrow \quad C_* \quad \nearrow$$

Lemma

$$F_{Ch_{\geq 0}} = \{ f \mid f_n \text{ is surjective, } \forall n \neq 0 \}$$

proof ||

$(GAC_{Ch_{\geq 0}})^{\pitchfork}$, so let's draw the right lifting property diagram!

by definition

$$\begin{array}{ccc} C_*\Lambda^n_k & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ C_*\Delta^n & \longrightarrow & Y \end{array}$$

What are we missing from $C_*\Lambda^n_k$ that is in $C_*\Delta^n$? ← an $n-1$-dim simplex

the kth face (one copy of $\mathbb{Z}$) and the n-dim simplices

we ~~know~~ have a decomposition as a graded abelian group

$$C_*\Delta^n = C_*\Lambda^n_k \oplus \mathbb{Z}[n-1] \oplus \mathbb{Z}[n]$$

ψ_{sm} $1 = s$; ψ_{id} $1 = t$

$$\partial t = (-1)^k s + \sum_{\substack{0 \leq i \leq n \\ i \neq k}} (-1)^i d_i t$$

top dimensional simplex

$\in C_*\Lambda^n_k$

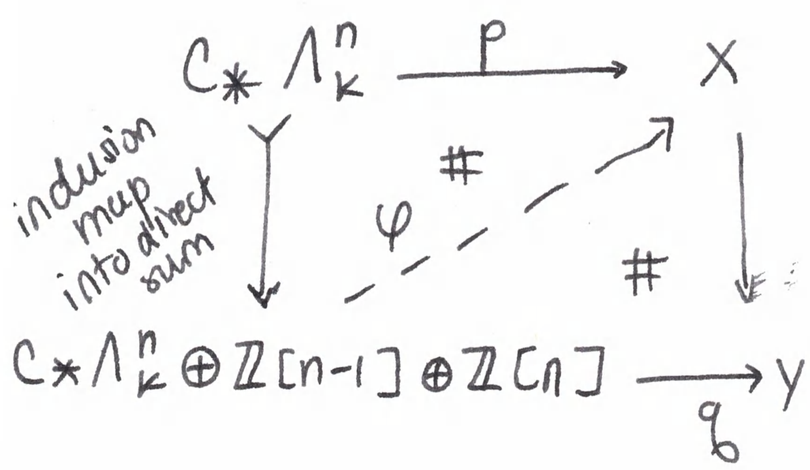
Recall $\mathbb{Z}[n-1]$

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow \underset{\text{deg } n-1}{\mathbb{Z}} \longrightarrow 0 \longrightarrow \cdots$$

place copy of $\mathbb{Z}$ in degree $n-1$

all other degrees are 0

In $Ch_{\geq 0}$



φ out of direct sum
by universal property,
define out of each summand

since $C_* \Lambda^n_k \xrightarrow{p} X$
is already defined as the top map

Map of chain complexes is in particular
a map of graded abelian groups

so, the map from $\mathbb{Z}[n-1]$
sends the generator in degree $n-1$
to the generator in X_{n-1}

so, we have: $\varphi = p \oplus x_{n-1} \oplus x_n$

x_{n-1} is a map that sends $s = 1 \in \mathbb{Z}[n-1]$ to $x_{n-1} \in X_{n-1}$

$x_{n-1} = \varphi(s)$

$x_n = \varphi(t)$

where $s = 1 \in \mathbb{Z}[n-1]$
$t = 1 \in \mathbb{Z}[n]$

φ is a chain map

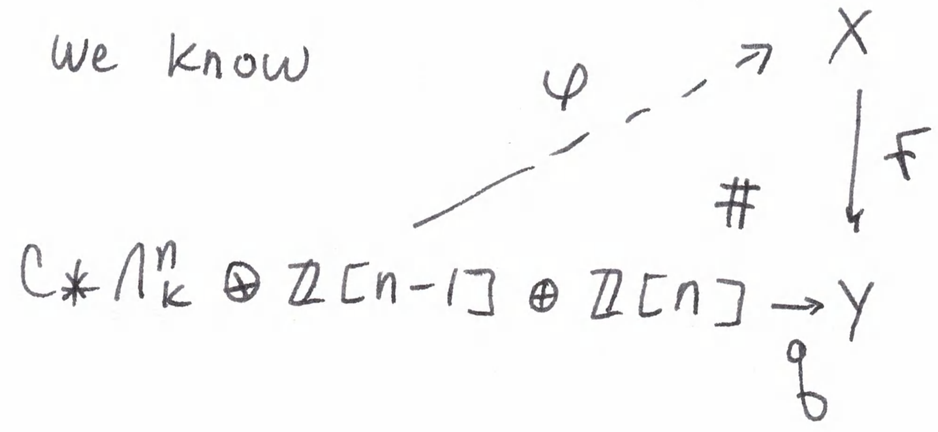
$$\partial(\varphi(t)) = \varphi(\partial t) = \varphi((-1)^k s + \sum (-1)^i d_i t)$$

$$\varphi(s) = (-1)^k \partial(\varphi(t)) - (-1)^k \varphi(\sum (-1)^i d_i t)$$

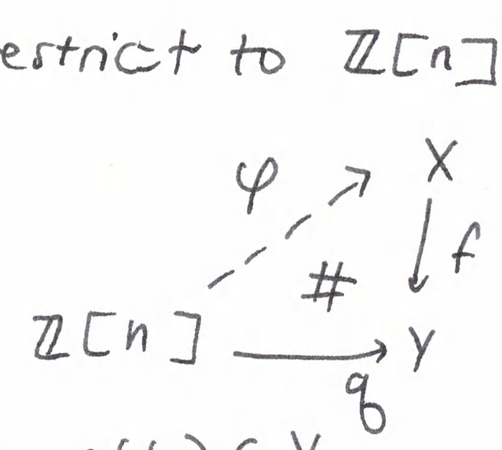
$\varphi(s) = x_{n-1}$ completely determined by x_n $= \varphi(t) = x_n$

so, if we know x_n, we can canonically / uniquely compute x_{n-1}

we know



, so, restrict to $\mathbb{Z}[n]$



$f(x_n) = q(t) \in Y_n$

x_n exists provided this equation has a solution.

But $q(t)$ can be an arbitrary element of Y_n

arbitrary element of Y_n must be in image of f, so we need f to be surjective! $\forall n \geq 1$.

Grandma's birthday!

Homological Algebra [11/14]

$$\mathbb{Z}[-] \dashv U : \mathrm{SSet} \to \mathrm{Ab}, \qquad N \dashv \Gamma : \mathrm{Ab} \to \mathrm{Ch}_{\geq 0}$$

Fibrations are chain maps which are surjective in nonzero degrees

Recall $A \xrightarrow{f} B$ in sSet,

f is a weak equivalence $\Leftrightarrow$ $f \in W$ iff

- $\pi_0 f$ isomorphism
- $\forall a \in A_0 : \pi_n(f,a) : \pi_n(A,a) \xrightarrow{\cong} \pi_n(B, f(a))$

$$\| \quad \pi_0 R\mathrm{Hom}_{\mathrm{sSet}_*}((S^n, *), (A,a))$$

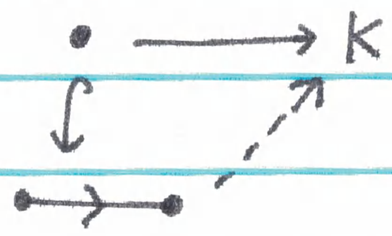
Proposition: (Kan)

Every simplicial group is a Kan complex

(Note: not necessarily an abelian simplicial group)

Proof | A simplicial set is Kan if it satisfies the horn filling condition.

