Mathematics 5324/5325 (Topology I/II)

Practice problems for the preliminary examination

1. Compute the homology and cohomology with coefficients in an arbitrary abelian group A, the cohomology ring with coefficients in an arbitrary commutative ring R, the Poincaré duality morphism, and the fundamental group for the following objects.

- An arbitrary triangulated picture.
- Orientable surface of genus $g \geq 0$, with $k \geq 0$ boundary components.
- Nonorientable surface of nonorientable genus $g \geq 1$, with $k \geq 0$ boundary components.
- The n-dimensional sphere, n-dimensional real projective space, and n-dimensional torus.
- The lens space $L(p, q)$.
- The infinite-dimensional sphere and real projective space.
- The complement of a link with n components in the 3-dimensional sphere.
- Bouquets (wedges) of any of the above.
- Products of any of the above.

2. For any collection of objects from the previous problem, determine which ones are weakly homotopy equivalent to each other, and which ones are not.

3. Determine which of the following topological spaces are (a) weakly homotopy equivalent and (b) homeomorphic to each other, and which ones are not:

- $[0, 1]$, $[0, 1)$, $(0, 1)$, S^1, S^n, $T^n = (S^1)^n$, $\mathbf{RP}^n$, $\prod_{0 \leq i \leq n} S^i$.

4. Can a subset of $\mathbf{R}^n$ be homeomorphic to S^n?

5. Set $S^k = \{x \in \mathbf{R}^{k+1} \mid \|x\| = 1\}$. Prove that there are no continuous maps $f\colon S^k \to S^{k-1}$ such that $f(-x) = -f(x)$.

6. Does the orientable surface of genus $g \geq 0$ admit a connected cover with $s \geq 0$ sheets? Express your answer in terms of g and s. Same question for nonorientable surfaces with g crosscaps.