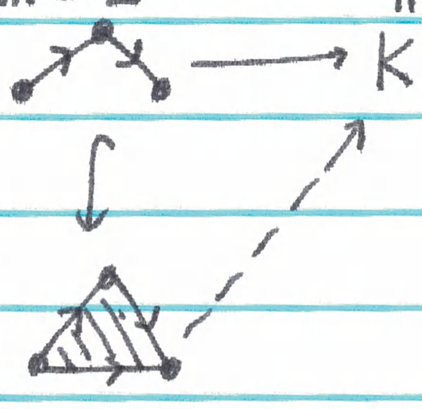
In dim=1



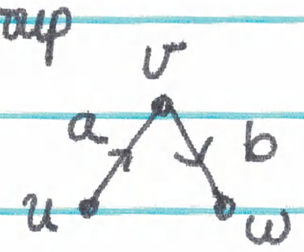
extension is trivial

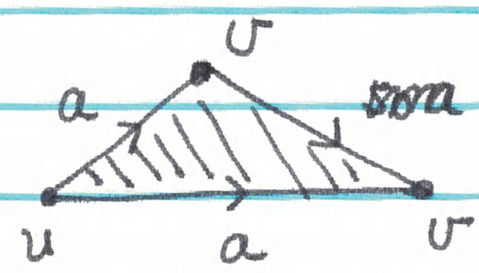
In dim=2

$UG \in \mathrm{sGroup}$

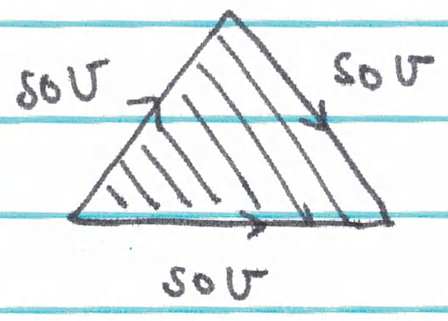


$u, v, w \in G_0$
$a, b \in G_1$

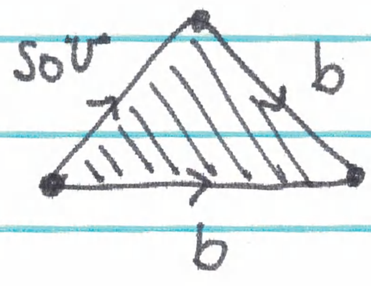




We have a degenate triangle from edge a



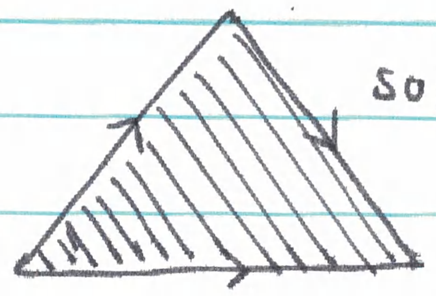
we have a degenerate triangle from vertex v.



and a degenerate triangle from edge b

→ since G is a group, we have

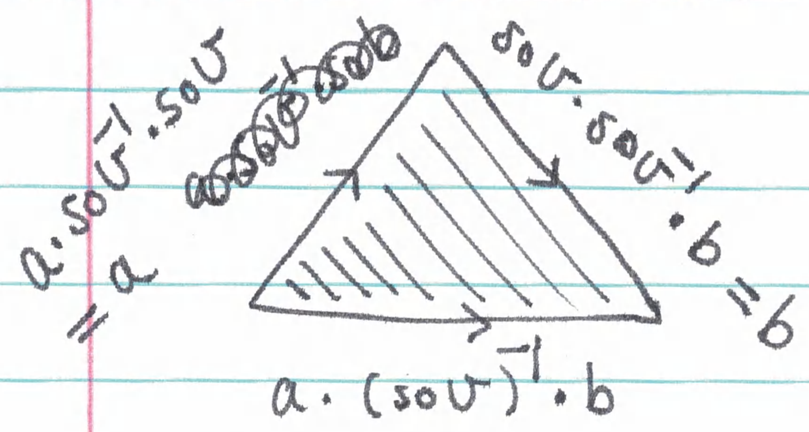
$(s_0 s_0 v)^{-1} = s_0 s_0 (v^{-1})$ since simplicial structure commutes with group operation

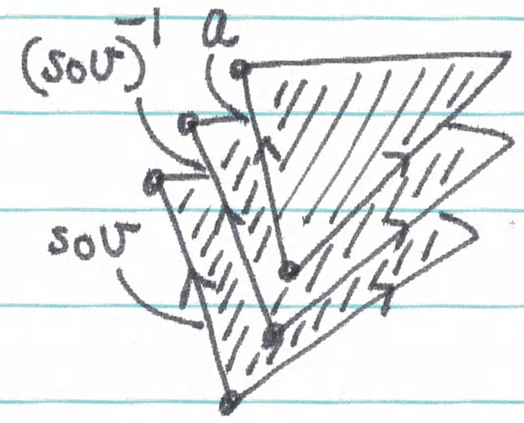


$s_0(v^{-1}) = (s_0 v)^{-1}$

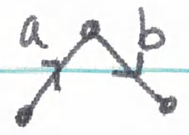
so, we can take $s_1 a \cdot (s_0 s_0 v)^{-1} \cdot s_0 b$

Recall, face maps are also homomorphisms of groups, so, we can compute separately on each:





In group, take product of these triangles by "stacking" them and multiplying

↖ Then this is defined as the horn filling of 

(and the same argument holds in all dimensions) ▨

Exercise: Show that $\mathcal{C}$ is a strict monoidal category with all objects invertible w.r.t. the monoidal product (all objects $\otimes$-invertible)

$\Leftrightarrow N\mathcal{C} \in$ sGroup

(The nerve is a simplicial group)

---

strict

$(a \otimes b) \otimes c$

$=\!=$

$a \otimes (b \otimes c)$

If $\mathcal{C}$ is strict, monoidal w/ invertible,

since $N\mathcal{C} \subset \mathcal{C}$,

$\forall$ elements of $N\mathcal{C}$ TBC later...

Not strict

$(a \otimes b) \otimes c$

$=\!=$

$a \otimes (b \otimes c)$

In a strict monoidal category, $v \otimes v^{-1} = id$

Remark: $A \xrightarrow{f} B$ sAb

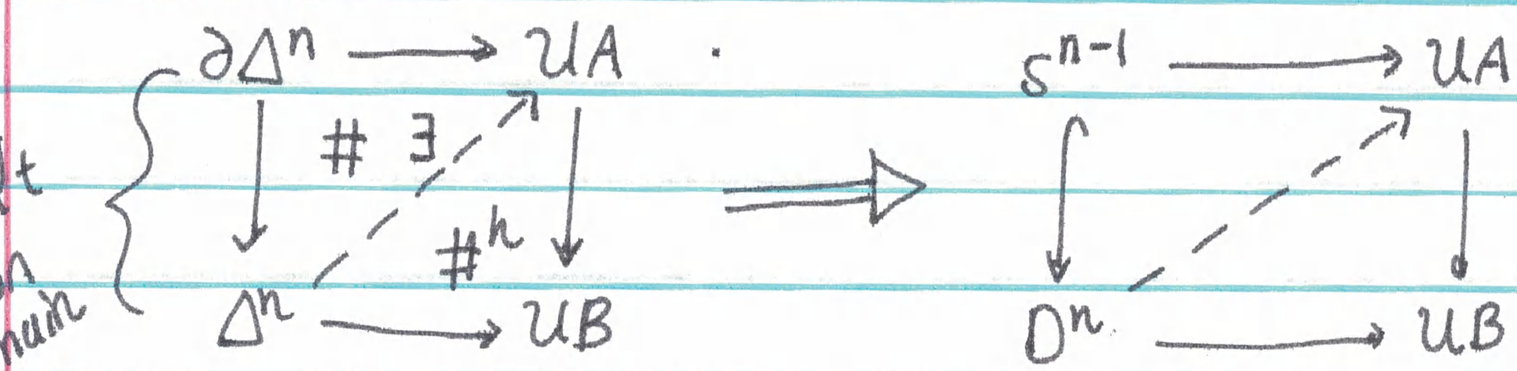
$f \in W \iff$ { $\pi_0 f$ isomorphism; $\forall a \in A_0$: $\pi_n(f,a): \pi_n(A,a) \xrightarrow{\cong} \pi_n(B, f(a))$ }

This relies on the property that everything is Kan!

same as previous except replace RHom with Hom.

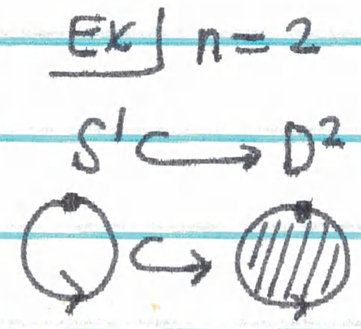
$\pi_0 \mathrm{Hom}_{sSet}((S^n, *), (A,a)) = \pi_n(A,a)$

Recall, simplicial Whitehead Theorem:

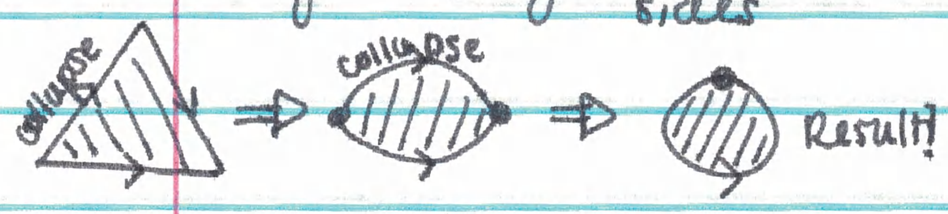


Replace with weakly equivalent cofibration whose domain ~~are~~ is S^{n-1} and range is n-dim disk D^n

Ex $n=2$

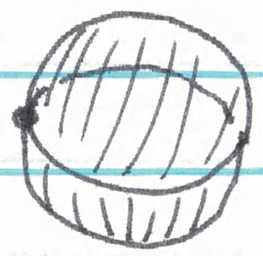


And the disk D^2 is obtained from a 2-simplex by collapsing two of the sides



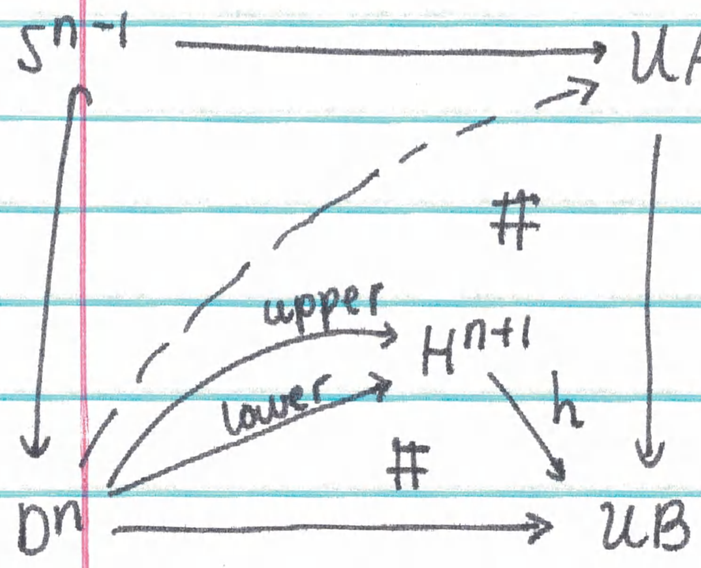
and we can "replace" the homotopy h in the lower triangle with

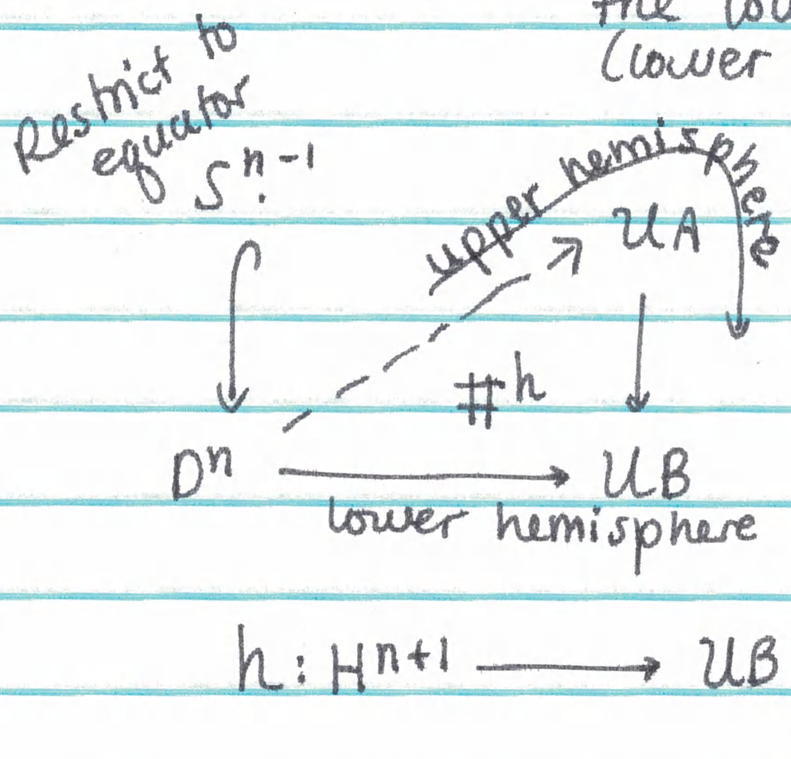
H^{n+1}



1 vertex, 1 edge, 2 faces, 1 volume

This is a homotopy between the upper face (upper hemisphere) to the lower face (lower hemisphere)





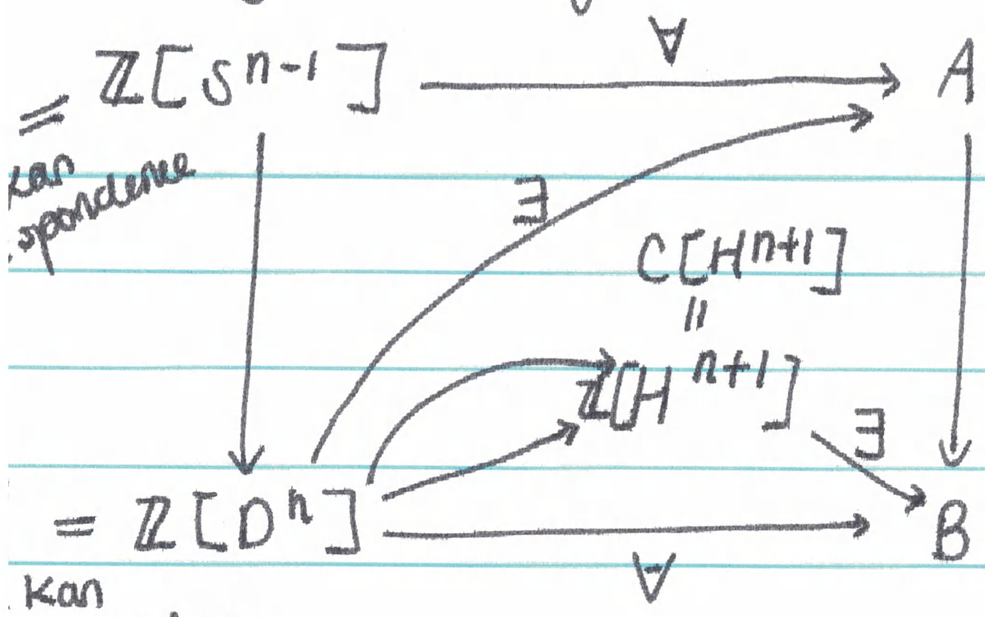
This is a homotopy relative boundary (boundary "equator" doesn't move!)

Take adjoint of diagram:

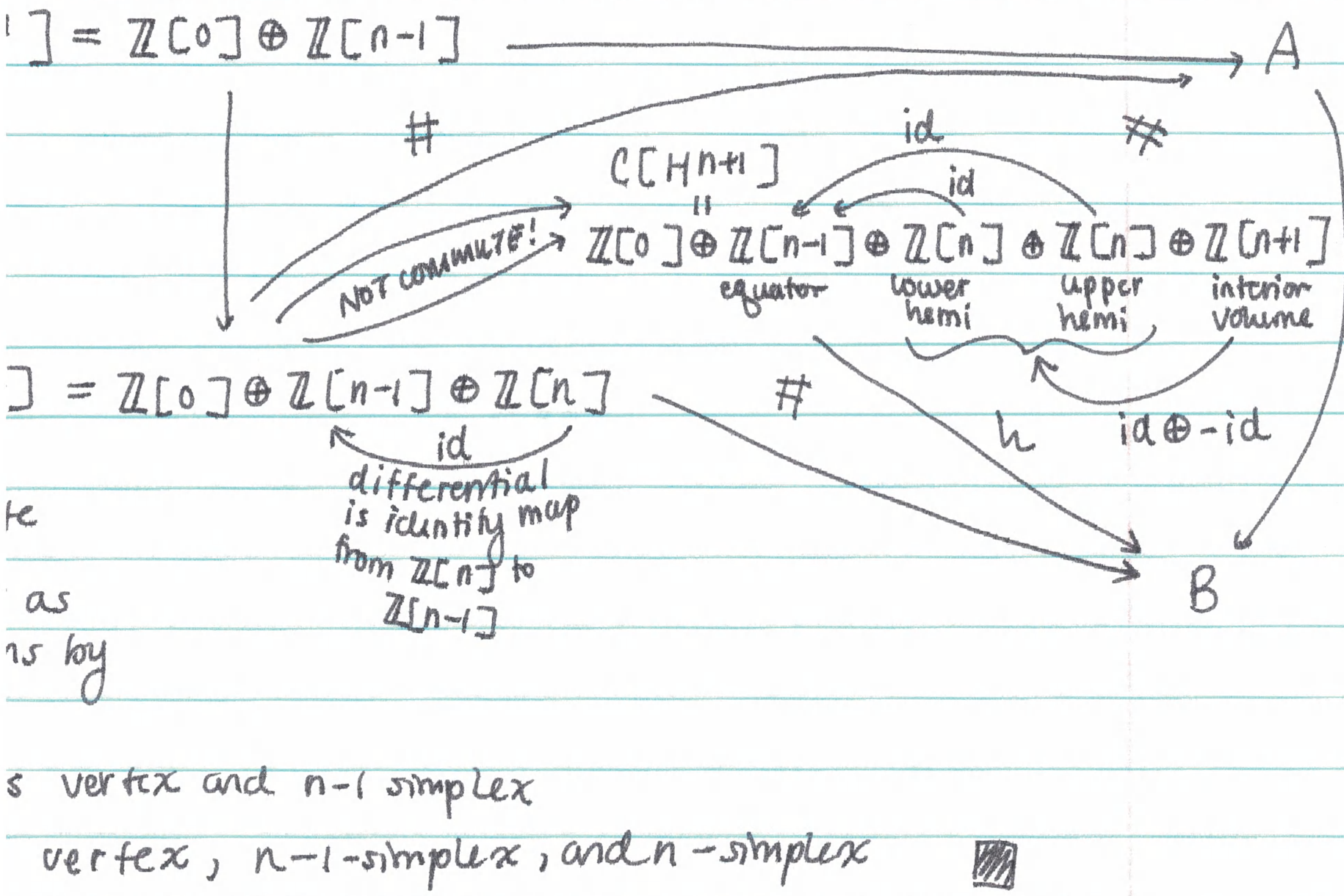
By Dold-Kan correspondence, we are really considering a map of chain complexes!

$A \longrightarrow B$

Ch≥0



for checking whether map
n two abelian groups is weak equivalence.
alently, by Dold-Kan, between to chain complex is weak equivalence)



he
as
ns by

s vertex and n-1 simplex

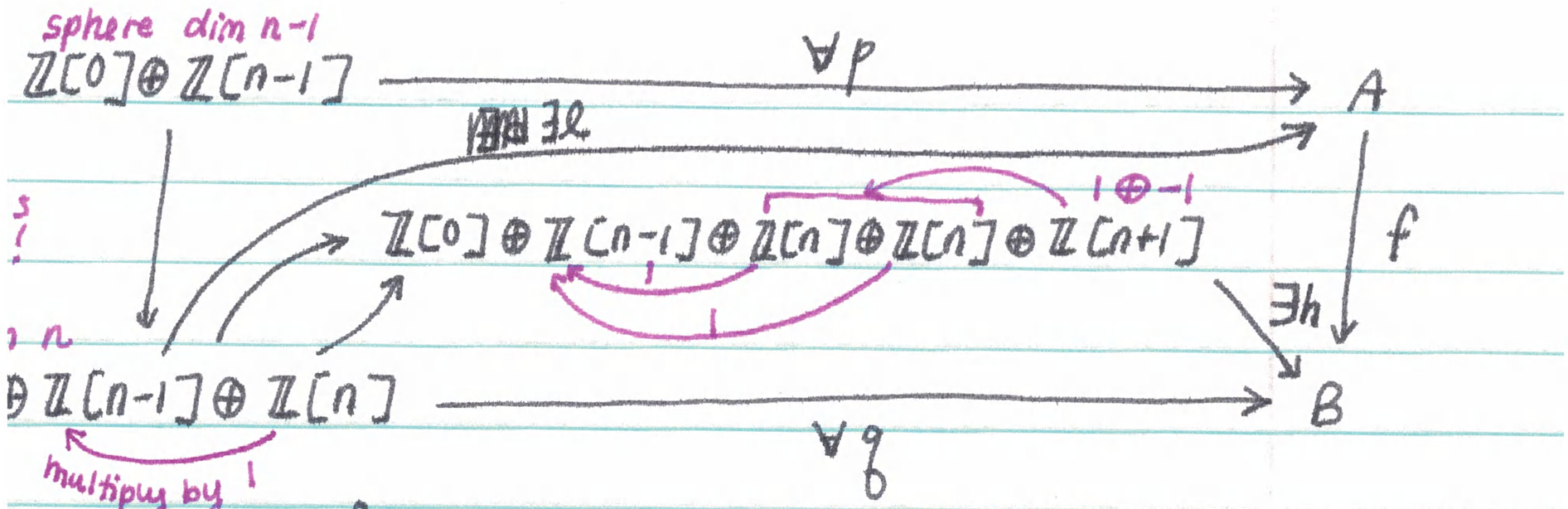
vertex, n-1-simplex, and n-simplex ▨

un previously

rtex,
-1 simplex (edge)
-simplices (faces)
-1 simplex (volume)



Recall from last time



ether map $A \xrightarrow{f} B$ is a quasi-isomorphism/weak equivalence

where $A, B \in$ ~~Ab~~ $Ch_{\geq 0}$

s of maps ℓ, h ~~are also~~ on $\mathbb{Z}[0]$ are uniquely
red by diagram commutativity.
essentially ignore them in the picture, leaving only
n and n-1 components)

define homomorphism $p: \mathbb{Z}[n-1] \longrightarrow A$
defined by a single element in degree n-1

$\Rightarrow p_{n-1} \in A_{n-1}^{\text{closed}}$

must be closed because... by definition of a chain map, the following diagram must commute

$$\begin{array}{ccc} 0 & \longrightarrow & \mathbb{Z} \\ {\scriptstyle p_{n-2}=0}\downarrow & \# & \downarrow{\scriptstyle p_{n-1}} \\ A_{n-2} & \longleftarrow & A_{n-1} \end{array}$$

$\partial_{n-1} p_{n-1} = 0$ so diagram commutes

-1] is a free chain complex on a ~~free~~ closed element in ~~degree~~ degree n-1

(p_{n-1} determines p uniquely!)

define homomorphism $q: \mathbb{Z}[n-1] \oplus \mathbb{Z}[n] \longrightarrow B$

$$\begin{array}{ccccc} 0 & \longleftarrow & \mathbb{Z} & \xleftarrow{\partial=1} & \mathbb{Z} \\ {\scriptstyle q_{n-2}=0}\downarrow & & {\scriptstyle q_{n-1}}\downarrow & & {\scriptstyle q_n}\downarrow \\ B_{n-2} & \xleftarrow{\partial_{n-1}} & B_{n-1} & \xleftarrow{\partial_n} & B_n \end{array}$$

$q_n \in B_n$

$q_{n-1} \in B_{n-1}$

By diagram, $q_{n-1} = \partial_n q_n$ (degree n-1 part determined by degree n part)

y, we saw that we
q_{n-1} to be a
element.

e $q_{n-1} = \partial_n q_n$

$_{n-1}) = \partial^2 q_2 = 0$

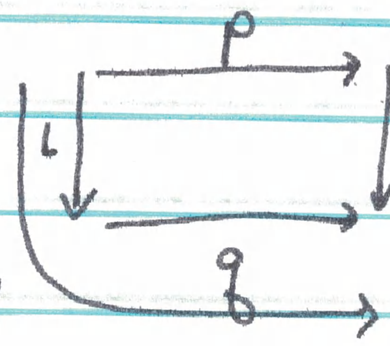
$_{-1}$ is closed element!

→ clude ($\mathbb{Z}[n-1] \longleftarrow \mathbb{Z}[n]$ is a free chain complex on a single element in degree n.

(q_n determines q_{n-1} and thus determines q uniquely)

e know how to define

os p and q



composition #1
$f \circ p$

composition #2
$q \circ l$

n #1] $f \circ p$ Recall, $p_{n-1} \in A_{n-1}$ closed

$= f \circ p_{n-1} = f(p_{n-1}) \in B_{n-1}$ closed

n #2] $q \circ l$ l is the restriction of $\mathbb{Z}[0] \oplus \mathbb{Z}[n-1] \oplus \mathbb{Z}[n]$ to $\mathbb{Z}[0] \oplus \mathbb{Z}[n-1]$,

so $q \circ l = q_{n-1} = \partial q_n \in B_{n-1}$ closed

$p_{n-1} \in A_{n-1}$ closed ($\partial p_{n-1} = 0$)

$q_n \in B_n$

$f(p_{n-1}) = \partial q_n$

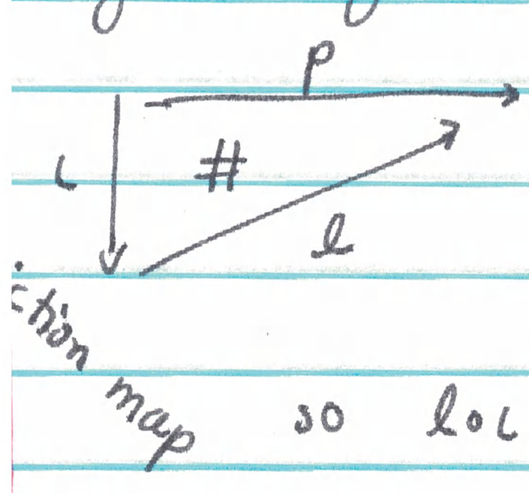
map l (from p and q known!)

$\mathbb{Z}[n-1] \oplus \mathbb{Z}[n] \longrightarrow A$

l is uniquely determined by an element in degree n

$l_n \in A_n$

(by same argument as before, $l_{n-1} = \partial_n l_n$)



$l_{n-1} = p_{n-1}$

$= \partial l_n$

so $l \circ l$ restricts to $n-1$ component

→

map $h: \mathbb{Z}[n-1] \oplus \mathbb{Z}[n] \oplus \mathbb{Z}[n] \oplus \mathbb{Z}[n+1] \longrightarrow B$

$\begin{pmatrix}0\\0\end{pmatrix}$ $\begin{pmatrix}1&1\\1&\oplus 1\end{pmatrix}$ $\begin{pmatrix}-1\\1&\oplus -1\end{pmatrix}$

$0 \leftarrow \mathbb{Z} \leftarrow \mathbb{Z}\oplus\mathbb{Z} \leftarrow \mathbb{Z}$ differentials same as in Topology!

$\downarrow h_{n-2}$ # $\downarrow h_{n-1}$ # $\downarrow h_n$ # $\downarrow h_{n+1}$

$B_{n-2} \leftarrow B_{n-1} \leftarrow B_n \leftarrow B_{n+1}$

$h_{n-1} \in B_{n-1}$ $h'_n \in B_n$ $h''_n \in B_n$ h_{n+1} picks some element in B_{n+1}

$h_{n+1} \in B_{n+1}$

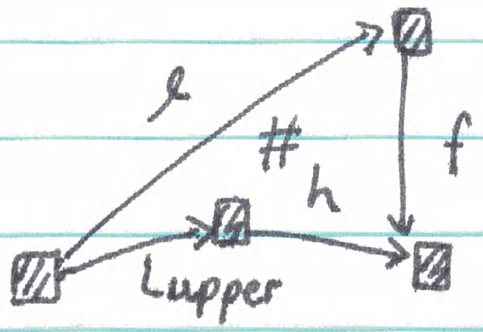
By commutativity $\partial h_{n-1} = 0$ // $\partial\partial h''_n$

By commutativity $\partial h'_n + \partial h''_n = h_{n-1}$

By commutativity $\partial h_{n+1} = h'_n - h''_n$

by abuse of notation, we use the name of the map for the name of this element

We have two elements in degree n whose difference is exact and whose sum determines the element in degree $n-1$



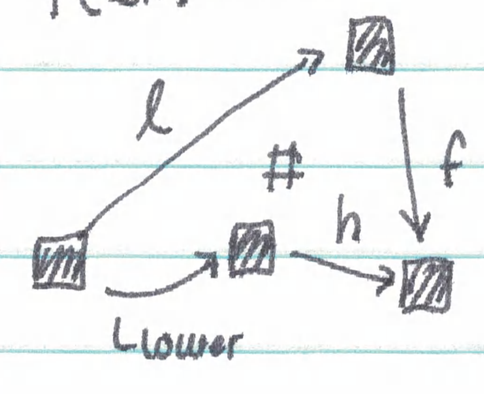
Retricting along l_{upper} "picks" the element corresponding to the first copy of $\mathbb{Z}[n]$

$h \circ l_{upper} = h'_n \in B_n$

$f \circ l = f(l_n) \in B_n$

BUT since the diagram commutes we know $f(l_n) = h'_n$

both $\in$ h by diagram and the diagram commutes



Restricting along l_{lower} "picks" the element corresponding to the second copy of $\mathbb{Z}[n]$

$h \circ l_{lower} = h''_n \in B_n$

but l_{lower} $\xrightarrow{h}$, $\xrightarrow{\#}$, $\xrightarrow{q}$

$h \circ l_{lower} = q$

so $h''_n = q_n \in B_n$

→ Thus, we already know what h'_n, h''_n are! Hence h is determined uniquely by (h_{n+1}) an element in degree $n+1$

narry
$$\begin{cases} l_n \in A_n \\ \partial l_n = p_{n-1} \\ h_{n+1} \in B_n \\ \partial h_{n+1} = f(l_n) - q_n \end{cases}$$

$f: A \longrightarrow B$ f chain map of chain complexes
$A, B \in Ch_{\geq 0}$

f is a quasi-isomorphism ($H(f)$ is an isomorphism)

iff $\forall p_{n-1} \in A_{n-1}$ such that $\partial_n p_{n-1} = 0$
$\forall q_n \in B_n$ such that $\partial_n q_n = f(p_n)$
$\exists l_n \in A_n$ such that $\partial_n l_n = p_n$
$\exists h_{n+1} \in B_{n+1}$ such that $\partial_{n+1} h_{n+1} = f(l_n) - q_n$

Proof injectivity on H_{n-1}
$[p] \in H_{n-1}(A)$

injectivity
$$\begin{cases} H_{n-1}(f)([p]) = 0 \in H_n(B) \\ \text{NTS: class of } p \text{ is } 0 \\ \text{But, this is exactly the input data we have!} \\ f(p) = \partial q \\ [p] = [\partial l] = 0 \end{cases}$$

surjectivity
$$\begin{cases} \underline{\text{surjectivity}}: \text{ on } H_n \\ [q] \in H_n(B) \\ \text{NTS: } [q] \text{ is in the image of some element!} \\ \partial q = 0 = f(0) \quad \text{so, take } p = 0 \\ \text{so } [l] \in H_n(A) \end{cases}$$

used application) time! Then, we can find $l \in A_n$ such that $\partial l = p = 0$

and we can find $h \in$ such that $H_n(f)([l]) = [f(l)] = [\partial q] = 0$. homology class of boundary $= 0$.

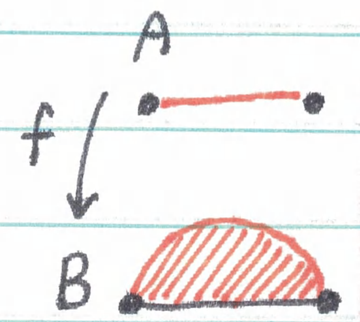
Homological Algebra [11/21]

na (continued...)

$f: A \longrightarrow B$, $A, B \in Ch_{\geq 0}$

f is a quasi-isomorphism $\Longleftrightarrow$ $\forall p \in A_{n-1}$, $\partial p = 0$

$\forall q \in B_n$, $\partial q = f(p)$



mmA

AQ
A!
n applied
nplexes)

$\exists l \in A_n$, $\partial l = p$

$\exists h \in B_{n+1}$, $\partial h = f(l) - q$

inued...) ($\Rightarrow$)

$[p] \in H_{n-1}(A)$ (a cycle already gives homology class)

$(f)([p]) \in H_{n-1}(B)$

$\| \; 0$

since we have an isomorphism between homology groups (assumed in this direction of proof)

$[p] \longrightarrow H_{n-1}(f)([p]) = 0$

$\Rightarrow [p] = 0$

class is zero $\Rightarrow$ $[p] = 0$ $\Rightarrow$ p is exact $\Rightarrow$ $\exists l \in A_n$ such that $\partial l = p$

To show existance of element h,
compute homology class of $[f(l) - q]$
(if $[f(l) - q] = 0$ then $f(l) - q$ is exact
hence, $\exists h$ such that $\partial h = f(l) - q$)

we want this:
$[f(l^*) - q] = 0$

so we need to find an element l with this property $\leftarrow$ and $\exists l^* \in A_n$, $\partial l^* = p$ l^* must also satisfy the previous property!

$\longrightarrow$

~~We want to do this such that~~ step (2/2) choose ℓ such that $\partial\ell' = p$

$H_n(f)$ is an isomorphism

so, $\exists a \in A_n$, $\partial a = 0$ such that

$[f(a)] = [f(\ell') - q]$

homology classes equal $\Rightarrow$ representatives differ by some exact element

$\exists h' \in B_{n+1}$, ~~$f(a) + \partial h' = f(\ell') - q$~~

$f(a) + \partial h' = f(\ell') - q$

rearrange $\partial h' = f(\ell' - a) - q$

Then $\ell = \ell' - a$

and $\partial\ell = \partial\ell' - \partial a = p$

so ℓ satisfies the previous condition $\partial\ell = p$ ✓

and also

$[f(\ell) - q] = [f(\ell' - a) - q]$

$= [f(\ell') - f(a) - q]$

$= [f(a) + \partial h - f(a)] = [\partial h] = 0$ ✓

step ④ + ε

Ch≥0 (we transferred a model structure from simplicial sets to abelian groups, and then to abelian groups)

W = We discovered that weak equivalences are quasi-isomorphisms

F = fibrations ≡ surjections in nonzero degrees.

AF = F ∩ W = acyclic fibrations

C = ? What are the cofibrations??

AC = C ∩ W

Once we know this, we will be able to derive Hom and Tensor product (to recover the classical Ext and Tor from Homological algebra) ⟶

→ cofibrations are generated by

$$C_*(\partial\Delta^n \longrightarrow \Delta^n) \quad n \geq 0$$

chains of boundary inclusions

acyclic cofibrations are generated by

$$C_*(\Lambda^n_k \longrightarrow \Delta^n) \quad n > 0 \quad 0 \leq k \leq n$$

chains of horn inclusions

ZONS (in Ch≥0)

ns are retracts of transfinite compositions of

anges of $C(\partial\Delta^n \longrightarrow \Delta^n)$

change

$$\begin{array}{ccc} C(\partial\Delta^n) & \longrightarrow & A \\ L \downarrow & \text{pushout} \ulcorner & \downarrow f \\ C(\Delta^n) & \longrightarrow & B \end{array}$$

L_k is an isomorphism if $k \neq n$

$\Rightarrow f_k$ is an isomorphism if $k \neq n$

(f is the cobase change of L)

The only interesting case is in degree n.

In degree n, $C_n(\Delta^n) = \mathbb{Z}$

$C_n(\partial\Delta^n) = 0$ (there are no nondegenerate simplices)

since,

$L_n : 0 \longmapsto \mathbb{Z}$

so $B_n = A_n \oplus \mathbb{Z}$

Ψ

Then, 1 ↗ $\partial 1 \in A_{n-1} = B_{n-1}$ arbitrary element

so,

cobase change adds copy of $\mathbb{Z}$ in degree n

(differential is arbitrary)

nite
sitions
ase changes

$$A \longrightarrow B$$

$B = A \oplus F$ (as graded abelian groups)

$F_k = \bigoplus \mathbb{Z}$ is a free abelian group, $\forall k$

The differentials of F can be set up arbitrarily, containing

→ elements of both A and F

$(\partial F \subset A \oplus F)$

ts
ns finite
sitions
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es

replace F by some retract F'

(since free modules are projective,
this amounts to saying
F' is a direct summand of F
(split F as direct sum of F' and F'';
throw away F''.)

In the case of integers, nothing happens,
$\mathbb{Z}$ because submodules of
free $\mathbb{Z}$-modules are free
because $\mathbb{Z}$ is a principal ideal
domain.

But when $\mathbb{Z}$ is replaced
by any ring $R \in CRing$ (commutative ring)
we get F' projective R-module!

→ That is why this structure is called
the projective model structure.

$R \in CRing$

COFIBRATIONS

(i) degreewise injections
(ii) coker $f = B/A$ is a degreewise projective R-module

ACYCLIC COFIBRATIONS

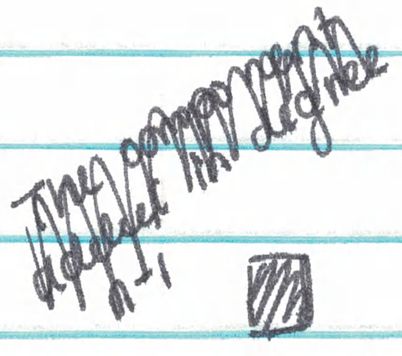
(i) and (ii') and additional condition
that $H(B/A) = 0$.

(this can be seen by replacing boundaries with horns)

B will be obtained from A as follows

$$B = A \oplus (\mathbb{Z}[n-1] \oplus \mathbb{Z}[n])$$
$$\Downarrow \qquad \Downarrow$$
$$1_{n-1} \qquad 1_n$$

and the differential
$(21)_n = 1_{n-1} + a$
$\in A_{n-1}$



Dold-Kan correspondence holds
in any idempotent complete
additive category

so we have a similar diagram

$$R[-] : {}_S\mathrm{Mod}_R \rightleftarrows \mathrm{Ch}^R_{\geq 0} : \Gamma$$

$$\mathrm{sSet} \xleftarrow{U} {}_S\mathrm{Mod}_R$$

composition

and $R[-] \to N$ (#) $\to C(-,R)$

is ... chains w/ coeffs in R

$\otimes_R : \mathrm{Ch}^R_{\geq 0} \times \mathrm{Ch}^R_{\geq 0} \longrightarrow \mathrm{Ch}^R_{\geq 0}$

$\otimes_R$ or product is a left Quillen bifunctor.

(recall, pushout product properties)

enough to verify pushout product properties for
ating cofibrations / acyclic cofibrations

simplification

the pushout product

$X_2) \square (Y_1 \to Y_2)$

$\to X_1 \otimes Y_2$

pout $\ulcorner$

$\to P$

$\to X_2 \otimes Y_2$

$\hookrightarrow$ 

pushout product!

NOTE! alternative generators for cofibrations

$\mathbb{Z}[n-1] \hookrightarrow \mathbb{Z}[n-1] \oplus \mathbb{Z}[n]$ (1)

and for acyclic cofibrations

$0 \hookrightarrow \mathbb{Z}[n-1] \oplus \mathbb{Z}[n]$ (1)

cobase changes of these will produce the same answer

$\to$

$\longrightarrow \quad \mathbb{Z}[n-1] \hookrightarrow \mathbb{Z}[n-1] \oplus \mathbb{Z}[n]$

for $X_1 \otimes Y_1$

$\mathbb{Z}[n_1-1] \quad \mathbb{Z}[n_2-1]$

sowe have

$$\mathbb{Z}[n_1+n_2-2] \hookrightarrow \mathbb{Z}[n_1+n_2-2] \oplus \mathbb{Z}[n_1+n_2-1]$$

(1)

ilarly, we can construct this pushout product diagram

$$\begin{array}{ccc} \mathbb{Z}[n_1+n_2-2] & \longrightarrow & \mathbb{Z}[n_1+n_2-2] \oplus \mathbb{Z}[n_1+n_2-1] \\ \downarrow & \ulcorner & \downarrow \\ \mathbb{Z}[n_1+n_2-2] \oplus \mathbb{Z}[n_1+n_2-1] & \longrightarrow & \mathbb{Z}[n_1+n_2-2] \oplus (\mathbb{Z} \oplus \mathbb{Z})[n_1+n_2-1] \oplus \mathbb{Z}[n_1+n_2] \end{array}$$

where $p = \mathbb{Z}[n_1+n_2-2] \oplus (\mathbb{Z} \oplus \mathbb{Z})[n_1+n_2-1]$

pushout

and pushout product is map $p \longrightarrow (11)$, and

colurnel of pushout product is just $\mathbb{Z}[n_1+n_2]$.

Homological Algebra [11/26]

Tor & Ext

Def! $A, B \in \mathrm{Mod}_R$ (R not $\mathbb{R}$!) $\quad R \in \mathrm{CRing}$

$\mathrm{Tor}_n^R(A,B) := H_n(A \overset{L}{\otimes}_R B)$ ← left derived functor

$\mathrm{Ext}^n_R(A,B) := H^n \hat{R}\mathrm{Hom}_R(A,B)$ ← derived Hom

Here, $\otimes_R : \mathrm{Ch}_R \times \mathrm{Ch}_R \longrightarrow \mathrm{Ch}_R$

$\mathrm{Hom}_R : \mathrm{Ch}_R \times \mathrm{Ch}_R \longrightarrow \mathrm{Ch}_R$

Recall, for noncommutative rings, we can derive

$\otimes_R : {}_Q\mathrm{Ch}_R \times {}_R\mathrm{Ch}_S \longrightarrow {}_Q\mathrm{Ch}_S$

$\mathrm{Hom}_R : {}_Q\mathrm{Ch}_R \times {}_S\mathrm{Ch}_R \longrightarrow {}_S\mathrm{Ch}_Q$

Ext & Tor know alot about derived tensor products and derived hom

Last time:

Tensor product is left Quillen bi-functor

$\otimes_R$ $\quad$ ℓQuillen

Hom is the adjoint of tensor product

Hom_R $\quad$ rQuillen (right Quillen bi-functor)

Thus, we can compute w/ cofibrant replacement

$A \overset{L}{\otimes}_R B = (\hat{Q}A) \otimes_R (\hat{Q}B)$

$\hat{Q}$ ← cofibrant replacement

wish

$\hat{R}\mathrm{Hom}(A,B) = \mathrm{Hom}(\hat{Q}A, B)$

↑ already fibrant (does not need to be derived)

(maps from B to 0 (abelian group terminal object) are always surjective)

Lemma "balancing Tor lemma"

$A \in Ch_R$
A is cofibrant

$B \xrightarrow{f} C$
f quasi-isomorphism ("quiso")

$\Rightarrow A \otimes f$ is a quasi-isomorphism ("quiso")

(tensoring w/ cofibrant objects is still quiso)

Proof Induction on A (continued below...)

Def M

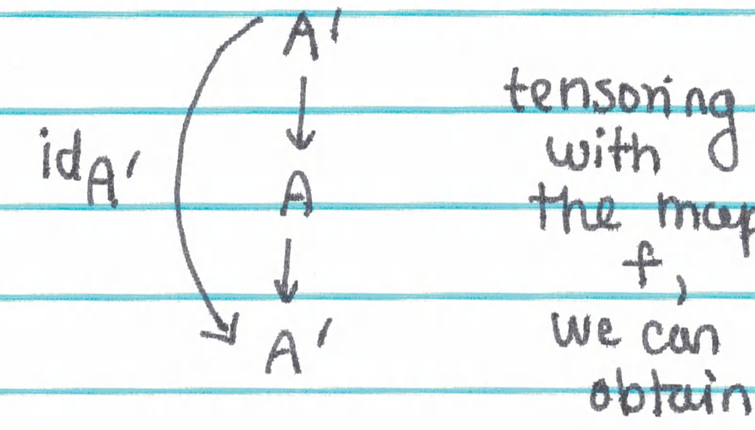
M is "flat" if $Cobj \otimes W \subset W$
(cofibrant objects ↑, weak equivalence)

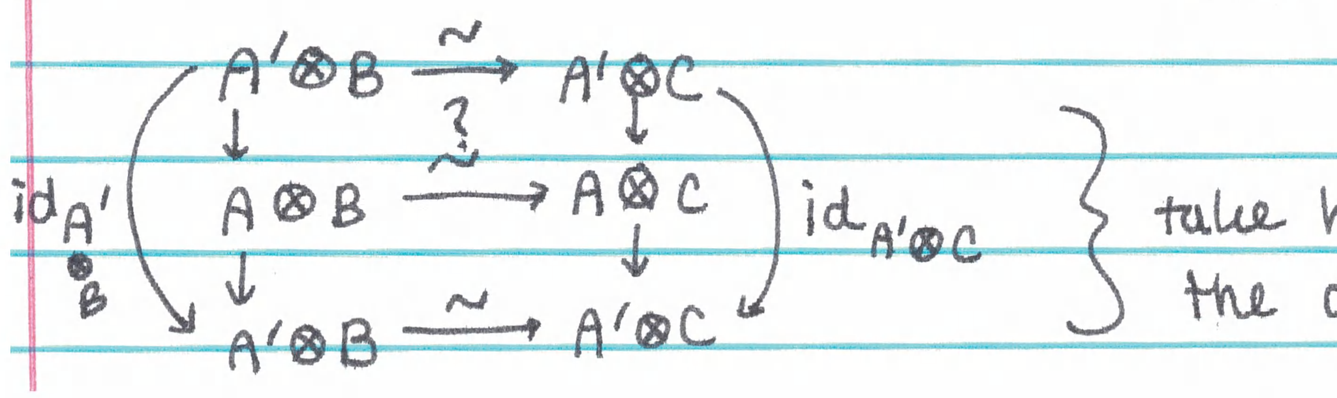
Ex $(Ch_R, \otimes)$
$(sSet, \times)$

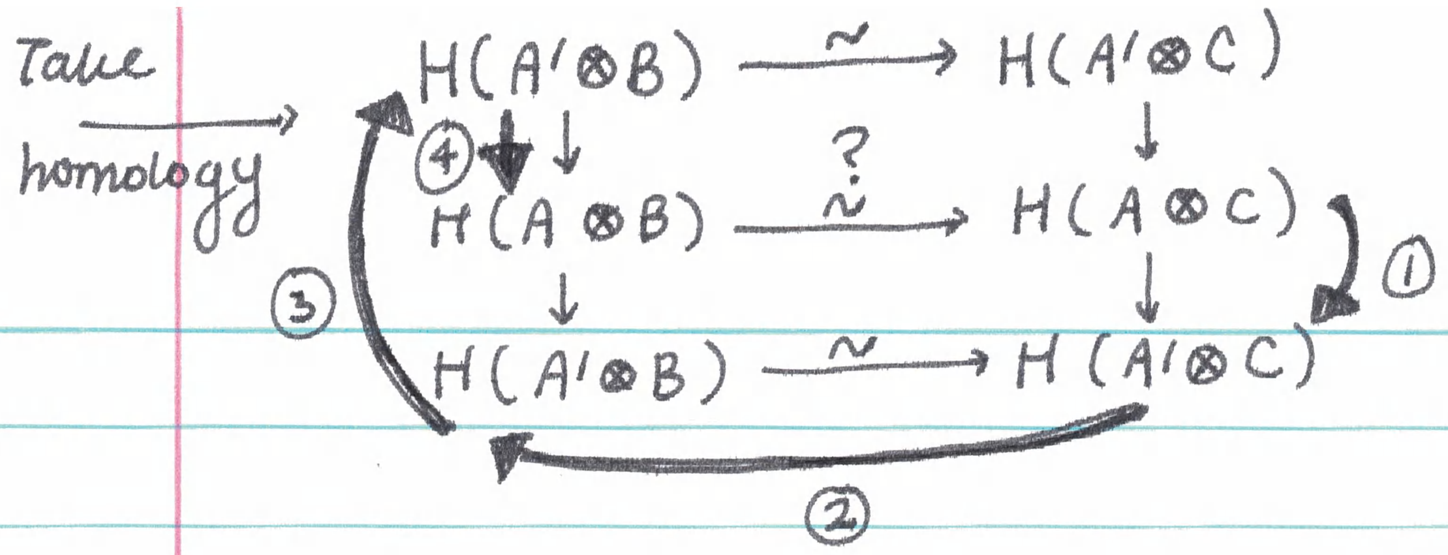
(proof continued...) Recall, A is a retract of a transfinite composition of cobase changes of

$R[n-1] \longrightarrow (R[n-1] \xleftarrow{id} R[n])$

(a) RETRACTS A is a retract of A' if you can "fit" them into the diagram







so, we want to show the middle map is an isomorphism

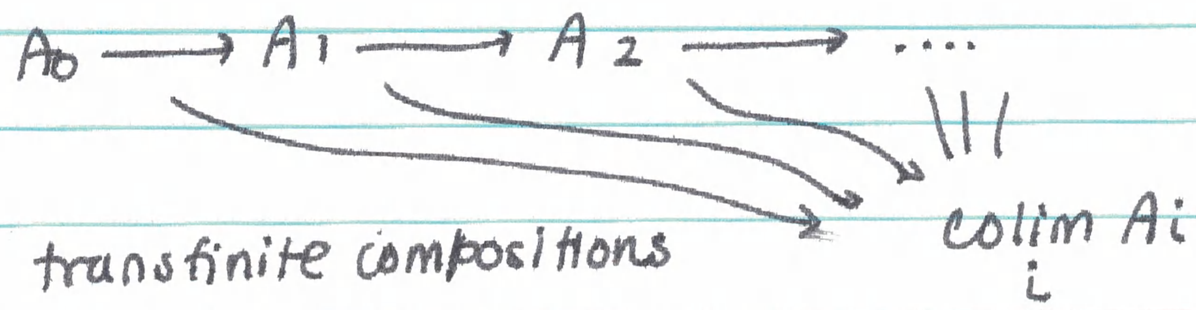
$$H(A\otimes B) \xrightarrow{\;?\;} H(A\otimes C)$$

we construct the inverse map

① → ② → ③ → ④

along the isomorphisms we know (top and bottom)

(b) TRANSFINITE COMPOSITIONS



If $A_i \otimes f$ is a quiso

we need to show $(\operatorname{colim}_i A_i)\otimes f$ is a quiso

this is equivalent

$$(\operatorname{colim}_i A_i)\otimes f \equiv \operatorname{colim}_i (A_i \otimes f)$$

so, we need to show $\operatorname{colim}_i (A_i\otimes f)$ is a quiso

allows fact that homology H functor preserves filtered colimits

Lemma For Ch$_R$

$$\begin{array}{ccc} \mathcal{D}_0 & \xrightarrow{\sim} & \mathcal{C}_0 \\ \downarrow & & \downarrow \\ \mathcal{D}_1 & \xrightarrow{\sim} & \mathcal{C}_1 \\ \downarrow & & \downarrow \\ \vdots & & \vdots \end{array}$$

lemma says that if $\mathcal{D}_i \xrightarrow{\sim} \mathcal{C}_i$ are quisos, then $\operatorname{colim}\mathcal{D} \longrightarrow \operatorname{colim}\mathcal{C}$ is a quiso

$$\operatorname{colim}\mathcal{D} = \mathcal{D}_\infty \longrightarrow \mathcal{C}_\infty = \operatorname{colim}\mathcal{C}$$

© cobase changes

We know
$A \otimes_R f$ quiso
we need to show
$A_1 \otimes_R f$ is a quiso

$$\begin{array}{ccc} R[n-1] & \longrightarrow & A_0 \\ \downarrow & & \downarrow \\ (R[n-1] \xleftarrow[id]{} R[n]) & \longrightarrow & A_1 \end{array}$$

homotopy pushout square

(NOTE FOR LATER: cofibrations produce homotopy pushout squares)

As a graded abelian group

$A_1 = A_0 \oplus R[n]$

∂

we know the map $A \otimes_R f : A_0 \oplus R[n] \longrightarrow$

$A_1 \otimes_R f : A_1 \otimes B \longrightarrow A_1 \otimes C$

$\| \qquad \|$

$A_0 \otimes B \oplus B[n] \longrightarrow A_0 \otimes C \oplus C[n]$

$A_0 \otimes f$ on the first summand; $f[n]$ on the second

and by Brute Force Method, we obtain result.

$R = \mathbb{Z}$

$A = \mathbb{Z}/m$ for some $m \in \mathbb{Z}$

compute Ext and Tor (where B is some other $\mathbb{Z}$-module, aka, abelian group)

$\hat{\mathbb{Q}} A = (\mathbb{Z} \xleftarrow[m]{} \mathbb{Z} \xleftarrow{} 0 \cdots)$
$\quad\quad 0 \qquad\quad 1$

quotient map ↓ ; multiplication by m

$A = (\mathbb{Z}/m \xleftarrow[0]{} 0 \xleftarrow{} 0 \cdots)$
$\quad\quad 0$

$$\begin{array}{ccc} \mathbb{Z} & \xleftarrow{m} & \mathbb{Z} \\ \downarrow q & \# & \downarrow q \\ \mathbb{Z}/m & \xleftarrow[0]{} & 0 \end{array} \longrightarrow$$

→ compute Homology induces isomorphism
(to verify weak equivalence)

We can now compute derived tensor product
(Balancing Tor Lemma)

- $A \overset{L}{\otimes} B = (\mathbb{Z} \xleftarrow{m} \mathbb{Z}) \otimes B = \mathbb{Z}\otimes B \xleftarrow{m} \mathbb{Z}\otimes B$
  when $B \in Ab$, $= B \xleftarrow[m]{} B$
  ($B \in Mod_{\mathbb{Z}}$)

tensor abelian group
with chain complex

~~$Tor_0(\mathbb{Z}/p \otimes B) \cong B/pB$~~
~~$Tor_1(\mathbb{Z}/p, B) \cong$~~

compute homology of chain complex $B \xleftarrow{m} B \leftarrow 0 \cdots$ just like in Topology!

$$\begin{cases} Tor_0(\mathbb{Z}/m, B) = B/mB = \mathbb{Z}/m \otimes B \\ Tor_1(\mathbb{Z}/m, B) = m\text{-torsion of } B \\ Tor_k(\mathbb{Z}/m, B) = 0, \ \forall k > 1 \end{cases}$$

$$\begin{cases} \hat{R}Hom = Hom(\mathbb{Z} \xleftarrow{m} \mathbb{Z}, B) \\ \quad = B \xleftarrow{m} B \\ \quad \deg -1 \quad \deg 0 \end{cases}$$

compute cohomology of chain complex $B \leftarrow B \leftarrow 0 \leftarrow \cdots$ just like in Topology!

$$\begin{cases} Ext^0(\mathbb{Z}/m, B) = m\text{-torsion} \\ \quad = Hom(\mathbb{Z}/m, B) \\ Ext^1(\mathbb{Z}/m, B) = B/mB \\ Ext^k(\mathbb{Z}/m, B) = 0, \ \forall k > 1 \end{cases}$$

...rious coincidence" occurs in degree 0 — true for all Rings R

e $\begin{cases} Tor_0(\mathbb{Z}/m, B) = \mathbb{Z}/m \otimes B & \text{(regular tensor product)} \\ Ext^0(\mathbb{Z}/m, B) = Hom(\mathbb{Z}/m, B) & \text{(regular hom)} \end{cases}$

ERCISE Prove $Tor_0 \cong \otimes_R$
VR (a) $Ext^0 \cong Hom_R$ (b) $Tor_k \cong 0$, $Ext^k \cong 0$ when $R = \mathbb{Z}$ and $k > 1$

Tor and Ext are a
Homology and cohomology Theory for R-modules
(they detect interesting characteristics about modules)

Example

$\mathbb{Z}[x] \longrightarrow \mathbb{Z}$ homomorphism of rings
$x \longmapsto 0$ turns $\mathbb{Z}$ into module over $\mathbb{Z}[x]$

compute tensor product $\mathbb{Z} \overset{L}{\otimes}_{\mathbb{Z}[x]} \mathbb{Z}$

cofibrantly replace $\mathbb{Z}$ ($\hat{\mathbb{Q}}\mathbb{Z}$) as a module over $\mathbb{Z}[x]$

$\hat{\mathbb{Q}}$

$\hat{\mathbb{Q}}\mathbb{Z}$ as a module over $\mathbb{Z}[x]$:

$$\mathbb{Z}[x] \xleftarrow{x} \mathbb{Z}[x] \xleftarrow{0} 0 \leftarrow \cdots$$

$\mathbb{Z}$ (Resolution of $\mathbb{Z}$)

so we get that

$$\mathbb{Z} \overset{L}{\otimes}_{\mathbb{Z}[x]} \mathbb{Z} = \mathbb{Z} \xleftarrow{0} \mathbb{Z} \leftarrow 0 \leftarrow \cdots$$

tensor over base ring just gives element of $\mathbb{Z}$

and $\operatorname{Tor}^{\mathbb{Z}[x]}_k(\mathbb{Z}, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & k=0, k=1 \\ 0, & k>1 \end{cases}$

Example $\mathbb{Z} \overset{L}{\otimes}_{\mathbb{Z}[x,y]} \mathbb{Z}$ "exciting example" (we have to make two replacements)

$\hat{\mathbb{Q}}(\mathbb{Z})$ as a module over $\mathbb{Z}[x,y]$

$$\mathbb{Z}[x,y] \xleftarrow{x \oplus y} \mathbb{Z}[x,y] \oplus \mathbb{Z}[x,y] \xleftarrow{-y \oplus x} \mathbb{Z}[x,y]$$

deg 0; deg 1; need elements of the form $-py \oplus px$